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Aggregation of Malmquist productivity indexes allowing for reallocation of resources

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Abstract

In this paper we consider aggregate (group) Malmquist productivity index measures which allow inputs to be reallocated within the group (when in output orientation). This merges the single period aggregation results allowing input reallocation of Nesterenko & Zelenyuk (2007) with the aggregate Malmquist productivity index results of Zelenyuk (2006) to determine aggregate Malmquist productivity indexes that are justified by economic theory, consistent with previous aggregation results, and which maintain analogous decompositions over time to the original measures. Such measures are of direct relevance to firms or countries who have merged (making input reallocation possible), allowing them to measure potential productivity gains and how these have been realised (or not) over time.

Keywords: Data envelopment analysis, Efficiency, Productivity, Index aggregation, Reallocation

1. Introduction

When considering the efficiency of organisations (be they branches, firms, countries etc.) it is useful to consider not only individual efficiencies but also the combined efficiency of a group (e.g. an industry) or a sub-group within a group. Measuring group efficiency dates back to Farrell (1957), who introduced the notion of structural efficiency. Important contributions were made by Førsund & Hjalmarsson (1979), who considered measuring the average decision making unit, Li & Ng (1995), who derived an aggregation scheme using shadow prices, Ylvinger (2000), who used weights derived from Data Envelopment Analysis, and Blackorby & Russell (1999) who found several impossibility results for efficiency

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aggregation. Recently, Färe & Zelenyuk (2003) developed an aggregation scheme for measuring group efficiency which is justified from an economic theory perspective. However, this aggregation scheme required the assumption that the current input allocation of the group was taken as given and could not be reallocated. This assumption was relaxed by Nesterenko & Zelenyuk (2007) who determined group efficiency measures allowing inputs to be reallocated between decision making units within the group. This is an important consideration for cases where such reallocation is possible - for example when the organisations considered are branches within a firm, when firms are merging within an industry, or when countries are merging into a monetary or customs union. It is also important to consider not just group efficiency at one time period, but to also look at how it changes over time. The Malmquist productivity index of Caves, Christensen & Diewert (1982) is a natural candidate for doing this, as it is popular for empirical use and has a number of interesting decompositions available. The aggregation scheme of Färe & Zelenyuk (2003) has already been extended intertemporally for the aggregation of Malmquist productivity indexes by Zelenyuk (2006). In our work, we seek to extend the group aggregation results allowing reallocation with the Malmquist productivity index to enable consideration of the change in group productivity over time, allowing reallocation of inputs across decision making units (when considering output orientation). Such measures are directly relevant to firms or countries who have merged, and can not only enable the measurement of potential productivity gains, but also measure how these have been realised (or not) over time. This information is of particular value to managers/leaders of merged firms/countries, or managers of branches within an organisation, as it allows them to measure the success or failure over time of endeavours to increase group output by reallocating inputs amongst individual units within the group.

Our paper is structured as follows. Section 2 introduces the key definitions for individual efficiency measures, and Sections 3 and 4 consider group efficiency measures, first assuming inputs cannot be reallocated and then relaxing this assumption. Section 5 then considers aggregation for Malmquist productivity indexes and group results assuming no reallocation of inputs. The main results are given in Section 6, which derives group Malmquist productivity indexes when relaxing the assumption of no reallocation, in such a way that the indexes decompose analogously to the original efficiency measures. Section 6 also discusses some important matters regarding practical implementation of these indexes. Finally, Section 7 concludes the study, highlighting possible areas for future research.

2. Individual Measures

Let us begin by considering individual efficiency measures for a group of K decision making units (DMUs), indexed $k = 1, \ldots, K$. DMUs could be individual branches, whole firms/organisations, whole industries or even whole countries, depending on the purpose of the study. A DMU k uses a vector $x^k = (x_1^k, \ldots, x_N^k)' \in \Re_+^N$ of N inputs to produce a vector $y^k = (y_1^k, \ldots, y_M^k)' \in \Re_+^M$ of M outputs. Let the individual technology of DMU k at a given time period τ be represented by the *technology set* T_τ^k, defined as:

$$\begin{aligned} T_\tau^k \equiv \{&(x^k, y^k) \in \Re_+^N \times \Re_+^M : \\ &\text{DMU } k \text{ can produce } y^k \text{ from } x^k \text{ using the technology in period } \tau\}. \end{aligned} \tag{1}$$

It can be equivalently characterised using the *output correspondence* for DMU k at period τ, $P_\tau^k : \Re_+^N \to 2^{\Re_+^M}$ where:

$$P_\tau^k(x^k) \equiv \{y^k \in \Re_+^M : (x^k, y^k) \in T_\tau^k\}, \quad x^k \in \Re_+^N. \tag{2}$$

The technology of each DMU is assumed to satisfy standard regularity axioms (see Färe & Primont (1995) for more details). Specifically ($\forall k = 1, \ldots, K$ and $\forall \tau$) we assume:

Axiom 1: The technology set T_τ^k is closed.
Axiom 2: The output set $P_\tau^k(x^k)$ is bounded $\forall x^k \in \Re_+^N$.
Axiom 3: There is no 'free lunch', that is, one cannot produce something from nothing. Formally, $(0_N, y^k) \notin T_\tau^k, \forall y^k \geq 0_M$ (i.e. $y^k \geq 0_M, y^k \neq 0_M$).
Axiom 4: It is possible to produce nothing. Formally, $0_M \in P_\tau^k(x^k), \;\; \forall x^k \in R_+^N$.
Axiom 5: Inputs and outputs are freely (strongly) disposable. Formally
$(x^0, y^0) \in T_\tau^k \Longrightarrow (x, y) \in T_\tau^k, \;\; \forall y \leqq y^0, \; \forall x \geqq x^0$.
Axiom 6: Output sets $P_\tau^k(x^k)$ are convex, $\forall x^k \in R_+^N$.

Axiom 6 is required to ensure duality results hold. We do not yet make the stronger assumption that technology sets T^k are convex, but will introduce it later when considering practical estimation.

We also involve the output-oriented Shephard (1970) *distance function* $D_\tau^k : \Re_+^N \times R_+^M \to \Re_+ \cup \{\infty\}$, defined as

$$D_\tau^k(x^k, y^k) \equiv \inf\{\theta : y^k/\theta \in P_\tau^k(x^k)\}. \tag{3}$$

This function completely characterises the technology in period τ in the sense that

$$D_\tau^k(x^k, y^k) \leq 1 \Leftrightarrow (x^k, y^k) \in T_\tau^k. \tag{4}$$

We also use it to define the Farrell-type *technical efficiency* (TE) measure of DMU k in period τ as

$$TE_\tau^k \equiv TE_\tau^k(x^k, y^k) \equiv 1/D_\tau^k(x^k, y^k), \;\; (x^k, y^k) \in \Re_+^{N+M}. \tag{5}$$

We also consider the dual characterisation of $P_\tau^k(x^k)$ given by the *revenue function*:

$$R_\tau^k(x^k, p) \equiv \max_y \{py : y \in P_\tau^k(x^k)\}, \;\; x^k \in \Re_+^N, \; p \in \Re_{++}^M, \tag{6}$$

given a price row-vector $p = (p_1, \ldots, p_M) \in \Re_+^M$ corresponding to the M outputs. Note that p is assumed to be the same for all DMUs, a necessary assumption for the aggregation results that follow, which we will discuss in more detail in section 3. Given the revenue function, the *revenue efficiency* (RE) for a DMU k at period τ is:

$$RE_\tau^k \equiv RE_\tau^k(x^k, y^k, p) \equiv \frac{R_\tau^k(x^k, p)}{py^k}, \;\; \text{for } py^k \neq 0, \; (x^k, y^k) \in \Re_+^{N+M}, \; p \in \Re_{++}^M. \tag{7}$$

It will always be the case that $RE^k(x^k, y^k, p) \geq TE^k(x^k, y^k)$, as a firm can be technically efficient without being revenue efficient, but a revenue efficient firm will always be technically efficient. The multiplicative residual that closes the inequality is called *allocative efficiency* (AE):

$$AE_\tau^k \equiv AE_\tau^k(x^k, y^k, p) \equiv RE_\tau^k(x^k, y^k, p)/TE_\tau^k(x^k, y^k), \;\; (x^k, y^k) \in \Re_+^{N+M}, \; p \in \Re_{++}^M, \tag{8}$$

which immediately provides the following decomposition, which holds for any period τ:

$$\begin{aligned} RE_\tau^k(x^k, y^k, p) = TE_\tau^k(x^k, y^k) \times AE_\tau^k(x^k, y^k, p),& \\ \forall (x^k, y^k) \in \Re_+^{N+M}, \; p \in \Re_{++}^M, \; py^k \neq 0,& \end{aligned} \tag{9}$$

which will be useful later.

3. Group Efficiency Without Reallocation

Following Färe & Zelenyuk (2003), group efficiency measures can be constructed to measure the efficiency of a group, taking current input endowments as given (which we will later relax in section 4 to allow reallocation of inputs between DMUs). Here we focus on aggregating all firms in a group, but it is possible to extend these results to aggregate separate subgroups of firms, and then consistently aggregate subgroups into larger groups (see Simar & Zelenyuk (2007)).

First consider a *group output set* for period τ, the sum of the individual output sets for a given period τ:

$$\overline{P}_\tau(x^1,\ldots,x^K) \equiv \sum_{k=1}^{K} P_\tau^k(x^k), \;\; x^k \in \Re_+^N, \; k=1,\ldots,K. \tag{10}$$

This group output set shows possible overall group output for a given input allocation amongst DMUs. It can be used to define a *group revenue function*, analogous to the individual revenue function:

$$\overline{R}_\tau(x^1,\ldots,x^K,p) \equiv \max_y\{py : y \in \overline{P}_\tau(x^1,\ldots,x^K)\}, \;\; x^k \in \Re_+^N, \; k=1,\ldots,K, \; p \in \Re_{++}^M, \tag{11}$$

and accompanying *group revenue efficiency* measure

$$\overline{RE}_\tau \equiv \overline{RE}_\tau(x^1,\ldots,x^K,\overline{Y},p) \equiv \frac{\overline{R}_\tau(x^1,\ldots,x^K,p)}{p\overline{Y}}, \;\; \text{for } p\overline{Y} \neq 0, \tag{12}$$

where $\overline{Y} = \sum_{k=1}^{K} y^k$ is the aggregate group output.
Note that this assumes all DMUs face common output prices, and so group revenue is maximised against the same prices as all individual revenue functions. In practice this could be the industry average price, the world price etc.

A crucial result here is the intertemporal extension by Zelenyuk (2006) of a result originally derived by Färe & Zelenyuk (2003), which we summarise here as a lemma:

Lemma 1. *Given regularity axioms 1-6, and the above definitions:*

$$\overline{R}_\tau(x^1,\ldots,x^K,p) = \sum_{k=1}^{K} R_\tau^k(x^k,p), \;\; x^k \in \Re_+^N, \; p \in \Re_{++}^M, \tag{13}$$

(see Färe & Zelenyuk (2003) for a proof)

and

$$\overline{RE}_\tau = \overline{TE}_\tau \times \overline{AE}_\tau, \quad \forall \tau, \tag{14}$$

where group revenue efficiency is:

$$\overline{RE}_\tau = \sum_{k=1}^{K} RE_\tau(x^k, y^k, p) \cdot S^k, \tag{15}$$

with weights

$$S^k = \frac{py^k}{p\overline{Y}}, \quad k = 1, \ldots, K, \tag{16}$$

and group technical efficiency is:

$$\overline{TE}_\tau = \sum_{k=1}^{K} TE_\tau^k(x^k, y^k) \cdot S^k, \quad k = 1, \ldots, K, \tag{17}$$

with S^k the same weights as for group revenue efficiency, and group allocative efficiency is:

$$\overline{AE}_\tau = \sum_{k=1}^{K} AE_\tau^k(x^k, y^k, p) \cdot S_{ae}^k, \tag{18}$$

with

$$S_{ae}^k = \frac{py_*^k}{\sum_{k=1}^{K} py_*^k} \text{ where } y_*^k = y^k TE_\tau^k(x^k, y^k), \quad k = 1, \ldots, K. \tag{19}$$

Intuitively, this means that the maximum revenue for the overall group of DMUs is equal to the sum of the individual maximum revenues. That is, if individual DMUs all maximise their revenues given their input endowments and facing the same output prices, then the sum will coincide with maximal group revenue defined by (11). This then allows group efficiencies to be expressed as a weighted sum of individual DMU efficiencies, and maintains a group level decomposition (14) analogous to the individual level decomposition (9).

It is worth noting that an advantage of this aggregation scheme is that it is not *ad hoc* but is derived from economic theory, taking as given output prices and the existing input allocation of each DMU, assuming the regularity axioms of production theory and the particular aggregation structure of the technology given by (10). In the next section we discuss relaxing the assumption of no reallocation of inputs presupposed by (10).

4. Group Efficiency With Reallocation

The assumption that inputs cannot be reallocated across DMUs was relaxed by Nesterenko & Zelenyuk (2007) (who in turn utilised some results from Li & Ng (1995)), whose results we summarise next. Unlike that paper, we define the results in terms of different time periods so we can use them in Malmquist productivity indexes later.

Nesterenko & Zelenyuk (2007) note that given the Färe & Zelenyuk (2003) definition of a group output set, (10), if DMUs were operating under non-constant returns to scale (or some have superior technologies) there may be unrealised output gains from reallocating inputs between DMUs. Such gains would be over and above those gains from all DMUs operating efficiently given their current input endowment. To measure this, consider a *group potential technology* which is the linear aggregation of individual DMU technologies in a given period τ:

$$T_\tau^g \equiv \sum_{k=1}^{K} T_\tau^k. \tag{20}$$

This technology aggregation structure was also proposed by Li & Ng (1995) and Blackorby & Russell (1999). It can be equivalently characterised by the *group potential output set*:

$$P_\tau^g(\overline{X}) = \{y : (\overline{X}, y) \in T_\tau^g\}, \tag{21}$$

where $\overline{X} \equiv \sum_{k=1}^{K} x^k$ is aggregate group input.

An important result from Nesterenko & Zelenyuk (2007) that we will use is summarised in lemma 2.

Lemma 2. *Given regularity axioms 1-6, and the above definitions:*

$$\overline{P}_\tau(x^1, \ldots, x^K) \subseteq P_\tau^g(\overline{X}), \tag{22}$$

(see Nesterenko & Zelenyuk (2007) for a proof)

and

$$RE_\tau^g = TE_\tau^g \times AE_\tau^g, \;\; \forall \tau, \tag{23}$$

where the group potential revenue efficiency is:

$$RE_\tau^g \equiv \frac{R_\tau^g(\overline{X}, p)}{p\overline{Y}}, \tag{24}$$

with the group potential revenue function:

$$R_\tau^g(\overline{X}, p) \equiv \max_y \{py : y \in P_\tau^g(\overline{X})\}, \tag{25}$$

and where group potential technical efficiency is:

$$TE_\tau^g \equiv \max_\theta \{\theta : \theta\overline{Y} \in P_\tau^g(\overline{X})\}, \tag{26}$$

and group potential allocative efficiency is:

$$AE_\tau^g \equiv RE_\tau^g / TE_\tau^g. \tag{27}$$

Intuitively, these measure group efficiency w.r.t the potential output set (21) rather than the group output set (10), and are otherwise defined directly analogously to the individual efficiency measures.

We will also use the concept of reallocative efficiency from Nesterenko & Zelenyuk (2007), again summarised in a lemma.

Lemma 3. *Given regularity axioms 1-6, and the above definitions:*

$$RRE_\tau^g = TRE_\tau^g \times ARE_\tau^g, \;\; \forall \tau, \tag{28}$$

where group revenue reallocative efficiency is:

$$RRE_\tau^g \equiv R_\tau^g(\overline{X}, p) / \overline{R}_\tau(x^1, \ldots, x^K, p), \tag{29}$$

with it immediately following that

$$RE_\tau^g = \overline{RE}_\tau \times RRE_\tau^g, \ \ \forall \tau, \tag{30}$$

and where group technical reallocative efficiency is

$$TRE_\tau^g \equiv TE_\tau^g / \overline{TE}_\tau, \tag{31}$$

and group allocative reallocative efficiency is

$$ARE_\tau^g \equiv AE_\tau^g / \overline{AE}_\tau. \tag{32}$$

Intuitively, each of these measure the difference for the group between all DMUs being individually efficient (given their individual input endowments) and the group being collectively efficient (allowing reallocation of inputs between DMUs).

Given (22), it will always be the case (for feasible input-output combinations) that $RRE_\tau^g \geq 1$ and $TRE_\tau^g \geq 1$, with each equal to unity only when all DMUs are individually and collectively efficient. However, we cannot say what ARE_τ^g will be - it could be less than unity if the current group output mix is closer to maximising group potential revenue rather than individual revenues, and greater than unity in the opposite case.

Finally, decompositions (14), (28) and (30) imply a final decomposition:

$$RE_\tau^g = \overline{TE}_t \times \overline{AE}_t \times TRE_\tau^g \times ARE_\tau^g, \ \ \forall \tau. \tag{33}$$

We want to determine group Malmquist productivity indexes of these measures over time that maintain analogous decompositions to these.

5. Change in Group Productivity Over Time Without Reallocation

We now turn to consider the change in group productivity over time, measured using the *Malmquist Productivity Index* (MPI). This inter-temporal extension was provided by Zelenyuk (2006), whose results we summarise here, to be further extended to allow for reallocation of inputs in section 6.

The productivity change from period s to period t can be measured by the Malmquist productivity index of Caves et al. (1982), which in output orientation can be defined for a DMU k as

$$M_{st}^k \equiv M_{st}^k(x_s^k, x_t^k, y_s^k, y_t^k) = \left[\left(\frac{TE_s^k(x_t^k, y_t^k)}{TE_s^k(x_s^k, y_s^k)} \cdot \frac{TE_t^k(x_t^k, y_t^k)}{TE_t^k(x_s^k, y_s^k)}\right)^{-1}\right]^{1/2}. \tag{34}$$

Note that it is now important to keep track of the time-subscripts of the input-output(-price) combinations, as these can now differ from the period of the function they are being measured by.

Following Zelenyuk (2006) the *revenue (or dual) MPI* from period s to t is:

$$\begin{aligned} RM_{st}^k &\equiv RM_{st}^k(x_s^k, x_t^k, y_s^k, y_t^k, p_s, p_t) \\ &= \left[\left(\frac{RE_s^k(x_t^k, y_t^k, p_t)}{RE_s^k(x_s^k, y_s^k, p_s)} \cdot \frac{RE_t^k(x_t^k, y_t^k, p_t)}{RE_t^k(x_s^k, y_s^k, p_s)}\right)^{-1}\right]^{1/2}, \end{aligned} \tag{35}$$

and the *allocative MPI* from period s to t is:

$$\begin{aligned} AM_{st}^k &\equiv AM_{st}^k(x_s^k, x_t^k, y_s^k, y_t^k, p_s, p_t) \\ &= \left[\left(\frac{AE_s^k(x_t^k, y_t^k, p_t)}{AE_s^k(x_s^k, y_s^k, p_s)} \cdot \frac{AE_t^k(x_t^k, y_t^k, p_t)}{AE_t^k(x_s^k, y_s^k, p_s)}\right)^{-1}\right]^{1/2}, \end{aligned} \tag{36}$$

with this decomposition holding for any input-output-price combination, any two periods s and t, and for all k:

$$RM_{st}^k = M_{st}^k \times AM_{st}^k, \tag{37}$$

which we will want an analogue of at the aggregate level.

It is worth reflecting briefly on what these measures mean, as the revenue and allocative MPIs appear to have been overlooked apart from Zelenyuk (2006), but are superior to the primal MPI when price information is available because they takes prices into account. The primal MPI measures the change in productivity from periods s to t, with the first fraction measuring the change w.r.t technology in period s and the second w.r.t technology in period t, without taking prices into account, which is useful when price information is unavailable. By contrast, the revenue (or dual) MPI measures the change in productivity taking into account price information, so we can consider not just whether the DMU is technically efficient but whether it is achieving maximum revenue. The decomposition (37) shows that this can be decomposed into a consideration of changes without and with prices, where the allocative MPI takes into account changes in the productivity of the allocation of outputs over the two periods. This can reflect both changes in the allocation itself as well as changes in the prices between periods, which will change which allocations are most revenue efficient.

Having defined the individual MPIs, we now summarise the results of Zelenyuk (2006) for the aggregate (group) MPIs in a lemma that we will use in deriving our results.

Lemma 4. *Given regularity axioms 1-6, and the above definitions:*

$$\overline{RM}_{st} = \overline{M}_{st} \times \overline{AM}_{st}, \tag{38}$$

which holds for any two periods s and t, and where the group revenue MPI from period s to t is:

$$\begin{aligned}\overline{RM}_{st} &\equiv \overline{RM}_{st}(x_s^1, \ldots, x_s^K, x_t^1, \ldots, x_t^K, \overline{Y}_s, \overline{Y}_t, p_s, p_t) \\ &= \left[\left(\frac{\sum_{k=1}^K RE_s^k(x_t^k, y_t^k, p_t) \cdot S_t^k}{\sum_{k=1}^K RE_s^k(x_s^k, y_s^k, p_s) \cdot S_s^k} \cdot \frac{\sum_{k=1}^K RE_t^k(x_t^k, y_t^k, p_t) \cdot S_t^k}{\sum_{k=1}^K RE_t^k(x_s^k, y_s^k, p_s) \cdot S_s^k} \right)^{-1} \right]^{1/2} \\ &= \left[\left(\frac{\overline{RE}_s(t)}{\overline{RE}_s(s)} \cdot \frac{\overline{RE}_t(t)}{\overline{RE}_t(s)} \right)^{-1} \right]^{1/2},\end{aligned} \tag{39}$$

with the last line a convenient shorthand which we will use for the rest of the paper. Note that the weights correspond to the period of the input-output-price combination, rather than the period of the technology.
Furthermore, the group technical MPI from period s to t is:

$$\overline{M}_{st} \equiv \overline{M}_{st}(x_s^1, \ldots, x_s^K, x_t^1, \ldots, x_t^K, \overline{Y}_s, \overline{Y}_t) = \left[\left(\frac{\overline{TE}_s(t)}{\overline{TE}_s(s)} \cdot \frac{\overline{TE}_t(t)}{\overline{TE}_t(s)} \right)^{-1} \right]^{1/2}, \tag{40}$$

with the weights the same as for the group revenue MPI, and the group allocative MPI from *period s to t is:*

$$\begin{aligned}\overline{AM}_{st} &\equiv \overline{AM}_{st}(x_s^1,\ldots,x_s^K,x_t^1,\ldots,x_t^K,\overline{Y}_s,\overline{Y}_t,p_s,p_t)\\ &= \left[\left(\frac{\overline{AE}_s(t)}{\overline{AE}_s(s)}\cdot\frac{\overline{AE}_t(t)}{\overline{AE}_t(s)}\right)^{-1}\right]^{1/2},\end{aligned} \tag{41}$$

where the weights here are

$$S_{ae,j}^k = \frac{p_j y_{*,j}^k}{\sum_{k=1}^K p_j y_{*,j}^k} \text{ where } y_{*,j}^k = y_j^k TE_\tau^k(x_j^k,y_j^k),\ k=1,\ldots,K,\ \tau,j=s,t. \tag{42}$$

Thus aggregate Malmquist productivity indexes can be constructed in such a way that the aggregate indexes decompose in a manner analogous to the individual indexes. Each of these group Malmquist productivity indexes are calculated from the individual efficiency scores, after aggregating them appropriately. This aggregation scheme, however, does not allow for reallocation of inputs between DMUs, which is important to allow for because there are potential output gains from such reallocation. This would be particularly relevant for DMUs considering a merger, which makes such reallocation possible and desirable - firms merging within an industry, countries merging into a monetary or customs union, etc. In the next section, we relax this restriction, which was imposed by deriving the aggregation scheme from the linear aggregation of DMU output sets.

6. Change in Group Productivity Over Time With Reallocation

We now merge the concepts presented above to define group (that is, aggregate) Malmquist productivity indexes for measuring the change in group potential productivity and group reallocative productivity over time. These form the main results of this paper. In deriving a coherent aggregation scheme, we want to ensure its consistency with economic theory and previous aggregation schemes, and in particular we want it to have decompositions that are analogous to the group measures.

Our first result proposes an aggregation scheme for group potential MPI, satisfying a decomposition analogous to (23) and maintaining consistency with previous aggregation results.

Proposition 1. *Given regularity axioms 1-6, and the above definitions, we have:*

$$RM_{st}^g = M_{st}^g \times AM_{st}^g, \tag{43}$$

which holds for any two periods s and t, where

$$RM_{st}^g \equiv RM_{st}^g(\overline{X}_s, \overline{X}_t, \overline{Y}_s, \overline{Y}_t, p_s, p_t) = \left[\left(\frac{RE_s^g(t)}{RE_s^g(s)} \cdot \frac{RE_t^g(t)}{RE_t^g(s)} \right)^{-1} \right]^{1/2}, \tag{44}$$

is the group potential revenue MPI from periods s to t, and

$$M_{st}^g \equiv M_{st}^g(\overline{X}_s, \overline{X}_t, \overline{Y}_s, \overline{Y}_t) = \left[\left(\frac{TE_s^g(t)}{TE_s^g(s)} \cdot \frac{TE_t^g(t)}{TE_t^g(s)} \right)^{-1} \right]^{1/2}, \tag{45}$$

is the group potential technical MPI from periods s to t, and

$$AM_{st}^g \equiv AM_{st}^g(\overline{X}_s, \overline{X}_t, \overline{Y}_s, \overline{Y}_t, p_s, p_t) = \left[\left(\frac{AE_s^g(t)}{AE_s^g(s)} \cdot \frac{AE_t^g(t)}{AE_t^g(s)} \right)^{-1} \right]^{1/2}, \tag{46}$$

is the group potential allocative MPI from periods s to t.

The proof of this proposition follows from substituting the decomposition of group potential revenue efficiency, (23), into the group potential revenue MPI, (44), for each period, and then rearranging to separate out the group potential technical and allocative MPI measures.

Intuitively, this proposition shows that we can define the group potential revenue MPI in the same style as the group MPI measures in section 5, and this can be decomposed into group potential technical and allocative MPI measures which are also in the same style as other group MPI measures. Similarly to the group technical MPI, the group potential technical MPI measures the change in group productivity from period s to t, but now allowing for reallocation of inputs between DMUs within the group. Likewise, the group potential revenue MPI measures the change in the group revenue productivity, allowing for reallocation of inputs, and (unlike the group potential technical MPI) takes into account price information. The decomposition allows the researcher to determine which productivity changes are due to changes in group technology or group technical efficiency, and which are due to changes in output prices or group allocative efficiency.

We also want to decompose the group potential revenue MPI into group revenue and group revenue reallocative MPI measures, analogous to decomposition (30) and consistent with previous results. This is achieved in proposition 2.

Proposition 2. *Given regularity axioms 1-6, and the above definitions, we have:*

$$RM^g_{st} = \overline{RM}_{st} \times RRM^g_{st}, \tag{47}$$

which holds for any two periods s and t, where $\overline{RM}_{st}$ is defined in (39) *and*

$$\begin{aligned} RRM^g_{st} \equiv RRM^g_{st}(\overline{X}_s, \overline{X}_t, \overline{Y}_s, \overline{Y}_t, p_s, p_t) &= \left[\left(\frac{RRE^g_s(t)}{RRE^g_s(s)} \cdot \frac{RRE^g_t(t)}{RRE^g_t(s)} \right)^{-1} \right]^{1/2} \\ &= \left[\left(\frac{\sum_{k=1}^K RRE^k_s(x^k_t, y^k_t, p_t) \cdot S^k_t}{\sum_{k=1}^K RRE^k_s(x^k_s, y^k_s, p_s) \cdot S^k_s} \cdot \frac{\sum_{k=1}^K RRE^k_t(x^k_t, y^k_t, p_t) \cdot S^k_t}{\sum_{k=1}^K RRE^k_t(x^k_s, y^k_s, p_s) \cdot S^k_s} \right)^{-1} \right]^{1/2}, \end{aligned} \tag{48}$$

is the group revenue reallocative MPI from periods s to t, with the last equality following from Nesterenko & Zelenyuk (2007, p.112) expressing group revenue reallocative efficiency as a weighted aggregation of individual revenue reallocative efficiencies, with $RRE^k \equiv RE^g/RE^k$ and the weights as for group revenue efficiency.

Similarly to proposition 1, the proof follows by substituting the decomposition of group potential revenue efficiency, (30), into the group potential revenue MPI, (44), and then rearranging to separate out the group revenue MPI and group revenue reallocative MPI measures.

In words, this allows us to decompose the group potential revenue MPI to see which productivity changes are due to changes in the group revenue MPI over time, and which are due to changes in the group revenue reallocative MPI. The latter measure captures the change in potential gains from input reallocation. Where inputs have been reallocated between the two periods to try and exploit the potential gains, their success should be indicated by a decline in the group revenue reallocative MPI (as there are less unexploited gains remaining) and a comparative increase in the group revenue MPI (from potential gains actually achieved). Thus, this decomposition could reveal actual gains in group revenue efficiency which would not be noticed if only the group potential revenue MPI was considered.

In turn, we can decompose the group revenue reallocative MPI into technical and allocative reallocative MPIs over time, analogous to (28):

Proposition 3. *Given regularity axioms 1-6, and the above definitions, we have:*

$$RRM_{st}^{g} = TRM_{st}^{g} \times ARM_{st}^{g}, \tag{49}$$

which holds for any two periods s and t, where

$$\begin{aligned} TRM_{st}^{g} &\equiv TRM_{st}^{g}(\overline{X}_s, \overline{X}_t, \overline{Y}_s, \overline{Y}_t) = \left[\left(\frac{TRE_s^g(t)}{TRE_s^g(s)} \cdot \frac{TRE_t^g(t)}{TRE_t^g(s)} \right)^{-1} \right]^{1/2} \\ &= \left[\left(\frac{\sum_{k=1}^{K} TRE_s^k(x_t^k, y_t^k) \cdot S_t^k}{\sum_{k=1}^{K} TRE_s^k(x_s^k, y_s^k) \cdot S_s^k} \cdot \frac{\sum_{k=1}^{K} TRE_t^k(x_t^k, y_t^k) \cdot S_t^k}{\sum_{k=1}^{K} TRE_t^k(x_s^k, y_s^k) \cdot S_s^k} \right)^{-1} \right]^{1/2}, \end{aligned} \tag{50}$$

is the group technical reallocative MPI from periods s to t, and

$$\begin{aligned} ARM_{st}^{g} &\equiv ARM_{st}^{g}(\overline{X}_s, \overline{X}_t, \overline{Y}_s, \overline{Y}_t, p_s, p_t) = \left[\left(\frac{ARE_s^g(t)}{ARE_s^g(s)} \cdot \frac{ARE_t^g(t)}{ARE_t^g(s)} \right)^{-1} \right]^{1/2} \\ &= \left[\left(\frac{\sum_{k=1}^{K} ARE_s^k(x_t^k, y_t^k, p_t) \cdot S_{ae,t}^k}{\sum_{k=1}^{K} ARE_s^k(x_s^k, y_s^k, p_s) \cdot S_{ae,s}^k} \cdot \frac{\sum_{k=1}^{K} ARE_t^k(x_t^k, y_t^k, p_t) \cdot S_{ae,t}^k}{\sum_{k=1}^{K} ARE_t^k(x_s^k, y_s^k, p_s) \cdot S_{ae,s}^k} \right)^{-1} \right]^{1/2}, \end{aligned} \tag{51}$$

is the group allocative reallocative MPI from periods s to t, with $TRE^k \equiv TE^g/TE^k$, $ARE^k \equiv AE^g/AE^k$ and the weights as before (see Nesterenko & Zelenyuk (2007, p.112)).

Again, the proof of this follows from substituting the decomposition of group revenue reallocative efficiency, (28), into the group revenue reallocative MPI, (48), and then rearranging to separate the group technical and allocative reallocative MPI measures.

Intuitively, this decomposition allows us to identify the source of changes in the group revenue reallocative MPI - to identify if they are due to changes in the group technical reallocative MPI, and/or changes in the group allocative reallocative MPI. The former could reflect that changes in the group technology allow additional potential gains from reallocation, while the latter could reflect whether reallocating inputs between DMUs to exploit potential gains has been successful or not, or reflect price changes leading to the group input-output combination being nearer to all DMU being individually efficient (given input endowments) or the group collectively being efficient (allowing for reallocation). It is also important to note that each component of (48) to (51) can be constructed from the individual efficiency measures.

By determining these group MPI measures such that they are consistent with previous aggregation results, and in particular have analogous decompositions to the group efficiency measures, it is also the case that other group decompositions not yet mentioned hold analogously for these MPI measures. In particular, both the group potential technical and allocative MPIs can be decomposed to reveal changes in the group MPIs and changes in the group reallocative MPIs.

Corollary 1. *Given regularity axioms 1-6, and the above definitions and propositions, we have:*

$$M_{st}^{g} = \overline{M}_{st} \times TRM_{st}^{g}, \tag{52}$$

and

$$AM_{st}^{g} = \overline{AM}_{st} \times ARM_{st}^{g}, \tag{53}$$

which both hold for any two periods s and t.

The proof of both of these follows from taking the group decompositions (31) and (32), rearranging in terms of group potential efficiency, substituting them into their respective group potential MPI measures, and rearranging to separate out the group MPIs from the group reallocative MPI measures.

The intuition is very similar to that for proposition 2, that is, these decompositions allow determination of the source of changes in group potential MPIs. Even if that measure does not change, the decomposition could reveal that reallocating inputs between DMUs to exploit potential gains has been successful if there are improvements in the group MPIs and comparative declines in the group reallocative MPIs. It also important to observe that each component of (52) and (53) can be constructed from the individual efficiency measures.

Finally, we also have the following decomposition of the group potential revenue MPI:

Corollary 2. *Given regularity axioms 1-6, and the above definitions and propositions, we have:*

$$RM^g_{st} = \overline{M}_{st} \times \overline{AM}_{st} \times TRM^g_{st} \times ARM^g_{st}, \tag{54}$$

which holds for any two periods s and t.

The proof of this follows from substituting the decompositions (38) and (49) into decomposition (47).

In words, this allows a fuller decomposition of the group potential revenue MPI, to more precisely identify the sources of productivity change between some periods s and t. As with other MPIs, the interpretation is that values greater than unity indicate an improvement over time, values less than unity indicate a decline, and values equal to unity indicate no overall change. The overall value can then be decomposed into other values, each of which is also interpreted relative to unity. For example, following decomposition (54), if the group potential revenue MPI has increased, was that due to improvements in the group technical MPI ($\overline{AM}_{st}$), the group allocative MPI ($\overline{M}_{st}$) or the group reallocative MPIs (TRM^g_{st} and ARM^g_{st})? These in turn can be further decomposed by applying existing individual level MPI decompositions to the group measures - for example the group technical MPI can be decomposed to differentiate between group technological change and group efficiency change (for individual level MPI decompositions see Färe, Grosskopf, Lindgren & Roos (1994a), Färe, Grosskopf, Norris & Zhang (1994b), Simar & Wilson (1998) to mention just a few, and see Zelenyuk (2006) for a decomposition of aggregate MPI).

Overall, we have been able to derive group potential and group reallocative Malmquist productivity indexes, and have done this in such a way that we maintain analogous decompositions of the group MPIs to those we had with the original measures. Unlike Zelenyuk (2006), our results allow for the reallocation of inputs between DMUs, and so are applicable in a wider variety of cases, especially measuring the productivity gains due to mergers.

A few important remarks are in order about these results:

Remark 1: Practical Implementation

At present, general methods to estimate the group potential technology allowing different firms to have different technologies (as in the theory outlined above) have not been developed. However, Nesterenko & Zelenyuk (2007) note that we can estimate it if we make two additional assumptions, both of which are usual for many methods, including Data Envelopment Analysis (DEA), a common estimation method for MPIs. These assumptions are that T^k is convex and identical across DMUs. With these assumptions the results of Li & Ng (1995) can be used to obtain

$$T_\tau^g = KT_\tau, \text{ where } T_\tau = T_\tau^k \text{ is convex}, \forall\, k = 1, \ldots, K, \ \forall\tau, \tag{55}$$

and so here we will have for any period τ,

$$P_\tau^g(\overline{X}) = KP_\tau(\tilde{x}), \tag{56}$$

where $\tilde{x} \equiv K^{-1} \sum_{k=1}^{K} x^k$, so $P_\tau(\tilde{x})$ is the output set of the average DMU in the group. Using these results, the group potential efficiencies are equal to the efficiency measures of the average DMU in the group, that is, we have:

$$RE_\tau^g(\overline{X}, \overline{Y}, p) = RE_\tau(\tilde{x}, \tilde{y}, p), \tag{57}$$

$$TE_\tau^g(\overline{X}, \overline{Y}) = TE_\tau(\tilde{x}, \tilde{y}), \tag{58}$$

$$AE_\tau^g(\overline{X}, \overline{Y}, p) = AE_\tau(\tilde{x}, \tilde{y}, p) = RE_\tau(\tilde{x}, \tilde{y}, p)/TE_\tau(\tilde{x}, \tilde{y}), \tag{59}$$

where $\tilde{y} \equiv K^{-1} \sum_{k=1}^{K} y^k$ and where RE, TE and AE are as defined in (7), (5) and (8) respectively, with superscript k dropped. Meanwhile, the other components of our MPI measures ($\overline{TE}_\tau$, $\overline{RE}_\tau$ etc.) can be constructed from individual efficiency measures, as clear from (15), (17) etc.

The proof of (57)-(59) can be found in Nesterenko & Zelenyuk (2007).

Remark 2: Price Independent Weights

In practice, a researcher does not always have access to corresponding output prices. The aggregation scheme presented here can still be used in this case, for example with shadow prices (as in Li & Ng (1995)). Alternatively, price independent weights can be used, which were originally developed by Färe & Zelenyuk (2003) and extended by Färe & Zelenyuk (2007) and Simar & Zelenyuk (2007). The principle used here is to make the additional assumption that the share of industry revenue from each output is a known constant, and then use the resulting output revenue shares to calculate price independent weights. Specifically, assume

$$\frac{p_m \overline{Y}_m}{\sum_{m=1}^{M} p_m \overline{Y}_m} = a_m,\ m = 1, ..., M, \tag{60}$$

where $\overline{Y}_m = \sum_{k=1}^{K} y_m^k$ and $a_m \in [0, 1]$ $(m = 1, ..., M)$ are constants (estimated or assumed) such that $\sum_{m=1}^{M} a_m = 1$. After obtaining such constants, let

$$\varpi_m^k = y_m^k / \overline{Y}_m, \tag{61}$$

be the industry share of DMU k in producing the mth output. The price independent weights for each DMU are then:

$$S^k = \sum_{m=1}^{M} a_m \varpi_m^k, \quad k = 1, \ldots, K, \tag{62}$$

which are the weighted sum of a DMU's share of industry output for each output, weighted by the industry revenue share of each output. A special case, applicable if the a_m were unavailable, is to assume them to be identical for all outputs, yielding an unweighted arithmetic average of output shares, as in Färe & Zelenyuk (2003). Note that Simar & Zelenyuk (2007) also extends this to the subgroup context to derive price independent weights for aggregating within and between subgroups.

Remark 3: Geometric Decomposition

The group efficiency MPIs presented here use a harmonic averaging scheme, while many researchers had been using an unweighted geometric average instead. Zelenyuk (2006) discusses the connection between these two, pointing out two basic messages. Firstly, the harmonic averaging is not *ad hoc* but comes from the weighting scheme justified by economic theory. Such a weighting scheme is important to take account of the economic importance of each DMU in the group, which an unweighted average does not. On the other hand, if a geometric weighting scheme were to use the same weights, Zelenyuk (2006) pointed out that both would have the same first order approximation around unity (a natural place to approximate productivity indexes). Monte Carlo simulations suggested that the difference between weighting schemes is relatively small for the range of productivity changes commonly encountered in practice. This suggests that if researchers prefer to use a geometric averaging scheme this is possible in practice, although the economic justification of the harmonic averaging appears to have more solid ground.

7. Conclusion

In this paper we have merged prior work on efficiency aggregation and productivity aggregation to derive an aggregation scheme for Malmquist productivity indexes that allows reallocation of inputs and outputs across DMUs. We have done this by constructing group potential and group reallocative MPI measures (following the definitions of group potential and group reallocative efficiency measures by Nesterenko & Zelenyuk (2007), and incorporating the prior Malmquist aggregation results of Zelenyuk (2006)). These group potential and reallocative MPIs allow measurement over time of changes in group potential productivity when reallocation of inputs amongst the group is possible. This is particularly relevant in considering the productivity gains due to restructuring branches within a firm, firms merging within an industry, countries merging into a monetary or customs union, etc., as reallocation of inputs between DMUs is possible in such scenarios. As our group Malmquist productivity indexes maintain analogous decompositions to the original efficiency measures, they can be decomposed in a variety of ways within and beyond existing MPI decompositions to give decision makers and researchers greater insights into the source of gains or losses in productivity over time. These new group Malmquist productivity indexes are not intended to replace existing results but to extend and encompass them.

A natural extension of this work would be to develop a methodology for bootstrapping the group Malmquist productivity indexes, to allow statistical inference to be performed on the results. This could be done by merging the ideas of Simar & Wilson (1999) and Daskovska, Simar & Van Bellegem (2010) with Simar & Zelenyuk (2007) and Simar & Wilson (2011). A second natural extension would be to construct group potential and reallocative productivity measures using other productivity indexes (for example Hicks-Moorsteen indexes, etc.) A third relevant extension would be to determine a methodology for estimating the group potential technology without requiring the assumption that all individual technologies are convex and identical between DMUs.

8. References

Blackorby, C., & Russell, R. R. (1999). Aggregation of efficiency indices. *Journal of Productivity Analysis*, *12*, 5–20.

Caves, D. W., Christensen, L. R., & Diewert, W. E. (1982). The economic theory of index numbers and the measurement of input, output, and productivity. *Econometrica*, *50*, 1393–1414.

Daskovska, A., Simar, L., & Van Bellegem, S. (2010). Forecasting the malmquist productivity index. *Journal of Productivity Analysis*, *33*, 97–107.

Färe, R., Grosskopf, S., Lindgren, B., & Roos, P. (1994a). Productivity developments in swedish hospitals: A malmquist output index approach. In A. Charnes, W. Cooper, A. Y. Lewin, & L. M. Seiford (Eds.), *Data Envelopment Analysis: Theory, Methodology and Applications* (pp. 253–272). Boston: Kluwer Academic Publishers.

Färe, R., Grosskopf, S., Norris, M., & Zhang, Z. (1994b). Productivity growth, technical progress, and efficiency change in industrialized countries. *The American Economic Review*, *84*, 66–83.

Färe, R., & Primont, D. (1995). *Multi-output production and duality: theory and applications*. New York: Kluwer Academic Publishers.

Färe, R., & Zelenyuk, V. (2003). On aggregate farrell efficiencies. *European Journal of Operational Research*, *146*, 615–620.

Färe, R., & Zelenyuk, V. (2007). Extending färe and zelenyuk (2003). *European Journal of Operational Research*, *179*, 594–595.

Farrell, M. J. (1957). The measurement of productive efficiency. *Journal of the Royal Statistical Society. Series A (General)*, *120*, 253–290.

Førsund, F. R., & Hjalmarsson, L. (1979). Generalized farrell measures of efficiency: An application to milk processing in swedish dairy plants. *The Economic Journal*, *89*, 294.

Li, S.-K., & Ng, Y. C. (1995). Measuring the productive efficiency of a group of firms. *International Advances in Economic Research*, *1*, 377–390.

Nesterenko, V., & Zelenyuk, V. (2007). Measuring potential gains from reallocation of resources. *Journal of productivity analysis*, *28*, 107–116.

Shephard, R. W. (1970). *Theory of cost and production functions* volume no. 4. Princeton, N.J: Princeton University Press.

Simar, L., & Wilson, P. W. (1998). *Productivity growth in industrialized countries*. CORE Discussion Papers 1998036 Université catholique de Louvain, Center for Operations Research and Econometrics (CORE). URL: http://ideas.repec.org/p/cor/louvco/1998036.html.

Simar, L., & Wilson, P. W. (1999). Estimating and bootstrapping malmquist indices. *European Journal of Operational Research*, *115*, 459–471.

Simar, L., & Wilson, P. W. (2011). Inference by the m out of n bootstrap in nonparametric frontier models. *Journal of Productivity Analysis*, *36*, 33–53.

Simar, L., & Zelenyuk, V. (2007). Statistical inference for aggregates of farrell-type efficiencies. *Journal of Applied Econometrics*, *22*, 1367–1394.

Ylvinger, S. (2000). Industry performance and structural efficiency measures: Solutions to problems in firm models. *European Journal of Operational Research*, *121*, 164–174.

Zelenyuk, V. (2006). Aggregation of malmquist productivity indexes. *European Journal of Operational Research*, *174*, 1076–1086.