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# Defining a new graph inefficiency measure for the Proportional Directional Distance Function and introducing a new Malmquist productivity index

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## Abstract

A natural multiplicative efficiency measure for the Constant Returns to Scale Proportional Directional Distance Function (pDDF) is derived, relating its associated linear program to that of the well-known output-oriented radial efficiency measurement model. Based on this relationship, a Malmquist index is introduced to show that, when it is based on the new efficiency measure associated with the pDDF, rather than on a radial efficiency measure associated with an oriented distance function, it becomes a Total Factor Productivity (TFP) index. This constitutes a new result, because heretofore the traditional Malmquist index has not been considered a TFP index. Additionally, a new decomposition of the Malmquist index is proposed that expresses productivity change as the ratio of two components, productivity change due to output change in the numerator and productivity change due to input change in the denominator. In an Appendix the efficiency measure is extended to include any returns to scale pDDF.

**Keywords.** Proportional Directional Distance Function, Efficiency Measure, Malmquist productivity index, Data Envelopment Analysis.

**JEL Codes.** C43, C61, O33

**Defining a new graph inefficiency measure for the Proportional Directional Distance Function and introducing a new Malmquist productivity index**

## 1. Introduction

Data envelopment analysis (DEA) is a mathematical programming, non-parametric technique commonly used to measure the relative performance of a set of homogeneous operating units, which use several inputs to produce several outputs. These operating units are usually called Decision Making Units (DMUs) in recognition of their autonomy in setting their input and output levels. DEA allows the estimation of technical efficiency scores[1] for all assessed units in a sample and, additionally, can be easily adapted for use in other contexts, e.g. when a panel of DMUs is available and the objective is to estimate productivity change over time.

In a general setting, a DMU is considered to be technically efficient if it is not possible to expand some of its outputs without requiring an increase in some of its inputs and/or to contract some of its inputs without requiring a reduction in some of its outputs. The potential for augmenting some outputs reflects output-oriented inefficiency, while the potential for reducing some inputs reflects input-oriented inefficiency. The choice between input or output orientation is contextual, depending on the situation being considered and, indeed, there exist in the DEA literature different measures created for use in these particular contexts. Moreover, when there is no specific reason to select either orientation, it is natural to resort to an approach that includes both output-expanding and input-saving components. In this case, the DEA models are usually known as graph or non-oriented in contrast to the oriented ones.

[1] The scores associated with any DEA model, that basically tell us if a specific DMU is efficient or inefficient, and, in the latter case, how far it is from being efficient, are always called "efficiency scores". In contrast, the measures - and associated programming models - that give rise to the efficiency scores are of two types: "efficiency measures", when the scores belong to the interval $]0,+1]$, and "inefficiency measures", when the scores belong to $[1,+\infty[$ .

In the case of oriented measures, input and output distance functions (Shephard, 1953, 1970) play the main role in most theoretical and empirical analysis. Each Shephard distance function has an associated efficiency score equal to the inverse of its value, that can be calculated, when dealing with a finite sample of units to be rated, through a specific linear program that corresponds to the so-called DEA CCR models (Charnes et al., 1978). The distinguishing feature of these two specific efficiency scores is, besides being oriented, that they are “multiplicative”, i.e., the projection of any point being evaluated is obtained by multiplying either its input or its output components by the corresponding input or output efficiency score. This is the key for the estimation of productivity change, as we will show below.

In the case of non-oriented measures things are rather different even though there exists a wide battery of tools. Examples of these are the Weighted Additive Model (Lovell and Pastor, 1995), the Directional Distance Function (DDF) (Chambers et al., 1998), the Range-Adjusted Measure (Cooper et al., 1999), the Bounded-Adjusted Measure (Cooper et al., 2011) and the Enhanced Russell Graph/Slacks-Based Measure (Pastor et al., 1999, Tone, 2001). In particular, two specific DDFs are related to Shephard’s input and output distance functions, when the analysis is forced to be oriented, and satisfy, at the same time, several interesting properties. For example, and as opposed to Shephard’s distance functions, a DDF implies a directional vector, thereby allowing for a flexible choice of orientation in both output and input dimensions. Additionally, being a distance function, the sign of the value of the DDF permits to identify whether a point of input-output space is inside, on, or outside the considered technology, a property that is useful, for example, when the aim is to measure productivity change over time. Also as opposed to Shephard’s distance functions, the efficiency score derived from any DDF is additive rather than multiplicative. Furthermore, Chambers et al. (1998) proved that the DDF under variable returns to scale (VRS) has a dual relationship with the profit function, allowing an additive decomposition of economic inefficiency into technical inefficiency (the value of the DDF) and price or allocative inefficiency in a natural way. Finally, in a DEA framework the DDF can be easily determined from a computational point of view since practitioners only need to resort to well-known and easy-to-use linear programming techniques.

In addition to the measurement of efficiency, DEA can be utilized for the estimation of productivity change. The usual approach, in this sense, is what we refer to as the Malmquist index (Malmquist, 1953), introduced by Caves et al. (1982), which is based on Shephard's output or input distance functions (but not both). The Malmquist index may be multiplicatively decomposed into components (see Färe et al. 1994 for an early attempt). More recently, and linked to the DDF, which is additive in nature, Chambers et al. (1996b) defined the Luenberger indicator, which is a difference-based indicator of directional distance functions that accounts for both output expansions and input contractions, and can be considered the first attempt to measure productivity change over time resorting to graph measures. Mirroring the Malmquist index, the Luenberger indicator also has been decomposed into components, beginning with Chambers et al. (1996b), but in an additive way. The efficiency scores associated with any DDF are additive

As it seems natural to expect, some attempts to compare and relate the Malmquist index to the Luenberger indicator have appeared in the literature since the introduction of the latter. Indeed, studying the relationship between these two measures has been the focus of much recent research. Boussemart et al. (2003), assuming Constant Returns to Scale (CRS), relate the Luenberger indicator based on the Proportional Directional Distance Function (pDDF), see Briec (1997), to the Malmquist index, based on Shephard's input distance function. They further relate certain Malmquist indexes with certain Luenberger indicators by means of first- and second-order approximations. Balk et al. (2008), assuming strong disposability of outputs and efficiency in each time period, establish a relationship between the Malmquist index and the Luenberger indicator, both based on output distance functions. Finally, Briec et al. (2012) provide a new relationship under the same assumptions as Balk et al. (2008) and prove a specific equality between the logarithm of the output-oriented Malmquist index and the output-oriented Luenberger indicator, resorting to the unit directional vector on the output side and defined on the logarithm of the original data.

Boussemart et al. (2003), Balk et al. (2008) and Briec et al. (2012) compare two worlds (multiplicative and additive), i.e. the Malmquist index and the Luenberger indicator, in a descriptive way. However, as far as we are aware, there has been no attempt to mix the two approaches, taking advantage of the good features of each. In this paper we do exactly that by defining, in a first stage, a new efficiency measure associated with the pDDF, which is used, in a second stage, for introducing and providing a novel decomposition of a new Malmquist productivity index.

In the first part of the paper we find an interesting relationship between the pDDF, in which the directional vector is defined as the actual input-output vector of the assessed DMU, and its input- and output-oriented versions under CRS. This relationship suggests how to associate a new multiplicative efficiency score with the pDDF. In the second part of the paper we define and decompose new Malmquist productivity indexes based on the new efficiency score. The indexes are the standard Adjacent Malmquist index, the Global Malmquist index (Pastor and Lovell, 2005) and the Biennial Malmquist index (Pastor et al., 2011), respectively. The reason for considering all three is to highlight the advantage of using any of them depending on the characteristics of the panel data being analyzed. As we will show, each of these indexes can be decomposed into the separate effects of output changes and input changes on productivity change, from which it follows directly that each can be interpreted as a Total Factor Productivity (TFP) index, which is a new, and perhaps controversial, finding for Malmquist indexes. Regarding this issue, Ang and Kerstens (2017) repeat O'Donnell's (2012) assertion that the CCD Malmquist index is not "multiplicatively complete," or, equivalently, is not a TFP index.[2] More to the point, they also claim that the Luenberger indicator is not "additively complete," and therefore not a TFP indicator. They also note that Briec and Kerstens (2004) showed that the Luenberger-Hicks-Moorsteen indicator is additively complete, and therefore a TFP indicator.

The remainder of the paper unfolds as follows. In Section 2 a new measure of technical efficiency associated with the pDDF is derived under the assumption of CRS. In Section 3 a new Malmquist index is introduced based on

[2] This has prompted Peyrache (2014) to refer to the index as the Malmquist technology index.

the new efficiency measure developed in Section 2. It is also shown that the new Malmquist index can be expressed as the ratio of an aggregate index of output change to an aggregate index of input change, and therefore interpreted as a productivity index. We extend this demonstration from the Adjacent Malmquist productivity index to its Global and Biennial versions, both of which satisfy the circularity property but which heretofore have not qualified as productivity indexes. Section 4 concludes with a summary of our findings, an extension to a hyperbolic efficiency measure as an alternative foundation for a Malmquist productivity index, and a suggestion of an important research agenda for which our new Malmquist productivity index may be appropriate. An Appendix extends the analysis of Section 2 to derive an inefficiency measure for the pDDF under any returns to scale.

## 2. Deriving an Inefficiency Measure for the pDDF under CRS

First of all, let us introduce the notation and assumptions that will be used in what follows. Additionally, some basic notions related to the production possibility set will be introduced.

Let us consider $n$ DMUs that use $m$ inputs to produce $s$ outputs, denoted as $\left(x_j, y_j\right)$, $j = 1,...,n$. It is assumed that $x_j = \left(x_{1j},...,x_{mj}\right) \in R_+^m$, $j = 1,...,n$, with at least one strictly positive component and $y_j = \left(y_{1j},...,y_{sj}\right) \in R_+^s$, $j = 1,...,n$, with at least one strictly positive component.[3]

The relative efficiency of each $DMU_0$ in the sample is traditionally assessed with reference to a technology, also called production possibility set, which is defined as follows:

[3] These requirements are all we need for working with a pDDF. For working with a general DDF, additional requirements are needed, the so-called “von Neumann conditions” considered in Färe and Primont (1995).They state, additionally, that no input or output is allowed to be zero for all units in the sample. In our particular case, if it fails, the corresponding input or output can be deleted without generating any type of inconsistency. Moreover, our two assumptions assure that we will keep at least one input and one output in the pDDF linear programming model. In the general case, if any of the two additional conditions fail, the considered DDF turns to be a program with infeasible restrictions whenever their associated directional vector components are nonzero unless its optimal value is 0, which means that any unit will be rated as efficient and, consequently, the considered DDF would not be useful at all, without any discriminatory power.

$$T = \{(x, y) \in R_+^{m+s} / \ x \text{ can produce } y\}. \tag{1}$$

Assuming CRS, $T$ is denoted as $T_c$, and can be empirically constructed from $n$ observations as follows (see Afriat, 1972, or Banker et al., 1984):

$$T_c = \left\{(x, y) \in R_+^{m+s} \middle/ \ x \geq \sum_{j=1}^{n} \lambda_j x_j, \ y \leq \sum_{j=1}^{n} \lambda_j y_j, \ \lambda_j \geq 0, j = 1, \ldots, n\right\}. \tag{2}$$

In order to determine in a DEA context the technical efficiency of a set of DMUs, several measures have been proposed in the literature. One of them, with relevant properties, is the directional distance function (DDF), defined as follows (Chambers et al., 1996a, 1998)[4]:

$$\begin{aligned}
\vec{D}\left(x_0, y_0; g_0^x, g_0^y\right) = \quad & \max \quad \beta_0 \\
& s.t. \\
& \qquad \sum_{j=1}^{n} \lambda_{j0} x_{ij} \leq x_{i0} - \beta_0 g_{i0}^x, \qquad && i = 1, \ldots, m \\
& \qquad \sum_{j=1}^{n} \lambda_{j0} y_{rj} \geq y_{r0} + \beta_0 g_{r0}^y, \qquad && r = 1, \ldots, s \\
& \qquad \lambda_{j0} \geq 0, \qquad && j = 1, \ldots, n
\end{aligned} \tag{3}$$

in which $g = \left(g_0^x, g_0^y\right) \in R_+^m \times R_+^s$, $g \neq 0_{m+s}$, is the considered directional vector and $\beta_0$ times $g$ measures the degree of technical inefficiency in the full input-output space of $DMU_0$, identified through the vector $\left(x_0, y_0\right)$ of inputs and outputs. The DDF projects any input and output vector onto the technological

---

[4] Luenberger (1992) introduced the concept of benefit function as a representation of the amount that an individual is willing to trade, in terms of a specific reference commodity bundle *g*, for the opportunity to move from a consumption bundle to a utility threshold. Luenberger also defined a so-called shortage function, which measures the distance in the direction of a vector *g* of a production plan from the boundary of the production possibility set. In other words, the shortage function measures the amount by which a specific plan is short of reaching the frontier of the technology. Later, Chambers et al. (1998) redefined the benefit function and the shortage function as efficiency measures, introducing to this end the DDF.

frontier in a pre-assigned direction given by the directional vector. For example, $g=\left(g_0^x,g_0^y\right)=\left(x_0,y_0\right)$ for assessing $DMU_0$ in the full input-output space corresponds to the definition of the proportional directional distance function (pDDF).

The pDDF was introduced by Briec (1997) and baptised in this way several years later by Boussemart et al. (2003). The pDDF, as any DDF, measures technical inefficiency. In this particular case, as Briec (1997) showed, $\vec{D}\left(x_0,y_0;g_0^x,g_0^y\right)=\beta_0^*$ at optimum is bounded by zero and one. It allows defining a corresponding technical efficiency measure simply as $1-\beta_0^*$, something that does not happen for a general directional vector $g=\left(g_0^x,g_0^y\right)$ since $\beta_0^*$ is not necessarily upper bounded[5]. Because DDFs are additive in nature, both the pDDF and its associated $1-\beta_0^*$ are also of the same nature. This is the reason why, in order to measure productivity change based on the pDDF, an (additive) Luenberger-type indicator was defined in the literature instead of resorting to a less well suited (multiplicative) Malmquist index.

However the Malmquist index continues to be the most widely used index to measure productivity change over time when panel data are available, despite the fact that, as currently formulated, it does not provide a measure of TFP change. For this reason, it seems interesting to define a new technical efficiency measure associated with the pDDF, and to convert it to a multiplicative measure. This last procedure will allow us to define a new Malmquist index based on the pDDF that does provide a measure of TFP change.

In order to deduce a (multiplicative) efficiency measure for the pDDF, we will show next that the linear program associated with the pDDF can be reformulated so as to obtain a linear program that coincides with the efficiency score of the radial CCR output-oriented model of Charnes et al. (1978)[6]. As a

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[5] It is not only the fact that $\beta_0^*$ is bounded but also that, in this particular case, $g=\left(x_0,y_0\right)$. Hence, $\beta_0^*$ works as a multiplicative efficiency score reducing inputs and augmenting outputs.

[6] Although we are aware that a similar reasoning will allow us to relate the pDDF to the input-oriented CCR model, and to the Shephard (1953) input distance function, we prefer our choice because most frequently Malmquist indexes are built upon output-oriented models.

by-product, we will relate the new efficiency measure to the corresponding Shephard output distance function (Shephard, 1970).

We focus our attention on the output-oriented CCR model (Charnes et al., 1978), or, equivalently, on the Shephard output distance function (Shephard, 1970), which in DEA has the formulation:

$$
\begin{aligned}
D\left(x_0,y_0\right)^{-1} = \quad & \max \ \phi_0 \\
& s.t. \\
& \sum_{j=1}^{n} \tau_{j0} x_{ij} \le x_{i0}, && i=1,\dots,m. \\
& \sum_{j=1}^{n} \tau_{j0} y_{rj} \ge \phi_0 y_{r0}, && r=1,\dots,s \\
& \tau_{j0} \ge 0, && j=1,\dots,n
\end{aligned}
\tag{4}
$$

At optimum, $\phi_0^* \ge 1$. Additionally, as is well known, the efficiency score of the CCR output-oriented model is exactly the inverse of the Shephard output distance function, i.e. $D\left(x_0,y_0\right)^{-1} = \phi_0^*$.

As for the pDDF[7], the technical inefficiency associated with this particular distance function is obtained from (3) after substituting $\left(g_0^x,g_0^y\right)$ by $\left(x_0,y_0\right)$ to obtain

$$
\begin{aligned}
pDDF\left(x_0,y_0\right) = \quad & \max \ \beta_0^p \\
& s.t. \\
& \sum_{j=1}^{n} \lambda_{j0} x_{ij} \le x_{i0} - \beta_0^p x_{i0} = \left(1-\beta_0^p\right) x_{i0}, && i=1,\dots,m \\
& \sum_{j=1}^{n} \lambda_{j0} y_{rj} \ge y_{r0} + \beta_0^p y_{r0} = \left(1+\beta_0^p\right) y_{r0}, && r=1,\dots,s \\
& \lambda_{j0} \ge 0, && j=1,\dots,n
\end{aligned}
\tag{5}
$$

It is worth mentioning at this point that $\left(1-\beta_0^{p*}\right)$ can be interpreted as the technical efficiency score associated with the inputs, whereas $\left(1+\beta_0^{p*}\right)$ may be understood as the technical efficiency score associated with the outputs.

[7] Assuming CRS, Zelenyuk (2014) has related $\beta_0^p$ to $\emptyset_0$ (and to its reciprocally related input-oriented measure). In particular, see his theorem at page 2187.

Since $\left(\beta_0^p, \lambda_0\right)$ with $\beta_0^p = 0$ and $\lambda_{j0} = 0$[8] is a feasible solution of model (5), it follows that $pDDF\left(x_0, y_0\right) = \beta_0^{p*} \geq 0$. We refer to $\beta_0^{p*}$ as the "proportional technical inefficiency" associated with the pDDF at $\left(x_0, y_0\right)$. It is well established that $\left(1-\beta_0^{p*}\right) > 0$ as a consequence of the assumptions made[9]. By dividing the input and output constraints by $\left(1-\beta_0^p\right)$ in (5) and replacing the *n* initial scalars $\lambda_{j0} \geq 0$ with the new set of related scalars $\tau_{j0} = \dfrac{\lambda_{j0}}{\left(1-\beta_0^p\right)} \geq 0$, we obtain the equivalent linear program:

$$
\begin{aligned}
pDDF\left(x_0, y_0\right) = \quad & \max \ \beta_0^p \\
& s.t. \\
& \sum_{j=1}^{n} \tau_{j0} x_{ij} \leq x_{i0}, && i = 1,\dots,m \\
& \sum_{j=1}^{n} \tau_{j0} y_{rj} \geq \frac{\left(1+\beta_0^p\right)}{\left(1-\beta_0^p\right)} y_{r0}, && r = 1,\dots,s \\
& \tau_{j0} \geq 0, && j = 1,\dots,n
\end{aligned}
\tag{6}
$$

Let us now reformulate model (6) through the change of variable $\Phi_0^p = \dfrac{\left(1+\beta_0^p\right)}{\left(1-\beta_0^p\right)}$. This change of variable[10] is reversible, in the sense that we can also express $\beta_0^p$ in terms of $\Phi_0^p$, i.e. $\beta_0^p = \dfrac{\Phi_0^p - 1}{\Phi_0^p + 1}$. Hence model (6) can be rewritten as follows:

---

[8] Since $DMU_0$ corresponds to a specific point of the considered finite sample, we know that there exists a certain $k \in [1, n]$ such that $\left(x_k, y_k\right) = \left(x_0, y_0\right)$.

[9] There exists an output constraint in (5) associated with a strictly positive output assuring that $\sum_{j=1}^{n} \lambda_j > 0$ and, consequently, any input constraint establishes that $0 < \sum_{j=1}^{n} \lambda_{j0} x_{ij} \leq \left(1-\beta_0^p\right) x_{i0}$, which finally implies that $\left(1-\beta_0^{p*}\right) > 0$.

[10] We have shown that $1 > \beta_0^{p*} \geq 0$. Consequently, $\Phi_0^{p*} \geq 1$, and all the terms for rewriting model (6) are well defined. By definition, $\beta_0^{p*} = 0$ is equivalent to $\Phi_0^{p*} = 1$. Let us observe that $\beta_0^{p*} = 0$ if, and only if, at least one input (or output) cannot be reduced (or augmented), i.e., if, and only if, the unit being rated belongs to the frontier of $T_c$.

$$\max \quad \frac{\Phi_0^p - 1}{\Phi_0^p + 1}$$

$$\begin{aligned}
&s.t. \\
&\sum_{j=1}^{n} \tau_{j0} x_{ij} \le x_{i0}, && i = 1,\dots,m \\
&\sum_{j=1}^{n} \tau_{j0} y_{rj} \ge \Phi_0^p y_{r0}, && r = 1,\dots,s \\
&\tau_{j0} \ge 0, && j = 1,\dots,n
\end{aligned} \qquad (7)$$

Consequently, the following relationship between the optimal value of $\Phi_0^p$ in (7), $\Phi_0^{p*}$, and the optimal value of $\beta_0^p$ in (6), $\beta_0^{p*}$, holds:

$$\Phi_0^{p*} = \frac{\left(1 + \beta_0^{p*}\right)}{\left(1 - \beta_0^{p*}\right)}. \qquad (8)$$

Model (7) is equivalent to model (6) and very similar to output-oriented model (4). In fact the only difference is its objective function. Let us now change model (7) slightly, by replacing its objective function.

$$\begin{aligned}
&\max \ \Phi_0^p \\
&s.t. \\
&\sum_{j=1}^{n} \tau_{j0} x_{ij} \le x_{i0}, && i = 1,\dots,m \\
&\sum_{j=1}^{n} \tau_{j0} y_{rj} \ge \Phi_0^p y_{r0}, && r = 1,\dots,s \\
&\tau_{j0} \ge 0, && j = 1,\dots,n
\end{aligned} \qquad (9)$$

Obviously new model (9) has exactly the same structure as the CCR output-oriented model, expressed above as model (4). In order to relate all the considered efficiency scores of the different considered models, starting with model (5), it would be highly interesting if variable $\Phi_0^p$ achieves the same optimal value in models (7) and (9). It is easy to realize that this is the case, because the objective function of (7) can be rewritten as $\frac{\Phi_0^p - 1}{\Phi_0^p + 1} = 1 - \frac{2}{\Phi_0^p + 1}$, which is a strictly increasing function in $\Phi_0^p$. Consequently, the optimal values of model (4) and of model (9), which are the same, are coincident with the optimal value of model (7). A first consequence is that $\Phi_0^{p*} = \phi_0^*$, and a second one is that model (9) gives rise to the same Malmquist index as model (4). An important property of model (9), which we exploit in Section 3, is that its efficiency score, $\Phi_0^{p*}$ is a function of the optimal value of the pDDF, $\beta_0^{p*}$, as given by equality (8).

The next result is a consequence of the equality $\Phi_0^{p*} = \phi_0^*$, using the original efficiency and inefficiency measures.

**Proposition 1.** The efficiency score associated with the output-oriented CCR model (4), $\phi_0^*$, and the proportional inefficiency under CRS in model (6), $\beta_0^{p*}$, are related as follows.

$$\phi_0^* = \frac{\left(1+\beta_0^{p*}\right)}{\left(1-\beta_0^{p*}\right)}. \tag{10}$$

Consequently we have been able to derive a new efficiency measure, $\Phi_0^{p*}$ **,** for the pDDF under CRS. The value of $\Phi_0^{p*}$ is coincident with that of $\phi_0^*$, the efficiency score of the output-oriented CCR model, which means that $\Phi_0^{p*}$ is a well-defined dimensionless number. Moreover, $\Phi_0^{p*}$ has exactly the same properties as $\phi_0^*$, which are listed next.

**Proposition 2.**

P1. $\Phi_0^{p*} \geq 1$[11].

[11] Property 1 tells us clearly that $\Phi_0^{p*}$ corresponds to an inefficiency measure that can be interpreted as an output-oriented inefficiency measure, even though it clearly corresponds to a graph model, namely pDDF. That means that, when comparing two units, the one that gets a larger efficiency score $\Phi_0^{p*}$ is more inefficient. As said before, it is also easy to get an input-oriented efficiency measure associated to the pDDF under CRS. For the sake of brevity we will not go further along this line**.**

P2. $\Phi_0^{p*}=1$ classifies unit $(x_0,y_0)\in R_+^{m+s}$ as efficient.

P3. $\Phi_0^{p*}$ is units invariant[12].

P4. $\Phi_0^{p*}$ is weakly monotonic[13].

Essentially, an additive efficiency score, $\beta_0^p$ in (5), which is a very common specification in the literature, has been converted to a proportional efficiency score, $\Phi_0^p$ in (7), which is new, and which enables us to equate the proportional efficiency scores $\Phi_0^p$ and $\phi_0$. Of crucial importance to our derivation of a new Malmquist productivity index, from (8) it follows that the new proportional efficiency score $\Phi_0^p$ combines both output and input orientations, whereas the traditional proportional efficiency score $\phi_0$ has an exclusively output orientation.

In the next section, we apply the new measure of technical efficiency for measuring productivity change over time.

## 3. New Malmquist productivity indexes associated with the pDDF

In this section we focus our attention on the definition of three new Malmquist productivity indexes based on the new efficiency measure $\Phi_0^{p*}$ associated with the pDDF in the previous sections. Given that the adjacent Malmquist index, as defined by Caves et al. (1982), is by far the most frequently used Malmquist index, let us start with it.

Regarding the notation for denoting observations in different periods of time, we will use hereinafter when needed $\left(x_j^h,y_j^h\right)$ with $h=t,t+1$. Additionally, $D^l\left(x_0^h,y_0^h\right)$ and $D_v^l\left(x_0^h,y_0^h\right)$ will denote the value of the Shephard output distance functions calculated under CRS and VRS, respectively, when unit $(x_0,y_0)$,

[12] "Units invariant" means that changing the units of measurement of any input or any output does not influence the value of $\Phi_0^{p*}$.

[13] "Weakly monotonic" means that if unit $(\hat{x}_0,\hat{y}_0)$ is located closer to the frontier than unit $(x_0,y_0)$, i.e., if $\hat{x}_0\le x_0$ and/or $\hat{y}_0\ge y_0$, then $\Phi_0^{p*}(\hat{x}_0,\hat{y}_0)\le\Phi_0^{p*}(x_0,y_0)$.

observed in period $h$, $h=t,t+1$, is projected onto the frontier of the reference technology in period $l$, $l=t,t+1$. In the same way, we will use the notation $\Phi_0^{p*l}\left(x_0^h,y_0^h\right)$, $\Phi_0^{p_v*l}\left(x_0^h,y_0^h\right)$, $\beta_0^{p*l}\left(x_0^h,y_0^h\right)$ and $\beta_0^{p_v*l}\left(x_0^h,y_0^h\right)$.[14]

The measurement of productivity change over time using frontier methods has claimed considerable attention in the literature that centers on the assessment of economic performance of DMUs. The most popular approach to evaluating productivity change, when market prices are not available, is the adjacent Malmquist index introduced by Caves et al. (1982) and popularized by Färe et al. (1994), who made it empirically tractable in a DEA framework, also allowing for the basic decomposition of productivity change into efficiency and technical changes. The adjacent Malmquist index is a ratio-based index that uses distance functions to represent technology and, in its most popular forms, adopt either an output expansion or an input contraction perspective. Here we are considering a completely different perspective, based on a graph measure.

Although the adjacent Malmquist index has been widely interpreted as a measure of productivity change over time, it has not been defined rigorously as a productivity change measure. In a multi-dimensional context, productivity is defined as the ratio of an aggregate output to an aggregate input. This definition naturally leads to productivity indexes that can be expressed in terms of the ratio of an output quantity index over an input quantity index. Accordingly, productivity change can be defined as the ratio of the corresponding productivity level at period *t*+1 to the corresponding productivity level at period *t*. Unfortunately, the adjacent Malmquist index cannot be expressed as a productivity index following the above definition (O'Donnell, 2012).

With the aim of overcoming this theoretical weakness of the adjacent Malmquist index, the Hicks–Moorsteen productivity index was introduced[15]. The Hicks-Moorsteen productivity index is defined as the ratio of an aggregate output quantity index to an aggregate input quantity index. More precisely, it measures the change in output quantities in the output direction using output

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[14] For the definition of the VRS inefficiency score and of the VRS multiplicative efficiency score associated with the pDDF, see the Appendix.
[15] The index was introduced by Bjurek (1996), who called it the Malmquist total factor productivity index, but was attributed by Diewert (1992) to Hicks (1961) and Moorsteen (1961).

distance functions and the change in input quantities in the input direction using input distance functions, instead of exclusively adopting an output perspective based on output distance functions or an input perspective based on input distance functions as the adjacent Malmquist index does.

However as we will show, the adjacent Malmquist index can be reformulated and interpreted as a TFP index.

### 3.1 The adjacent Malmquist productivity index

The traditional definition of the output-oriented Malmquist index is based on the following ratio that utilizes the CRS technology of period $l$, $l = t, t+1$, as reference (see Grifell-Tatjé and Lovell, 1995).

$$M^{l}\left(x_0^t, y_0^t, x_0^{t+1}, y_0^{t+1}\right) = \frac{D^{l}\left(x_0^{t+1}, y_0^{t+1}\right)}{D^{l}\left(x_0^t, y_0^t\right)}, \; l = t, t+1. \tag{11}$$

A value of $M^{l}\left(x_0^t, y_0^t, x_0^{t+1}, y_0^{t+1}\right) > 1$ has usually been interpreted as an improvement in productivity from period $t$ to $t$+1, $M^{l}\left(x_0^t, y_0^t, x_0^{t+1}, y_0^{t+1}\right) < 1$ as a decline and $M^{l}\left(x_0^t, y_0^t, x_0^{t+1}, y_0^{t+1}\right) = 1$ as stagnation, each from the perspective of the technology prevailing in period $l$.

We next introduce a new Malmquist productivity index by analogy with (11). Since the relation between the Shephard output distance function and its corresponding DEA efficiency score is $D\left(x_0, y_0\right)^{-1} = \phi_0^*$, the new Malmquist index based on the efficiency measure associated with the pDDF, $\Phi_0^{p*}$, and exploiting (9), would be (see Førsund, 1990):

$$M_{\Phi}^{l}\left(x_0^t, y_0^t, x_0^{t+1}, y_0^{t+1}\right) = \frac{\Phi_0^{p*l}\left(x_0^t, y_0^t\right)}{\Phi_0^{p*l}\left(x_0^{t+1}, y_0^{t+1}\right)}, \; l = t, t+1, \tag{12}$$

which may be decomposed using (8) as

$$M_{\Phi}^{I}=\frac{\dfrac{\left(1+\beta_0^{p*I}\left(x_0^t,y_0^t\right)\right)}{\left(1-\beta_0^{p*I}\left(x_0^t,y_0^t\right)\right)}}{\dfrac{\left(1+\beta_0^{p*I}\left(x_0^{t+1},y_0^{t+1}\right)\right)}{\left(1-\beta_0^{p*I}\left(x_0^{t+1},y_0^{t+1}\right)\right)}}=\frac{\dfrac{\left(1+\beta_0^{p*I}\left(x_0^t,y_0^t\right)\right)}{\left(1+\beta_0^{p*I}\left(x_0^{t+1},y_0^{t+1}\right)\right)}}{\dfrac{\left(1-\beta_0^{p*I}\left(x_0^t,y_0^t\right)\right)}{\left(1-\beta_0^{p*I}\left(x_0^{t+1},y_0^{t+1}\right)\right)}}=\frac{Y_{\Phi}^{I}\left(x_0^t,y_0^t,x_0^{t+1},y_0^{t+1}\right)}{X_{\Phi}^{I}\left(x_0^t,y_0^t,x_0^{t+1},y_0^{t+1}\right)}. \qquad (13)$$

By relationship (9), $M^I\left(x_0^t,y_0^t,x_0^{t+1},y_0^{t+1}\right)=M_{\Phi}^I\left(x_0^t,y_0^t,x_0^{t+1},y_0^{t+1}\right)$. However, the Malmquist index based on the non-oriented efficiency measure $\Phi_0^{p^*}$ linked to the pDDF allows us to go one step further and express the traditional Malmquist index as a TFP index, i.e. the ratio of an output quantity index to an input quantity index in the last term in expression (13). This is a new result.

From (13), it is possible to define the adjacent Malmquist index, as usual, as the geometric mean of $M^t\left(x_0^t,y_0^t,x_0^{t+1},y_0^{t+1}\right)$ and $M^{t+1}\left(x_0^t,y_0^t,x_0^{t+1},y_0^{t+1}\right)$:

$$M^{t,t+1}\left(x_0^t,y_0^t,x_0^{t+1},y_0^{t+1}\right)=\left[M^t\left(x_0^t,y_0^t,x_0^{t+1},y_0^{t+1}\right)\cdot M^{t+1}\left(x_0^t,y_0^t,x_0^{t+1},y_0^{t+1}\right)\right]^{1/2}. \qquad (14)$$

(14) may be decomposed into efficiency change (EC), technical change (TC) and scale efficiency change (SEC) as (see Ray and Desli, 1997, and Lovell, 2003):

$$M^{t,t+1}\left(x_0^t,y_0^t,x_0^{t+1},y_0^{t+1}\right)=\underbrace{\frac{D_v^{t+1}\left(x_0^{t+1},y_0^{t+1}\right)}{D_v^t\left(x_0^t,y_0^t\right)}}_{EC}\cdot\underbrace{\left(\frac{D_v^t\left(x_0^t,y_0^t\right)}{D_v^{t+1}\left(x_0^t,y_0^t\right)}\cdot\frac{D_v^t\left(x_0^{t+1},y_0^{t+1}\right)}{D_v^{t+1}\left(x_0^{t+1},y_0^{t+1}\right)}\right)^{1/2}}_{TC}\cdot$$
$$\underbrace{\left[\frac{D^t\left(x_0^{t+1},y_0^{t+1}\right)/D_v^t\left(x_0^{t+1},y_0^{t+1}\right)}{D^t\left(x_0^t,y_0^t\right)/D_v^t\left(x_0^t,y_0^t\right)}\cdot\frac{D^{t+1}\left(x_0^{t+1},y_0^{t+1}\right)/D_v^{t+1}\left(x_0^{t+1},y_0^{t+1}\right)}{D^{t+1}\left(x_0^t,y_0^t\right)/D_v^{t+1}\left(x_0^t,y_0^t\right)}\right]^{1/2}}_{SEC} \qquad (15)$$

By analogy with the adjacent Malmquist index, it is possible to define an adjacent Malmquist productivity index based on $\Phi_0^{p^*}$ as:

$$M_{\Phi}^{t,t+1}\left(x_0^t,y_0^t,x_0^{t+1},y_0^{t+1}\right)=\left[M_{\Phi}^t\left(x_0^t,y_0^t,x_0^{t+1},y_0^{t+1}\right)\cdot M_{\Phi}^{t+1}\left(x_0^t,y_0^t,x_0^{t+1},y_0^{t+1}\right)\right]^{1/2}, \qquad (16)$$

which may be also decomposed into efficiency change, technical change and scale efficiency change:

$$M_{\Phi}^{t,t+1} = \underbrace{\frac{\Phi_0^{p_V *t}\left(x_0^t, y_0^t\right)}{\Phi_0^{p_V *t+1}\left(x_0^{t+1}, y_0^{t+1}\right)}}_{EC_{\Phi}} \cdot \underbrace{\left(\frac{\Phi_0^{p_V *t+1}\left(x_0^t, y_0^t\right)}{\Phi_0^{p_V *t}\left(x_0^t, y_0^t\right)} \cdot \frac{\Phi_0^{p_V *t+1}\left(x_0^{t+1}, y_0^{t+1}\right)}{\Phi_0^{p_V *t}\left(x_0^{t+1}, y_0^{t+1}\right)}\right)^{1/2}}_{TC_{\Phi}} \cdot$$

$$\underbrace{\left[\frac{\Phi_0^{p_V *t}\left(x_0^{t+1}, y_0^{t+1}\right) \big/ \Phi_0^{p*t}\left(x_0^{t+1}, y_0^{t+1}\right)}{\Phi_0^{p_V *t}\left(x_0^t, y_0^t\right) \big/ \Phi_0^{p*t}\left(x_0^t, y_0^t\right)} \cdot \frac{\Phi_0^{p_V *t+1}\left(x_0^{t+1}, y_0^{t+1}\right) \big/ \Phi_0^{p*t+1}\left(x_0^{t+1}, y_0^{t+1}\right)}{\Phi_0^{p_V *t+1}\left(x_0^t, y_0^t\right) \big/ \Phi_0^{p*t+1}\left(x_0^t, y_0^t\right)}\right]^{1/2}}_{SEC_{\Phi}} \qquad (17)$$

Moreover, each term in (17) can be decomposed into an effect due to output change and an effect due to input change by exploiting (13). Additionally, it is worth mentioning that, although the traditional Malmquist productivity index and the new Malmquist index coincide in value (by (9)) to measure productivity change, the three terms in their corresponding decompositions can differ.

Summarizing, (13) and (17) show that the new adjacent Malmquist productivity index decomposes in two different ways. (13) shows that it decomposes into the ratio of an output quantity index to an input quantity index, enabling us to interpret it as a TFP index. (17) shows that it decomposes into the product of three economic drivers. Both decompositions exploit Proposition 1, which relates efficiency scores obtained from a pDDF to those obtained from an output-oriented radial model.

### 3.2 The global and biennial Malmquist productivity indexes

The adjacent Malmquist index is not circular, and its two base-period components can provide different measures of productivity change. This drawback is generally accepted in inter-temporal studies, but it poses serious problems in inter-spatial studies (Balk, 2008). For these reasons, alternative solutions have been proposed in the literature in order to overcome this downside.

One of these solutions is the global Malmquist index (Pastor and Lovell, 2005), which satisfies circularity but is not a total factor productivity index. The global Malmquist index is defined as in (11) but with underlying reference technology based on a global benchmark technology that uses all the data for all the DMUs and periods to be estimated. There being just one global benchmark technology, there is no need to take the geometric mean of adjacent indexes, as in (14) and (16). The global technology extends the contemporaneous technology $T_C$ in (2) to incorporate all time periods by way of $T_c^G$ = conv$\{T_c^1 \cup \ldots \cup T_c^T\}$, there being T time periods in the panel. A global Malmquist index is defined on $T_c^G$ as

$$M_c^G(x^t,y^t,x^{t+1},y^{t+1}) = \frac{D_c^G(x^{t+1},y^{t+1})}{D_c^G(x^t,y^t)}. \qquad (18)$$

This index is circular, gives a single measure, allows technical regress and is always feasible. Of course, in a natural way, it is possible to express (18) as a global Malmquist productivity index based on $\Phi_0^{p^*}$ in (7) and (8), following the same procedures as in Section 3.1. This index inherits the same desirable properties as the original global Malmquist index, most importantly the decomposability property (13), which allows us to refer to it as a global Malmquist productivity index.

Nevertheless, the global Malmquist productivity index obtained from (18) by basing it on $\Phi_0^{p^*}$ needs to be recomputed when a new time period is added to the data set. In contrast, the biennial Malmquist index (Pastor et al., 2011) overcomes this problem by using as a reference technology one constructed from observations in periods $t$ and $t+1$, so that $T_c^B$ = conv$\{T_c^t, T_c^{t+1}\}$ and a biennial Malmquist index is defined on $T_c^B$ as

$$M_c^B(x^t,y^t,x^{t+1},y^{t+1}) = \frac{D_c^B(x^{t+1},y^{t+1})}{D_c^B(x^t,y^t)}. \qquad (19)$$

Again there is just one technology, $T_c^B$, and there is no need to resort to the geometric mean convention. It is also possible to express (19) as a biennial Malmquist productivity index based on $\Phi_0^{p^*}$ in (7) and (8), following the same procedures as in Section 3.1. This index also inherits the same desirable properties as the original biennial Malmquist index, including the decomposability property (13) that allows us to refer to it as a biennial Malmquist productivity index. The relationship of (19), with overlapping biennial technologies, to DEA window analysis (Cooper et al. 2000) should be apparent.

## 4. Conclusions

The CCD Malmquist index, in any of its adjacent, global or biennial versions, has heretofore not been interpreted as a productivity index because it has not been expressed as the ratio of a well-behaved output quantity index to a well-behaved input quantity index. Our first contribution is to demonstrate that this inability is due entirely to the standard practice of basing the index on efficiency measures derived from Shephard's radial distance functions, either output-oriented or input-oriented but not both. It is logically impossible to base a productivity index, which incorporates both output change and input change, on distance functions with a single orientation. Our second contribution is to demonstrate that if the index is based on a new efficiency measure associated with a proportional directional distance function it can be interpreted as a productivity index. The intuition underlying our second contribution derives from the fact that a proportional directional distance function simultaneously scales outputs up and inputs down, which leads naturally to a productivity index expressed as the ratio of an output quantity index to an input quantity index.

We have used a proportional directional distance function to express the CCD Malmquist index as a productivity index, but the door is open to discover other functions that can lead to the same result. To provide but one example, the hyperbolic graph efficiency measure proposed by Färe et al. (1985) generates a CCD Malmquist total factor productivity index because it satisfies two basic conditions. First, it is simultaneously output-oriented and input-oriented. And secondly, we are able to transform the original formulation of that model to obtain new "equivalent" models until we end up with the classical CCR output-oriented model, which is obviously a multiplicative model. The second condition is absolutely essential for obtaining a TFP index.

We conjecture that our results can be applied to a situation in which an input vector is used to produce a desirable output vector and at the same time to generate a vector of environmentally undesirable byproducts. Shen et al. (2017) use a proportional directional distance function to estimate green productivity change. They compress three quantity vectors to two, by conditioning on the input vector, and they define green productivity change in terms of increases in desirable outputs and decreases in undesirable byproducts. Du et al. (2018) and Li and Wu (2018) follow a similar procedure, and they also insert the proportional directional distance function into a CCD Malmquist index to obtain a Malmquist-Luenberger index incorporating undesirable byproducts.

**Appendix. Defining an inefficiency measure for the pDDF under any returns to scale.**

The linear program associated with the pDDF under non-constant returns to scale is formulated by adding to model (5) a single additional constraint involving only the lambda variables. There are three possibilities: variable returns to scale (VRS), non-increasing returns to scale (NIRS) and non-decreasing returns to scales (NDRS). The first is most frequently used, because it provides the foundation for measuring scale efficiency. The constraint to be added to model (5) under VRS is $\sum_{j=1}^{n} \lambda_j = 1$, also known as the "convexity constraint"[16]. Having in mind the usual decomposition of the Malmquist index, which is addressed in Section 3, the VRS case is developed in what follows.

The model associated with the pDDF under VRS is

$$
\begin{aligned}
pDDF_v\left(x_0, y_0\right) = \quad & \max \ \beta_0^{p_v} \\
& s.t. \\
& \sum_{j=1}^{n} \lambda_{j0} x_{ij} \le x_{i0} - \beta_0^{p_v} x_{i0} = \left(1 - \beta_0^{p_v}\right) x_{i0}, \forall i \\
& \sum_{j=1}^{n} \lambda_{j0} y_{rj} \ge y_{r0} + \beta_0^{p_v} y_{r0} = \left(1 + \beta_0^{p_v}\right) y_{r0}, \forall r \\
& \sum_{j=1}^{n} \lambda_{j0} = 1 \\
& \lambda_{j0} \ge 0, \qquad \forall j
\end{aligned}
\tag{A1}
$$

In general, $\beta_0^{p_v*}$ of (A1) will differ from $\beta_0^{p*}$ of (5) due to the presence of the convexity constraint, and since the feasible set of (A1) is a subset of the feasible set of (5), we can assert that $0 \le \beta_0^{p_v*} \le \beta_0^{p*}$. Hence the efficiency scores satisfy $1 - \beta_0^{p_v*} \ge 1 - \beta_0^{p_v*} > 0$.

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[16] The constraint to be added under NIRS is $\sum_{j=1}^{n} \lambda_j \le 1$, and under NDRS $\sum_{j=1}^{n} \lambda_j \ge 1$.

Since we are going to propose a new VRS model that will allow us to measure scale efficiency along the direction associated with the pDDF, we do not consider the idea of obtaining a possible connection between model (A1) and the VRS version of the output-oriented radial model, known as the output-oriented BCC model (Banker et al., 1984). Alternatively, our proposal is simply to define directly an efficiency measure for the pDDF under VRS, by analogy with the efficiency measure derived under CRS in Section 2, after solving model (A1) and determining its optimal value $\beta_0^{p_V *}$.

**Definition A1.**

The efficiency measure $\Phi_0^{p_V *}$ associated with the pDDF under VRS, based on the optimal value of model (A1), $\beta_0^{p_V^*}$, is defined as

$$\Phi_0^{p_V *} = \frac{1+\beta_0^{p_V *}}{1-\beta_0^{p_V *}}. \tag{A2}$$

Since we have shown that $1-\beta_0^{p_V *} \geq 1-\beta_0^{p^*} > 0$, $\Phi_0^{p_V *}$ is a well-defined dimensionless efficiency measure. Curiously enough, we are able to prove that $\Phi_0^{p_V *}$ under VRS satisfies exactly the same properties as $\Phi_0^{p^*}$ under CRS.

**Proposition A1.** $\Phi_0^{p_V *}$ satisfies the following properties.

P1. $\Phi_0^{p_V *} \geq 1$.

P2. $\Phi_0^{p_V *} = 1$ classifies unit $(x_0, y_0) \in R_+^{m+s}$ as efficient.

P3. $\Phi_0^{p_V *}$ is units invariant.

P4. $\Phi_0^{p_V *}$ is weakly monotonic.

**Proof.**

**P1.** Since Footnotes 9 and 10 are also valid for model (A1), we conclude that $0 \le \beta_0^{p_v*} < 1$. Consequently, $1+\beta_0^{p_v*} \ge 1$ and $0 \ge -\beta_0^{p_v*} > -1$ or, equivalently, $1 \ge 1-\beta_0^{p_v*} > 0$. Therefore, $\Phi_0^{p_v*} = \dfrac{1+\beta_0^{p_v*}}{1-\beta_0^{p_v*}} \ge 1+\beta_0^{p_v*} \ge 1$.

**P2.** $\Phi_0^{p_v*} = 1$ is equivalent to saying that $\beta_0^{p_v*} = 0$, which in turn holds when at least one input or one output of the point being rated cannot be reduced at all by the directional vector, i.e., the point belongs to the frontier.

**P3.** In the VRS pDDF, the restrictions associated with inputs and outputs in model (A1) are the same as the corresponding restrictions in model (5), while the added convexity constraint is independent of inputs and outputs. Hence, since model (5) is units invariant, which means that none of its constraints changes after performing a change of units of measurement of the corresponding input or output, the desired result follows.

**P4.** Let us consider two different monotonically related units $(x_0, y_0)$ and $(\hat{x}_0, \hat{y}_0)$, i.e., $\hat{x}_0 \le x_0$ and $\hat{y}_0 \ge y_0$. Let us prove that $\Phi_0^{p_v*}(\hat{x}_0, \hat{y}_0) \le \Phi_0^{p_v*}(x_0, y_0)$ which means exactly that $\Phi_0^{p_v^*}$ is a weakly monotonic efficiency measure. Since the transit from $(\hat{x}_0, \hat{y}_0)$ to $(x_0, y_0)$ can be performed as a sequence of at most *m*+*s* steps, let us focus on the k-th input value that is different, let's say $\hat{x}_{k0} = x_{k0} - p_k, p_k > 0$, where $k = 1,...,m$, and let us compare $(\hat{x}_0, \hat{y}_0)$ with the intermediate unit $(\tilde{x}_0, \tilde{y}_0) = \left(\hat{x}_{10}, ..., \hat{x}_{(k-1)0}, x_{k0}, \hat{x}_{(k+1)0}, ..., \hat{x}_{m0}, \hat{y}_0\right)$. Let us start showing that the optimal solution of program (A1) when rating unit $(\hat{x}_0, \hat{y}_0)$, $\left(\hat{\beta}_0^{p_v*}, \hat{\lambda}_0^*\right)$, is a feasible solution of program (A1) when rating unit $(\tilde{x}_0, \tilde{y}_0)$, and, therefore, since (A1) maximizes $\tilde{\beta}_0^{p_v*}$, we conclude that $\hat{\beta}_0^{p_v*} \le \tilde{\beta}_0^{p_v*}$, or equivalently, $\Phi_0^{p_v*}(\hat{x}_0, \hat{y}_0) \le \Phi_0^{p_v*}(\tilde{x}_0, \tilde{y}_0)$, the desired result. Let us consider the constraint associated with input *k* in model (A1) when rating $(\hat{x}_0, \hat{y}_0)$ and realize that $\hat{x}_{k0} = x_{k0} - p_k < x_{k0}$. It follows that $\sum_{j=1}^{n} \hat{\lambda}_{j0}^{*} \hat{x}_{kj} \leq \left(1 - \hat{\beta}_0^{p_v *}\right) \hat{x}_{k0} < \left(1 - \hat{\beta}_0^{p_v *}\right) x_{k0}$, and considering that all the other constraints of model (A1) are just the same, we can assert that $\left(\hat{\beta}_0^{p_v *}, \hat{\lambda}_0^{*}\right)$ is feasible when rating unit $\left(\tilde{x}_0, \tilde{y}_0\right)$, which concludes the proof. ■

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