Centre for Efficiency and Productivity Analysis

Working Paper Series
No. WP03/2018

Some Mathematical and Historical Clarification on Aggregation in Efficiency and Productivity Analysis and Connection to Economic Theory

Valentin Zelenyuk

Date: June 2018

School of Economics
University of Queensland
St. Lucia, Qld. 4072
Australia

ISSN No. 1932 - 4398

Some Mathematical and Historical Clarifications on Aggregation in Efficiency and Productivity Analysis and Connection to Economic Theory

Valentin Zelenyuk*

June 19, 2018

Abstract

In this note we clarify a few important aspects about aggregation in efficiency and productivity analysis. By doing so we also sketch a brief historical map on how the area of aggregation in efficiency and productivity analysis has been developing to where it is now and its connection to classic studies in economic theory.

Keywords: Aggregation, Scale Efficiency, Industry Efficiency, Duality.

JEL Classification: D24, C43, L25

Acknowledgment: I thank Rolf Färe and Erwin Diewert for their fruitful feedback.

*School of Economics and Centre for Efficiency and Productivity Analysis (CEPA) at The University of Queensland, Australia; *address*: 530, Colin Clark Building (39), St Lucia, Brisbane, Qld 4072, AUSTRALIA; e-mail: v.zelenyuk@uq.edu.au; tel: + 61 7 3346 7054;

1 Introduction

The question of measuring industry (or sub-industry or more generally a group or aggregate) efficiency and productivity is a very important question often faced both by theorists as well as practitioners. This question of aggregation was approached in the very early works in the area, most prominently by Farrell (1957) who, in his seminal work, introduced a new concept which he dubbed as the 'Structural Efficiency of an Industry'.

This aggregation question was then also scrutinized by many scholars from different perspectives, such as Førsund and Hjalmarsson (1979), Li and Ng (1995), Blackorby and Russell (1999), Färe and Zelenyuk (2003), Briec et al. (2003), Li and Cheng (2007) Färe et al. (2004), Färe et al. (2004), Färe and Zelenyuk (2005), Bogetoft and Wang (2005), Zelenyuk (2006), Mussard and Peypoch (2006), Färe and Zelenyuk (2007), Cooper et al. (2007a), Simar and Zelenyuk (2007), Nesterenko and Zelenyuk (2007), Färe et al. (2008), Pachkova (2009), Kuosmanen et al. (2010), Raa (2011), Mayer and Zelenyuk (2014a,b, 2017), Färe and Karagiannis (2014), Karagiannis (2015), Karagiannis and Lovell (2015), to mention a few.

It is worth noting that the biggest forum for this important sub-area of research was, as for many key works in the area of efficiency and productivity analysis, provided by the OR literature, and most prominently by the European Journal of Operational Research. The primary goal of this article is to clarify several key aspects that appears to have been overlook in the current literature.

Fundamentally, most of this literature is grounded on the seminal essays of Koopmans (1957) and, at least indirectly, inspired and influenced by other classical works in economic theory such as Debreu (1951), and especially works on aggregation by Gorman (1953, 1959), Theil (1954), Diewert (1974b, 1978, 1980, 1983) and related works in econometrics (e.g., see reviews by Stoker (1993) and Blundell and Stoker (2005) and references therein).

There is also an interesting connection to the game theoretic literature that has been largely overlooked by the scholars in productivity and efficiency analysis, yet appears to have a great potential for future research. The secondary, yet also as important, goal of this article therefore is to point out the connections of the aggregation in the productivity and efficiency literature to the economic theory and game theory literature.

2 Definitions of Aggregate Technologies

In a nutshell, the main premise of aggregation theories in economics is that an aggregation must have economic rational: Besides mathematical coherence, a proposed relationship between disaggregated objects and their aggregates should reflect some economic meaning, in the sense that the disaggregate objects are adequately represented by their aggregates, preserving some (though not always all) fundamental aspects, justified through some economic theory reasoning. Typically, this is achieved by postulating some economic principles, axioms, assumptions and then deriving the aggregation results from them. A simple example that illustrates the point is the case of aggregation of efficiency scores measures between 0 and 1, regardless of an economic weight (e.g., share on the market)—simple averaging there may yield radically different conclusions than averaging that accounts for an economic weight of individuals. Different economic weights can also yield different conclusions and therefore the key is to derive various systems of aggregating weights that can be justified by certain assumptions or beliefs.

A common way to approach the aggregation question in efficiency and productivity analysis is to start with a definition of what is the aggregate technology and go from there, defining various objects (correspondences, functions, efficiency measures, etc.) based on such aggregate technology and then deriving relationships to their disaggregate analogues. Different definitions of aggregate technology may (and typically do) result in different aggregate measures of efficiency and of productivity and therefore it is imperative to clearly understand the differences (and the resulting consequences) and the relationships between these different definitions.

It is natural to define aggregate technology as a summation of individual technologies. A very general way to do so formally is to assume that the aggregate technology can be characterized as

$$\overline{\Psi}_n := \sum_{\oplus k=1}^{n} \Psi^k \tag{2.1}$$

where Ψ^k is a technology set characterizing production possibilities of an individual (e.g., firm) $k \in \{1, ..., n\}$ being aggregated. While each particular industry or even a firm may have its own Ψ^k, in general terms it is often formally defined in generic terms as

$$\Psi^k = \left\{(x, y) \in \mathbb{R}_+^N \times \mathbb{R}_+^M : \text{ firm } k \text{ can produce } y \text{ from } x\right\}, \tag{2.2}$$

where $x = (x_1, \ldots, x_N)' \in \mathbb{R}_+^N$ is a vector of N inputs that a firm uses to produce a vector of M outputs, denoted by $y = (y_1, \ldots, y_M)' \in \mathbb{R}^M_+$.

Importantly, note that the summation in (2.1) is not the usual summation used for scalars, vectors or matrices, but the summation for sets, i.e., Minkowski summation (we have indicated this by $\oplus$ sign). To be more precise, recall that Minkowski summation of any two sets A and Z is defined as[1]

$$A \oplus Z := \{a + z \ : \ \forall a \in A, \ \forall z \in Z\}.$$

The idea of using Minkowski summation for aggregation of technology sets in production theory context can be found as early as Debreu (1951) and Koopmans (1957). In the modern efficiency analysis context, the use of this concept goes back to at least Li and Ng (1995) and Blackorby and Russell (1999), and has been more recently investigated by Briec et al. (2003), Nesterenko and Zelenyuk (2007), Färe et al. (2008), Peyrache (2013, 2015), Mayer and Zelenyuk (2014a,b, 2017), to mention a few.

An alternative (yet closely related) approach is to apply Minkowski summation (over k) for the output sets, defined as $P^k(x) := \{y \ : \ (x, y) \in \Psi^k\}$, rather than the entire technology, i.e.,

$$\overline{P}_n(x^1, ..., x^n) := \sum_{\oplus k=1}^{n} P^k(x^k) \tag{2.3}$$

which also gives an aggregate technology characterization, although this does not allow for reallocation of inputs across individuals being aggregated (Färe and Zelenyuk (2003)).

Yet another approach is to apply Minkowski summation (over k) just for the input requirement sets, defined as $L^k(y) := \{x \ : \ (x, y) \in \Psi^k\}$, i.e.,

$$\overline{L}_n(y^1, ..., y^n) := \sum_{\oplus k=1}^{n} L^k(y^k), \tag{2.4}$$

which is also an aggregate technology, yet this does not allow for reallocation of output plans across individuals being aggregated (Färe et al. (2004)).

The last two approaches help to derive and justify theoretically a generalized version of the concept of 'Structural Efficiency of an Industry' proposed by Farrell, for the output orientation and input orientation, respectively, and their multi-output-multi-input analogues and which then relate them to the aggregate efficiency measures defined

[1]E.g., see Krein and Smulian (1940), Schneider (1993) and references therein.

on the more general aggregate technology, as in (2.1). The resulting aggregation approaches have systems of weights (and the aggregating functions) that are justified by and derived from economic principles as well as possess intuitive economic meaning.

3 Properties of the Aggregate Technology

An important issue to note for general context where Ψ^k varies with k, is that it is critical to assume that none of Ψ^k is an empty set. This is particularly important because when n is large and especially if $n \to \infty$ (as in the case we will consider below) where such an event may not be totally unlikely, unless explicitly assumed otherwise. Indeed, if it happens that at least for one of k we have $\Psi^k = \emptyset$, the theory of Minkowski summation implies that we also get $\overline{\Psi}_n = \emptyset$. Intuitively, if absolutely infeasible technology is combined (in the sense of Minkowski summation) with any feasible technologies, the aggregate result is still an absolutely infeasible technology. A way to rule out this degenerate case (as well as many other peculiar and inconvenient cases) is to assume that each technology satisfies a list of regularity axioms of production theory, e.g., such as those advocated by Shephard (1953) and more recently in Färe and Primont (1995), and we do so here.

It is natural that the properties of the aggregate technology are influenced and in some sense inherited from the properties of individual technologies being aggregated, yet this subject appears to have never been fully clarified precisely with respect to the axioms of production theory and so the goal of this section is to clarify this aspect.

While there is apparently no list of axioms (or regularity conditions) of production theory to which all researchers unanimously agree as necessary and sufficient (mainly because both necessity and sufficiency is often dictated by a context under study), this list usually includes the following axioms (for all k):

A1: ‘Producing nothing is feasible’, i.e.,

$$(x, 0_M) \in \Psi^k,\ \forall x \in \mathbb{R}^N_+. \tag{3.1}$$

A2: ‘No free lunch’, i.e.,

$$(0_N, y^k) \notin \Psi^k,\ \ \forall y^k \geq 0_M.$$

A3: The output correspondence $P^k(x^k)$ is bounded $\forall x^k \in \mathbb{R}^N_+$.

A4: The technology set Ψ^k is closed.

A5: Outputs and inputs are freely (strongly) disposable, i.e.,

$$(x^0, y^0) \in \Psi^k \Longrightarrow (x, y) \in \Psi^k, \;\; \forall y \leqq y^0, \; \forall x \geqq x^0, \; y \geq 0$$

A6: (As an alternative to A5) Outputs and inputs are weakly (or more precisely, proportionately) disposable, i.e.,

$$(x^0, y^0) \in \Psi^k \Longrightarrow (\lambda x, \theta y) \in \Psi^k, \;\; \forall \lambda \in [1, \infty), \; \forall \theta \in (0, 1],$$

Intuitively, A1 means that a firm k can choose an option to produce nothing even if they use some positive inputs. While very simple, this type of singularity condition is a very important feature of economic reality, postulating that a firm has an option or a right to have no output even if it used some inputs.[2] This axiom also has an important implication to the aggregation theory as will be clarified in the next section.

A2 states that nothing (zero input) cannot produce something (any positive output). A3 basically says that only finite amounts of output can be produced from finite amounts of inputs. A4 is mainly a technical regularity condition ensuring that all the sequences in Ψ^k have accumulation points inside Ψ^k and, together with A3, also implies that $P^k(x^k)$ is a compact set $\forall x^k \in \mathbb{R}^N_+$, thus guaranteeing existence of an optima of a continuous function on $P^k(x^k)$ and other bounded regions of Ψ^k. A5 says that, for a given technology, if it was possible to produce some level of outputs with certain amounts of inputs (e.g., because the extra outputs can be freely disposed), then it should be possible to produce any lower amounts of outputs with the same inputs, or the same amounts of outputs but with greater amounts of inputs (e.g., because the extra inputs can be freely disposed). This axiom is quite natural, yet not always the case because some outputs (e.g., undesirable outputs) or some inputs (congesting inputs) may not be disposed freely but only in a combination with some other outputs or inputs and A6 is one of the most popular variations of weak disposability that allows the modeling of such phenomena, restricting the disposability to be in equiproportionate manner.

When such axioms are imposed on the individual technologies, what will they mean for the aggregate technologies, defined in (2.1), (2.3) and (2.4)? This question is clarified in the next proposition.

[2]This axiom also allows for cases where a certain minimal level of input is required to get a positive output, yet in that case the lower inputs will still belong to the technology sets when in combination with zero output.

Proposition. For any $i \in \{1, 2, 3, 4, 5, 6\}$, if the axiom A$i$ is satisfied by Ψ^k, $\forall k$ then so do the aggregate technologies based on (2.1), (2.3) and (2.4). However, if the axiom Ai is not satisfied by Ψ^k for at least one of k being aggregated, then this may prevent the entire aggregate technologies based on (2.1), (2.3) and (2.4) also not satisfy that axiom.

This result follows from the definition of Minkowski summation. Here, it is worth noting that in general, summation of closed sets is not automatically guaranteed to be closed if those sets are unbounded (as are the technology sets and the input requirement sets). In economics literature, it appears that this was first pointed out by Debreu (1951), who also clarified that a sufficient condition to guarantee 'closedness' of a sum of closed but unbounded sets would be to assume that all the sets under summation can be contained in some closed, convex, pointed cone, which is typically the case in production analysis.

Similar results can also be derived for many other properties. It seems worth clarifying the implications of constant returns to scale (i.e., if and only if $\delta\Psi^k = \Psi^k, \forall\delta > 0$, hereafter CRS), non-increasing returns to scale (i.e., if and only if $\delta\Psi^k \subseteq \Psi^k, \forall\delta \in (0, 1]$, hereafter NIRS) and non-decreasing returns to scale (i.e., if and only if $\delta\Psi^k \subseteq \Psi^k, \forall\delta > 1$) at disaggregate technologies onto the aggregate technologies. This is clarified in the next proposition.

Proposition. If Ψ^k, $\forall k$ satisfies the property of CRS or NIRS or NDRS then so do the aggregate technologies based on (2.1), (2.3) and (2.4), respectively. However, if CRS is not satisfied by Ψ^k for at least one of k being aggregated, then this may halt the entire aggregate technologies based on (2.1), (2.3) and (2.4) also not satisfy that property.

A similar result can be derived regarding the property of convexity (of Ψ or $P(x)$ or $L(y)$), which is sometimes imposed or obtained by construction or as a consequence of other assumptions, although usually not considered as a critical axiom of production theory. We will come back to convexity property at the end of this paper.

4 The Relationships between Different Aggregations

4.1 Reallocation versus No Reallocation

The relationships between (2.1) on the one hand and (2.3) or (2.4) on the other hand were investigated in various papers, e.g., Nesterenko and Zelenyuk (2007), Raa (2011) and more recently in Mayer and Zelenyuk (2014a,b, 2017). In a nutshell, these works clarified that

$$\overline{P}_n(x^1, ..., x^n) \subseteq \mathcal{P}_n\left(\sum_{k=1}^{n} x^k\right) := \{y \ : \ (\sum_{k=1}^{n} x^k, y) \in \overline{\Psi}_n\}$$

and

$$\overline{L}_n(y^1, ..., y^n) \subseteq \mathcal{L}_n\left(\sum_{k=1}^{n} y^k\right) := \{x \ : \ (x, \sum_{k=1}^{n} y^k) \in \overline{\Psi}_n\}$$

and then pointed out that the gap between $\overline{P}_n(x^1, ..., x^n)$ and $\mathcal{P}_n\left(\sum_{k=1}^{n} x^k\right)$ represents the potential gains in ability to produce greater outputs due to reallocation of inputs across the firms $k \in \{1, ..., n\}$, while the gap between $\overline{L}_n(y^1, ..., y^n)$ and $\mathcal{L}_n\left(\sum_{k=1}^{n} y^k\right)$ represents the potential gains in terms of possibility of using lower inputs due to reallocation of output plans across the firms $k \in \{1, ..., n\}$.

It is worth noting that the potential benefit of the reallocation can be very substantial even if individual technologies are the same and even if they exhibit CRS. To see this vividly, consider a simple example of two firms, each having the same CRS technology given by $\Psi = \{(x, y) \ : \ y \leq (x_1 x_2)^{1/2}\}$ and are endowed with different allocations of inputs, e.g., $(x_1^A, x_2^A) = (1, 100)$ and $(x_1^B, x_2^B) = (100, 1)$. In this case, $\overline{P}_n(x^1, x^2) = \{y = y^A + y^B \ : \ 0 \leq y^A \leq 10, \ \ 0 \leq y^B \leq 10\} = [0, 20]$. Meanwhile, if the firms cooperate by allowing reallocation of inputs, then their production possibilities are characterized by $\mathcal{P}_n\left(\sum_{k=1}^{n} x^k\right) = \{y \ : \ 0 \leq y \leq (101 \times 101)^{1/2} = 101\} = [0, 101]$. Thus, the potential benefit of reallocation can be very dramatic and the extent of it depends heavily on the degree of heterogeneity in allocations across the individuals (besides possible heterogeneity in technologies), and even if all have the same technology, even if it exhibits CRS.

Of course, having a potential does not mean all of it can be easily (or at all) realized, which in practice may depend on many aspects, including the unpredictable consequences of clashes of personalities of managers of firms, potentially fueled by imperfect and asymmetric information and mistakes, eventually leading to some unrealized potential[3]—an inefficiency that one can try to measure. In principle, various efficiency measures suggested in the literature can be used for such measurement, although most of the attention so far appears to have been on the Farrell-type measures. For example, Nesterenko and Zelenyuk (2007), Raa (2011) and Mayer and Zelenyuk (2014a,b, 2017) have explored the question of how to measure these gaps (or reallocative inefficiencies) from various perspectives using Farrell-type measures of productivity indexes based on them.[4]

An interesting future direction of research here is to explore how the reallocation efficiency measures can be linked to the bargaining and cooperative games in the game theoretic approach inspired by von Neumann (1945), Shapley and Shubik (1966), etc. and, possibly, to the concept of Shapley value. After all, the problem of efficiency and productivity analysis can typically be formulated as a programming problem, which in turn can be equivalently represented as a game theoretic formulation (Dantzig (1951b); David Gale (1951); Karlin (1959)).

4.2 The Union of Sets and Minkowski Sum of Sets

First of all, it should be clear that Minkowski summation of technology sets is very different from the union of technology sets (or a convex-hull of such a union), which sometimes is used for defining the so-called grand frontier (or meta-frontier). In general, under the basic axioms of production theory A1-A5 (or A6 instead of A5), the union of individual technology sets is always a sub-set in the Minkowski sum of these sets, i.e.,

$$\cup_{k=1}^{n} \Psi^{k} \quad \subseteq \quad \overline{\Psi}_{n} \tag{4.1}$$

an similarly, we also have

$$\cup_{k=1}^{n} P^{k}(x^{k}) \quad \subseteq \quad \overline{P}_{n}(x^{1}, ..., x^{n}) \tag{4.2}$$

while

$$\overline{L}_{n}(y^{1}, ..., y^{n}) \quad \subseteq \quad \cup_{k=1}^{n} L^{k}(y^{k}). \tag{4.3}$$

Recently, another aggregate technology that involved the union of sets was pro-

[3]E.g., see discussions in Greenwald and Stiglitz (1986); Leibenstein (1966); Leibenstein and Maital (1992); Thaler and Sunstein (2009)

[4]Also see Bogetoft and Wang (2005) for related discussion.

posed by Peyrache (2013), who called it "the industry technology (as different from the aggregate and the firm technology)", and defined it as[5]

$$\widetilde{\Psi} \;:=\; \cup_{S=1}^{\infty}\overline{\Psi}_S \tag{4.4}$$

It is useful to clarify the relationship of (2.1) with (4.4). It is worth mentioning here that (4.4) was originally introduced when focussing on a specific type of technology: when $\Psi^k = \Psi,\ \forall k$ is characterized via Activity Analysis Model or Data Envelopment Analysis (DEA) formulation for the technology set. For the sake of generality, here we will first focus on a very general case, aiming to clarify what it means for a *true* technology that satisfies standard regularity axioms of production theory, independently of an estimator (DEA, SFA, etc.) employed to estimate it. Then we will consider the specific case of the DEA-estimator.

4.2.1 Relationship between $\overline{\Psi}_S$ and $\widetilde{\Psi}$ in a general case

When describing the relationship between the industry technology and the aggregate technology, its is sometimes viewed that their main difference is that in the definition of aggregate technology (2.1) the number of firms is fixed, while in the definition of industry technology (4.4) this number is variable. It is important to clarify that it is the opposite. Indeed, the number of firms in $\widetilde{\Psi}$ is 'integrated out' (in the sense of unions of the sets) over the entire domain of S, i.e., over $\mathbb{N}$, and so $\widetilde{\Psi}$ does not depend on S anymore. On the other hand, in $\overline{\Psi}_S$, the number of firms shall be considered as a variable which $\overline{\Psi}_S$ depends upon, in a similar sense as the sample mean depends on the sample size, and so we indicate this with the subscript (S or n), while we dropped the subscript for $\widetilde{\Psi}$.

More importantly, note that the axiom A1, which is a very natural axiom in production theory, implies that the origin is always a point in the technology set, i.e.,

$$(0, 0) \in \Psi^k,\ \forall k. \tag{4.5}$$

This condition was also advocated by Dantzig (1951a), who called it 'Null activity', also requiring it to be feasible for firms. This condition also appears in many models of economic theory, including economic growth models (as part of 'Inada conditions').

[5]The original definition also involved time subscript, which we drop here because it does not play a role in our derivations and discussions.

In its turn, $(0,0) \in \Psi^k$ provides a very important regularization for the aggregation theory that follows directly from the definition of the Minkowski summation, namely

$$\Psi^k \subseteq \Psi^k \oplus \Psi^j, \forall k, j$$

and therefore, using mathematical induction,

$$\Psi^j \subseteq \sum_{\oplus k=1}^{n} \Psi^k \subseteq \sum_{\oplus k=1}^{n'} \Psi^k, \forall j = 1, ..., n';\ n \leq n' \tag{4.6}$$

This result is very intuitive and, in a sense, can also be viewed as a manifestation of the idea of free disposability at the aggregate level, stating that the aggregation of individuals should not be destructive in the sense that whatever was possible to produce individually (or by a smaller group) must still be feasible to produce as a group (or by an expanded group) of these individuals.

In turn, (4.6) implies that

$$\begin{aligned}
\cup_{S=1}^{n} \left(\sum_{\oplus k=1}^{S} \Psi^k \right) &= \Psi^1 \cup (\Psi^1 \oplus \Psi^2) \cup (\Psi^1 \oplus \Psi^2 \oplus \Psi^3) \cup ... \cup (\Psi^1 \oplus \Psi^2 \oplus ... \oplus \Psi^n) \\
&= \Psi^1 \oplus \Psi^2 \oplus ... \oplus \Psi^n \\
&= \sum_{\oplus k=1}^{n} \Psi^k,
\end{aligned}$$

using the "industry technology" as defined in (4.4), and yet for finite natural numbers n and S, this is equivalent to the Koopman's aggregate technology defined in (2.1).

Furthermore, within the range, when $n \to \infty$, as the original definition (4.4) requires, we also have

$$\cup_{S=1}^{\infty} \left(\sum_{\oplus k=1}^{S} \Psi^k \right) = \lim_{n\to\infty} \cup_{S=1}^{n} \left(\sum_{\oplus k=1}^{S} \Psi^k \right) = \lim_{n\to\infty} \sum_{\oplus k=1}^{n} \Psi^k = \sum_{\oplus k=1}^{\infty} \Psi^k,$$

i.e., the "industry technology" defined in (4.4) is also equivalent to (2.1) with infinite number of individuals being aggregated.

The main reason for splitting the discussion into the last two (finite and infinite) cases is because the latter case was not defined explicitly for (2.1) and, in fact, may require proper normalization to be handled as will be clear below.

In a special, yet important case when all technologies are equal (as is often assumed in theoretical or applied analysis) and for simplicity when it is convex, i.e., when $\Psi^k = \Psi,\ \forall k$ and $(x', y'),\ (x^o, y^o) \in \Psi\ \Rightarrow \delta(x', y') + (1-\delta)(x^o, y^o) \in \Psi,\ \forall \delta \in [0, 1]$, then by using Minkowski summation theory, we have a very useful result, stating that

$$\sum_{\oplus k=1}^{n} \alpha_k \Psi = \alpha \Psi \tag{4.7}$$

where $\alpha_k \in \mathbb{R}_+$ and $\alpha = \sum_{k=1}^{n} \alpha_k$.[6]

Intuitively, in the production context, the scalars α_k can be understood as scalability coefficients, reflecting a possibility to scale (up or down) any feasible allocation in Ψ, e.g., Ψ can be understood as the benchmark technology, while α_k can be understood as the degree with which an individual (firm, region, country) k is able to replicate this benchmark technology.

Furthermore, in the special case when $\alpha_k = 1,\ \forall k$, from (4.7) we have $\sum_{\oplus s=1}^{n} \Psi = n\Psi$. In practice, since n is always finite, $\sum_{\oplus s=1}^{n} \Psi^s$ is well-defined and can be used in practical estimations as those we mention in the next section. In the case when $n \to \infty$ the aggregate technology is given by $\lim_{n\to\infty} \left(\sum_{\oplus s=1}^{n} \Psi\right) = \Psi \times \lim_{n\to\infty}(n) = \Psi \times \infty$, and so one may need to use normalization by a quantity of the order $O(n)$ for $\overline{\Psi}_n$ or for the objects defined upon it, to prevent them from exploding into infinity, as is done in the case of deriving asymptotic results for the efficiency measures defined on the aggregate technologies (e.g., see Simar and Zelenyuk (2017) for related discussions).

To summarize, under standard and fairly general regularity conditions on technology, the industry technology defined in (4.4) is equivalent to the well-established concept of aggregate technology (2.1) due to Koopmans (1957), i.e.,

$$\widetilde{\Psi}_n \ := \ \overline{\Psi}_n \tag{4.8}$$

and

$$\widetilde{\Psi} \ := \ \overline{\Psi}_\infty \tag{4.9}$$

and in the latter case a normalization by a quantity of the order $O(n)$ may be required to avoid its explosion to infinity.

[6]E.g., see Schneider (1993) and references therein. In production theory, this result goes back to at least Li and Ng (1995).

4.2.2 The relationship between $\overline{\Psi}_S$ and $\widetilde{\Psi}$ in the case of DEA

DEA is a very popular approach that provides a consistent estimator of a true technology.[7] Its early roots and inspiration go back to works of Leontief (1925) and von Neumann (1945)[8], who in fact focused on aggregate context (the whole economy) rather than individuals. Their ideas were shaped up further by Dantzig (1949, 1951a), Koopmans (1951a,b) and Shephard (1953), and most prominently by Farrell (1957) and Charnes et al. (1978) and many others after them. While books have been written on various types of DEA in various contexts,[9] its most basic form for estimating Ψ can be concisely stated as follows

$$\hat{\Psi} = \left\{ (x, y) \in \mathbb{R}^N_+ \times \mathbb{R}^M_+ \ : \ \sum_{k=1}^{n} z^k x^k \leqq x, \ \sum_{k=1}^{n} z^k y^k \geqq y, \ \delta_l \leq \sum_{k=1}^{n} z^k \leq \delta_u, \ z^k \geq 0 \right\} \tag{4.10}$$

where $(z^1, ..., z^n)$ is a vector of intensity variables that spans the set $\hat{\Psi}$ by enveloping the input-output data $\{(x^k, y^k)\}_{k=1}^n$, with a form that depends on the restrictions imposed upon these variables. In particular, if $\delta_l = 0$ and $\delta_u = +\infty$ then (4.10) gives the DEA estimator of the true technology under the assumption of constant returns to scale (DEA-CRS). If $\delta_l = 0$ and $\delta_u = 1$ then (4.10) gives the DEA estimator of the true technology under the assumption of non-increasing returns to scale (DEA-NIRS). If $\delta_l = 1$ and $\delta_u = 1$ then (4.10) gives the DEA estimator of the true technology under the assumption of variable returns to scale (DEA-VRS, which we denote as $\hat{\Psi}_{VRS}$).[10]

Although involving data on all firms $k \in \{1, ..., n\}$, (4.10) is an estimator for a technology for an individual k (rather than the aggregate technology), with the assumption that all individuals $k = 1, ..., n$ share access to the same technology (though possibly with different levels of efficiency).

The credit for the first DEA estimation of an efficiency measure based on aggregate technology goes back to at least Maindiratta (1990), who also appears to be the first to suggest using prioritized integer linear programming for this problem. The approach of Maindiratta (1990) was later revisited and refined by Ray and Hu (1997); Ray and

[7]For more details on statistical properties of DEA, see a review by Simar and Wilson (2015) and references therein.

[8]This is the English version of his article published in 1938.

[9]E.g., see Charnes et al. (1994), Ray (2004), Cooper et al. (2007b) and Cooper et al. (2011), to mention just a few.

[10]While DEA-CRS form was considered by Farrell (1957) and Charnes et al. (1978), the non-CRS form goes back to at least Färe et al. (1983) and Banker et al. (1984), with some earlier ideas from Afriat (1972).

Mukherjee (1998) and more recently by Peyrache (2013, 2015). An alternative (though closely related) approach was taken by Li and Ng (1995), Bogetoft and Wang (2005), Nesterenko and Zelenyuk (2007) and more recently refined and connected to the theory of industrial organization by Raa (2011).

The results on the equivalence of $\widetilde{\Psi}_n$ and $\overline{\Psi}_n$ established in the previous section apply directly to DEA-CRS and DEA-NIRS because $(0, 0)$ is guaranteed to be in the estimated technology set by construction. It will also apply to DEA-VRS if the point $(0, 0)$ is included in the original data or as an out-of-sample information due to the axiom A1. Otherwise, DEA-VRS gives an estimate of technology that is inconsistent with (3.1), a key production theory axiom based on common sense, and in particular such an estimator will not guarantee that $(0, 0) \in \hat{\Psi}_{VRS}$, even if $(0, 0) \in \Psi$, unless $(0, 0)$ is an observation in the data set.

In practice, even if $(0, 0)$ is in the data set then researchers usually discard it (explicitly or implicitly) when doing computations of efficiency scores in order to ensure convergence of LP algorithms.[11] This however does not mean that $(0, 0)$ is not in the *true* technology set Ψ. Indeed, the DEA-VRS formulation is not a true technology but just an estimator of it, with an artifact (or perhaps a defect) that it is being sharply 'abrupt' at the observation level with the lowest inputs, pretending that the technology set does not exist below that level, even if the common sense (and axiom A1) suggests that all the lower combinations of inputs, including zero input, shall be feasible for a firm at least in combination with the zero output.

This undesirable property of DEA-VRS that violates a key production axiom is known, yet rarely acknowledged. A very simple cure to this problem is to add the theoretically required feasible points *ex post*, i.e., amend it as follows $\hat{\Psi}^*_{VRS} := \hat{\Psi}_{VRS} \cup \{(\mathbb{R}^N_+, 0)\}$.

Clearly, $\hat{\Psi}^*_{VRS}$ may be non-convex (unless $(0, 0)$ was in the data set used to get $\hat{\Psi}_{VRS}$) and here a researcher may want to choose what property is more imperative to keep as a maintained assumption: convexity of Ψ vs. the basic axiom A1. An opinion of the author of this paper is that the axiom A1 is more important to satisfy than requiring convexity of the entire technology set, which appears to be too strict. Indeed, even basic microeconomic theory textbooks typically have pictures of the technology set looking more like a logistic function rather than having an abrupt shape as in the DEA-VRS case. Meanwhile, if a researcher strongly believes in convexity of the true

[11] See von Neumann (1945) and Karlin (1959).

technology set *a priori*, and also wants to satisfy the basic axiom A1, then the resulting DEA-VRS will become equivalent with DEA-NIRS.

On the other hand, if a researcher believes in increasing returns to scale at some lower level of input and that it then changes to constant returns to scale at some level (at least one point) and then continues into decreasing returns to scale, then one clearly has in mind a non-convex technology and the DEA-VRS estimator would be an appropriate one to use for the *observed* area of technology. With the acknowledgment that the frontier of technology does not stop at the lowest input but continues towards the origin, and includes the origin, whether smoothly (but that area is not identified by DEA-VRS) or along the input space with zero output, i.e., the final technology estimate is $\hat{\Psi}^*_{VRS} := \hat{\Psi}_{VRS} \cup \{(\mathbb{R}^N_+, 0)\}$, which will satisfy the production theory axioms, and in particular the common sense that $(0, 0) \in \Psi$, and in turn will ensure that (4.4) is equivalent to the concept of aggregate technology (2.1) of Koopmans (1957).

Finally, it is also worth noting that this view on convexity is coherent with a common view of convexity in general economic theory, e.g., succinctly described by Koopmans (1961), while commenting on the discussion of Farrell (1959), stating that:

> "As explained by Farrell, the convexity assumptions imply perfect divisibility of all commodities, constant or decreasing returns to scale in production, and a diminishing marginal rate of substitution in consumption." (see p. 478-479 in Koopmans (1961)).

Somehow, this understanding appears to have eventually been lost for the context of DEA-VRS and hopefully this article will help clarifying it.

5 Convexity, Non-Convexity and Approximate Convexity

Convexity is an important property that frequently appears in various models of economic theory. It is particularly important for many (although not all) the results in duality theory in economics. It is important to emphasize, however, that many of these results still hold without convexity of the technology sets (Ψ^k), because many require convexity of only the input requirement sets or convexity of the output sets, and not necessarily a convexity of the entire technology.[12] For this and other reasons, convexity

[12] See Shephard (1953, 1970) as well as Diewert (1974a, 1982); Färe and Primont (1995).

of a technology set is usually viewed as not a critical assumption and so non-convexity of Ψ^k may often appear in practice (e.g., as in the case of using $\hat{\Psi}^*_{VRS}$). A natural question is then, what is happening to the aggregate technologies defined in (2.1), (2.3) and (2.4) when all or some of the individual technologies are convex or not? This question is clarified in the next proposition.

Proposition. If individual technologies are such that

1. If Ψ^k is convex $\forall k$ then the aggregate technology based on (2.1) (and also on (2.3) and on (2.4)) is also convex.

2. If $P^k(x^k)$ is convex $\forall k$ then the aggregate technology based on (2.3) is also convex (though the one based on (2.1) or (2.4) may be non-convex).

3. If $L^k(y^k)$ is convex $\forall k$ then the aggregate technology based on (2.4) is also convex (though the one based on (2.1) or (2.3) may be non-convex).

On the other hand, if for at least one of k being aggregated, the convexity is not satisfied for Ψ^k or $P^k(x^k)$ or $L^k(y^k)$, then the corresponding aggregate technology (i.e., (2.1), (2.3) or (2.4), respectively) may also not satisfy convexity.

The last part of this proposition might sound disappointing. However, there is another important result in mathematics—the Shapley-Folkman lemma[13]—that says that the Minkowski sum of non-convex sets can be viewed as approximately convex when the number of summands is larger than the number of dimensions. That is, even if Ψ^k or $P^k(x^k)$ or $L^k(y^k)$ is not convex for some or even all of k, the aggregate technology sets defined as Minkowski summations in (2.1) or (2.3) or (2.4), can still be viewed as approximately convex (if $n > N+M$ or $n > M$ or $n > N$, respectively, which is typically the case in practice).

Moreover, by the same Shapley-Folkman lemma, even if Ψ^k or $P^k(x^k)$ or $L^k(y^k)$ is not convex for some or even all of k, their Minkowski summations, (2.1), (2.3) and (2.4) are asymptotically convex when $n \to \infty$, in the sense that a measure of 'non-convexity' converges to zero while n increases to ∞.

Finally, recall that if Ψ^k is convex and also satisfies A1 and additivity (i.e., $(x', y') \in \Psi^k$ and $(x^o, y^o) \in \Psi^k$ implies $(x^o, y^o)+(x', y') \in \Psi^k$), then Ψ^k is a cone, i.e., exhibits

[13] E.g., see Starr (2008) for details.

CRS.[14] Combining this with previous proposition one can immediately conclude that if for all k we have Ψ^k is convex, satisfies A1 and additivity, then the aggregate technology $\overline{\Psi}_n$ defined in (2.1) exhibits CRS.

It seems fair to say that the additivity assumption might be viewed as a too strong assumption on a disaggregated level (for an individual technology). On the aggregate level, however, this assumption is coherent with how $\overline{\Psi}_n$ is defined in (2.1). Along with the 'approximate convexity' result from Shapley-Folkman lemma, this gives a justification for the CRS assumption at the aggregate level, for $\overline{\Psi}_n$.

6 Concluding remarks

In this article we clarified various aspects of aggregate technologies that were recently employed in theoretical studies on aggregation in productivity and efficiency analysis, especially in this journal.

First, we clarified that satisfaction of the basic axioms of production theory by all the individual technologies is a sufficient condition for the aggregate technology to satisfy these axioms.

We also clarified relationship between aggregate technologies based on Minkowski summation and those based on unions. Here, we also showed that under the standard regularity axioms of production theory, the aggregate technology concept (4.4) is equivalent to the aggregate technology concept (2.1).

Third, we clarified that even though a non-convexity for just one individual being aggregated can make the aggregate technology non-convex, due to the Shapley-Folkman lemma it is still asymptotically convex (even if all individual technologies are not convex), while for finite number of individuals aggregated, the aggregate technology can be viewed as 'approximately convex', when the number of individuals aggregated exceeds the dimension of the technology set.

Here it is also interesting (from historical point of view) to note that Shapley's work on dealing with non-convexities was also influenced, *inter alia*, by works of gurus of productivity and efficiency analysis—Farrell and Koopmans, e.g., due to Farrell (1959, 1961) and Koopmans (1961) and the discussion they inspired.[15] It would therefore be not surprising, yet quite outstanding, if there was still great research potential in further synthesis of the area of efficiency analysis (including recent developments with

[14] This result in economics goes back to at least Debreu (1951).

[15] E.g., see Shapley and Shubik (1966) and Starr (2008) for more details and references.

the aggregate perspective) and the area of game theory, especially cooperative game theory. The roots of such a synthesis can be seen starting from the seminal works of von Neumann (1945); Dantzig (1951b); David Gale (1951); Karlin (1959) and more recently explored in Hao et al. (2000); Nakabayashi and Tone (2006); Liang et al. (2008); Lozano (2012), to mention a few.[16]

[16] For more works in this area, Cook et al. (2010) and Lozano et al. (2016).

References

Afriat, S. N. (1972). Efficiency estimation of production functions. *International Economic Review*, 13(3):568–598.

Banker, R. D., Charnes, A., and Cooper, W. W. (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Management Science*, 30(9):1078–1092.

Blackorby, C. and Russell, R. R. (1999). Aggregation of efficiency indices. *Journal of Productivity Analysis*, 12(1):5–20.

Blundell, R. and Stoker, T. M. (2005). Heterogeneity and aggregation. *Journal of Economic Literature*, 43(2):347–391.

Bogetoft, P. and Wang, D. (2005). Estimating the potential gains from mergers. *Journal of Productivity Analysis*, 23(2):145–171.

Briec, W., Dervaux, B., and Leleu, H. (2003). Aggregation of directional distance functions and industrial efficiency. *Journal of Economics*, 79(3):237–261.

Charnes, A., Cooper, W., and Rhodes, E. (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research*, 2(6):429–444.

Charnes, A., Cooper, W. W., Lewin, A. Y., and Seiford, L. M., editors (1994). *Data Envelopment Analysis: Theory, Methodology, and Applications*. Amsterdam, NL: Springer.

Cook, W. D., Liang, L., and Zhu, J. (2010). Measuring performance of two-stage network structures by DEA: A review and future perspective. *Omega*, 38(6):423–430.

Cooper, W., Huang, Z., Li, S., Parker, B., and Pastor, J. (2007a). Efficiency aggregation with enhanced Russell measures in data envelopment analysis. *Socio-Economic Planning Sciences*, 41(1):1–21.

Cooper, W. W., Seiford, L. M., and Tone, K. (2007b). *Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA-Solver Software*. New York, NY: Springer.

Cooper, W. W., Seiford, L. M., and Zhu, J., editors (2011). *Handbook on Data Envelopment Analysis*. New York, NY: Springer.

Dantzig, G. B. (1949). Programming of interdependent activities: Ii mathematical model. *Econometrica*, 17(3/4):200–211.

Dantzig, G. B. (1951a). The programming of interdependent activities: Mathematical model. In Koopmans, T., editor, *Activity Analysis of Production and Allocation*. New York, NY: Wiley.

Dantzig, G. B. (1951b). A proof of the equivalence of the programming problem and the game problem. In Koopmans, T., editor, *Activity Analysis of Production and Allocation*. New York, NY: Wiley.

David Gale, Harold W. Kuhn, A. W. T. (1951). Linear programming and the theory of games. In Koopmans, T., editor, *Activity Analysis of Production and Allocation*. New York, NY: Wiley.

Debreu, G. (1951). The coefficient of resource utilization. *Econometrica*, 19(3):273–292.

Diewert, W. E. (1974a). Applications of duality theory. In Intriligator, M. D. and Kendrick, D. A., editors, *Frontiers in Quantitative Economics*, volume 2, pages 106–171. Amsterdam, NL: Elsevier.

Diewert, W. E. (1974b). A note on aggregation and elasticities of substitution. *The Canadian Journal of Economics*, 7(1):12–20.

Diewert, W. E. (1978). Superlative index numbers and consistency in aggregation. *Econometrica*, 46(4).

Diewert, W. E. (1980). *Aggregation Problems in the Measurement of Capital*, pages 433–528. Chicago, IL: University of Chicago Press.

Diewert, W. E. (1982). Duality approaches to microeconomic theory. In *Handbook of Mathematical Economics*, volume 2, pages 535–599. Netherlands, NL: Elsevier.

Diewert, W. E. (1983). The measurement of waste within the production sector of an open economy. *The Scandinavian Journal of Economics*, 85(2):159–179.

Färe, R., Grosskopf, S., and Logan, J. (1983). The relative efficiency of Illinois electric utilities. *Resources and Energy*, 5(4):349–367.

Färe, R., Grosskopf, S., and Zelenyuk, V. (2004). Aggregation of cost efficiency: Indicators and indexes across firms. *Academia Economic Papers*, 32(3):395–411.

Färe, R., Grosskopf, S., and Zelenyuk, V. (2008). Aggregation of Nerlovian profit indicator. *Applied Economics Letters*, 15(11):845–847.

Färe, R. and Karagiannis, G. (2014). A postscript on aggregate Farrell efficiencies. *European Journal of Operational Research*, 233(3):784–786.

Färe, R. and Primont, D. (1995). *Multi-Output Production and Duality: Theory and Applications*. New York, NY: Kluwer Academic Pubishers.

Färe, R. and Zelenyuk, V. (2003). On aggregate Farrell efficiencies. *European Journal of Operational Research*, 146(3):615–620.

Färe, R. and Zelenyuk, V. (2005). On Farrell's decomposition and aggregation. *International Journal of Business and Economics*, 4(2):167–171.

Färe, R. and Zelenyuk, V. (2007). Extending Färe and Zelenyuk (2003). *European Journal of Operational Research*, 179(2):594–595.

Farrell, M. J. (1957). The measurement of productive efficiency. *Jornal of the Royal Statistical Society. Series A (General)*, 120(3):253–290.

Farrell, M. J. (1959). The convexity assumption in the theory of competitive markets. *Journal of Political Economy*, 67(4):377–391.

Farrell, M. J. (1961). On convexity, efficiency, and markets: A reply. *Journal of Political Economy*, 69(5):484–489.

Førsund, F. R. and Hjalmarsson, L. (1979). Generalised Farrell measures of efficiency: An application to milk processing in Swedish dairy plants. *The Economic Journal*, 89(354):294–315.

Gorman, W. M. (1953). Community preference fields. *Econometrica*, 21(1):63–80.

Gorman, W. M. (1959). Separable utility and aggregation. *Econometrica*, 27(3):469–481.

Greenwald, B. C. and Stiglitz, J. E. (1986). Externalities in economies with imperfect information and incomplete markets. *The Quarterly Journal of Economics*, 101(2):229–264.

Hao, G., Wei, Q. L., and Yan, H. (2000). A game theoretical model of DEA efficiency. *The Journal of the Operational Research Society*, 51(11):1319–1329.

Karagiannis, G. (2015). On structural and average technical efficiency. *Journal of Productivity Analysis*, 43(3):259–267.

Karagiannis, G. and Lovell, C. A. K. (2015). Productivity measurement in radial DEA models with a single constant input. *European Journal of Operational Research*, 251(1):323–328.

Karlin, S. (1959). *Mathematical methods and theory in games, programming, and economics*, volume 2. Reading, MA: Addison Wesley.

Koopmans, T. (1951a). *Activity Analysis of Production and Allocation.* New York, NY: Wiley.

Koopmans, T. (1957). *Three Essays on The State of Economic Science.* New York, NY: McGraw-Hill.

Koopmans, T. C. (1951b). Analysis of production as an efficienct combination of activities. *Activity Analysis of Production and Allocation*, 13:33–37.

Koopmans, T. C. (1961). Convexity assumptions, allocative efficiency, and competitive equilibrium. *Journal of Political Economy*, 69(5):478–479.

Krein, M. and Smulian, V. (1940). On regulary convex sets in the space conjugate to a Banach space. *Annals of Mathematics*, 41(2):556–583.

Kuosmanen, T., Kortelainen, M., Sipiläinen, T., and Cherchye, L. (2010). Firm and industry level profit efficiency analysis using absolute and uniform shadow prices. *European Journal of Operational Research*, 202(2):584–594.

Leibenstein, H. (1966). Allocative efficiency vs. "X-efficiency". *American Economic Review*, 56(3):392–415.

Leibenstein, H. and Maital, S. (1992). Empirical estimation and partitioning of X-inefficiency: A data-envelopment approach. *American Economic Review*, 82(2):428–433.

Leontief, W. (1925). Die bilanz der russischen volkswirtschaft - eine methodologische untersuchung. *Weltwirtschaftliches Archiv*, 22(2):338–344.

Li, S.-k. and Cheng, Y.-s. (2007). Solving the puzzles of structural efficiency. *European Journal of Operational Research*, 180(2):713–722.

Li, S.-K. and Ng, Y. C. (1995). Measuring the productive efficiency of a group of firms. *International Advances in Economic Research*, 1(4):377–390.

Liang, L., Wu, J., Cook, W. D., and Zhu, J. (2008). The DEA game cross-efficiency model and its Nash equilibrium. *Operations Research*, 56(5):1278–1288.

Lozano, S. (2012). Information sharing in DEA: A cooperative game theory approach. *European Journal of Operational Research*, 222(3):558–565.

Lozano, S., Hinojosa, M. A., Mármol, A. M., and Borrero, D. V. (2016). DEA and cooperative game theory. In *Handbook of Operations Analytics Using Data Envelopment Analysis*, pages 215–239. New York, NY: Springer.

Maindiratta, A. (1990). Largest size-efficient scale and size efficiencies of decision-making units in data envelopment analysis. *Journal of Econometrics*, 46(1):57 – 72.

Mayer, A. and Zelenyuk, V. (2014a). Aggregation of Malmquist productivity indexes allowing for reallocation of resources. *European Journal of Operational Research*, 238(3):774–785.

Mayer, A. and Zelenyuk, V. (2014b). An aggregation paradigm for Hicks-Moorsteen productivity indexes. CEPA Working Paper No. WP01/2014.

Mayer, A. and Zelenyuk, V. (2017). Aggregation paradigm of individual efficiency measures and productivity indices. forthcoming.

Mussard, S. and Peypoch, N. (2006). On multi-decomposition of the aggregate Malmquist productivity index. *Economics Letters*, 91(3):436–443.

Nakabayashi, K. and Tone, K. (2006). Egoist's dilemma: a DEA game. *Omega*, 34(2):135–148.

Nesterenko, V. and Zelenyuk, V. (2007). Measuring potential gains from reallocation of resources. *Journal of Productivity Analysis*, 28(1–2):107–116.

Pachkova, E. V. (2009). Restricted reallocation of resources. *European Journal of Operational Research*, 196(3):1049–1057.

Peyrache, A. (2013). Industry structural inefficiency and potential gains from mergers and break-ups: A comprehensive approach. *European Journal of Operational Research*, 230(2):422–430.

Peyrache, A. (2015). Cost constrained industry inefficiency. *European Journal of Operational Research*, 247(3):996–1002.

Raa, T. T. (2011). Benchmarking and industry performance. *Journal of Productivity Analysis*, 36(3):285–292.

Ray, S. C. (2004). *Data Envelopment Analysis: Theory and Techniques for Economics and Operations Research*. New York, NY: Cambridge University Press.

Ray, S. C. and Hu, X. (1997). On the technically efficient organization of an industry: A study of u.s. airlines. *Journal of Productivity Analysis*, 8(1):5–18.

Ray, S. C. and Mukherjee, K. (1998). Quantity, quality, and efficiency for a partially super-additive cost function: Connecticut public schools revisited. *Journal of Productivity Analysis*, 10(1):47–62.

Schneider, R. (1993). *Convex Bodies: The Brunn-Minkowski Theory*. New York, NY: Cambridge University Press.

Shapley, L. S. and Shubik, M. (1966). Quasi-cores in a monetary economy with nonconvex preferences. *Econometrica*, 34(4):805–827.

Shephard, R. W. (1953). *Cost and Production Functions*. Princeton, NJ: Princeton University Press.

Shephard, R. W. (1970). *Theory of Cost and Production Functions*. Princeton studies in mathematical economics. Princeton, NJ: Princeton University Press.

Simar, L. and Wilson, P. W. (2015). Statistical approaches for nonparametric frontier models: A guided tour. *International Statistical Review*, 83(1):77–110.

Simar, L. and Zelenyuk, V. (2007). Statistical inference for aggregates of Farrell-type efficiencies. *Journal of Applied Econometrics*, 22(7):1367–1394.

Simar, L. and Zelenyuk, V. (2017). Central limit theorems for aggregate efficiency. *Operations Research*. Forthcoming.

Starr, R. M. (2008). Shapley-Folkman theorem. In Durlauf, S. N. and Blume, L. E., editors, *The New Palgrave Dictionary of Economics*, pages 317–318. Basingstoke, UK: Palgrave Macmillan.

Stoker, T. M. (1993). Empirical approaches to the problem of aggregation over individuals. *Journal of Economic Literature*, 31(4):1827–1874.

Thaler, R. H. and Sunstein, C. R. (2009). *Nudge: Improving Decisions About Health, Wealth, and Happiness*. London, UK: Penguin.

Theil, H. (1954). *Linear aggregation of economic relations*. Contributions to economic analysis. North-Holland Pub. Co.

von Neumann, J. (1945). A model of general equilibrium. *Review of Economic Studies*, 13(1):1–9.

Zelenyuk, V. (2006). Aggregation of Malmquist productivity indexes. *European Journal of Operational Research*, 174(2):1076–1086.