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AN AGGREGATION PARADIGM FOR HICKS-MOORSTEEN PRODUCTIVITY INDEXES

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Abstract

In this paper we consider the problem of aggregation of Hicks-Moorsteen productivity indexes. The aggregate indexes are derived in a manner which is justified by economic theory, consistent with previous aggregation results, and maintains analogous decompositions to the original measures. These aggregate Hicks-Moorsteen productivity indexes allow researchers to consider the change in group productivity over time, with and without allowing full reallocation of inputs and outputs. We illustrate these indexes with an application to the productivity of countries, demonstrating the difference between the group index and the frequently used simple average.

Keywords: Productivity, Index Numbers, Aggregation, Hicks-Moorsteen index

JEL Classification: C14, C43, D24

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1 Introduction

One of the most popular methods of measuring productivity changes over time is the Malmquist Productivity Index (MPI), introduced in the seminal work of Caves *et al.* (1982). Recently, more attention has been given to an alternative approach, the Hicks-Moorsteen Productivity Index (HMPI), introduced by Diewert (1992) and Bjurek (1996). The HMPI is considered to have appealing theoretical and practical properties, including that it always has feasible solutions and always has a total factor productivity (TFP) interpretation (for more details see O'Donnell (2012a,b)). Like the MPI, the HMPI uses the Shephard (1953, 1970) distance functions to calculate the resulting productivity change for an individual decision making unit (DMU), without including price information.

To describe the productivity change of a group of DMUs as a whole, such as an industry consisting of firms, researchers have generally used simple averages of individual indexes. Recently, aggregation results in the closely related field of efficiency analysis highlight the importance of using a weighted mean (e.g. see Färe & Zelenyuk (2003)). Zelenyuk (2006) developed a similar aggregation approach for the MPI. The goal of this work is to develop such aggregation results for the Hicks-Moorsteen productivity index.

Moreover, in some groups of DMUs, reallocation of inputs and outputs between DMUs is possible (such as different branches within an organisation, when firms have merged within an industry, or when countries are in an economic union where resource reallocation is an aim). In this case, aggregation results should consider as a benchmark the group with resources reallocated optimally between DMUs. Nesterenko & Zelenyuk (2007) derived aggregate efficiency measures using this benchmark, and Mayer & Zelenyuk (2013) extended this approach to the MPI, but no such results exist for the HMPI. In this paper, then, we seek to develop a theoretically justified aggregation scheme for Hicks-Moorsteen productivity indexes, with and without allowing full reallocation of resources, that relates to the individual measures.

Our paper is structured as follows. Section 2 presents the individual efficiency and productivity measures. Section 3 presents the key aggregation results for the group efficiency measures, first restricting reallocation and then allowing full reallocation. Section 4 presents the main results, deriving the aggregate HMPIs allowing full reallocation, and then decomposing these into components with and without allowing reallocation. Section 5 considers practical issues: estimation, price-independent weights, and geometric weighting. Section 6 presents an empirical example of country productivity, contrasting the aggregate HMPIs with

the simple averages of individual indexes. Section 7 concludes and considers directions for further research.

2 Individual Efficiency and Productivity Measures

Let us first consider individual efficiency measures for a group of K DMUs, indexed $k = 1, \dots, K$. Depending on the purpose of study these DMUs could be individual branches, whole firms or organisations, whole industries, or even whole countries. A DMU k uses a vector $x^k = (x_1^k, \dots, x_N^k)' \in \mathfrak{R}_+^N$ of N inputs to produce a vector $y^k = (y_1^k, \dots, y_M^k)' \in \mathfrak{R}_+^M$ of M outputs. Let the individual technology of DMU k at a given time period τ be represented by the *technology set* T_τ^k , defined as:

$$T_\tau^k \equiv \{(x^k, y^k) \in \mathfrak{R}_+^N \times \mathfrak{R}_+^M : \text{DMU } k \text{ can produce } y^k \text{ from } x^k \text{ using the technology in period } \tau\}. \quad (1)$$

T_τ^k can be equivalently (and sometimes more conveniently) characterised by the *output set*, which for DMU k at period τ is $P_\tau^k : \mathfrak{R}_+^N \rightarrow 2^{\mathfrak{R}_+^M}$ where:

$$P_\tau^k(x^k) \equiv \{y^k \in \mathfrak{R}_+^M : (x^k, y^k) \in T_\tau^k\}, \quad x^k \in \mathfrak{R}_+^N. \quad (2)$$

It can likewise be equivalently characterised by the *input requirement set* for DMU k at period τ , $L_\tau^k : \mathfrak{R}_+^M \rightarrow 2^{\mathfrak{R}_+^N}$ where:

$$L_\tau^k(y^k) \equiv \{x^k \in \mathfrak{R}_+^N : (x^k, y^k) \in T_\tau^k\}, \quad y^k \in \mathfrak{R}_+^M. \quad (3)$$

The technology of each DMU is assumed to satisfy standard regularity axioms (see Färe & Primont (1995) for more details). Specifically ($\forall k = 1, \dots, K$ and $\forall \tau$) we assume:

Axiom 1: The technology set T_τ^k is closed.

Axiom 2: The output set $P_\tau^k(x^k)$ is bounded $\forall x^k \in \mathfrak{R}_+^N$.

Axiom 3: There is no ‘free lunch’, that is, one cannot produce something from nothing.

Formally, $(0_N, y^k) \notin T_\tau^k, \quad \forall y^k \geq 0_M$ (i.e. $y^k \geq 0_M, y^k \neq 0_M$).

Axiom 4: It is possible to produce nothing. Formally, $0_M \in P_\tau^k(x^k), \quad \forall x^k \in \mathfrak{R}_+^N$.

Axiom 5: Inputs and outputs are freely (strongly) disposable.

Formally $(x^0, y^0) \in T_\tau^k \implies (x, y) \in T_\tau^k, \quad \forall y \leq y^0, \quad \forall x \geq x^0$.

Axiom 6: Output sets $P_\tau^k(x^k)$ are convex, $\forall x^k \in \mathfrak{R}_+^N$.

Axiom 7: Input requirement sets $L_\tau^k(y^k)$ are convex, $\forall y^k \in \mathfrak{R}_+^M$.

Axioms 6 and 7 are required to ensure that duality results hold. For obtaining theoretical results we do not make the stronger assumption that technology sets T^k are convex, but will introduce it later when considering practical estimation.

Efficiency measures can be presented using output or input orientation, and both are used in constructing the HMPI, so we will present both throughout, starting each time with the output-oriented results. We use the *output-oriented Shephard (1970) distance function* $OD_\tau^k : \mathfrak{R}_+^N \times \mathfrak{R}_+^M \rightarrow \mathfrak{R}_+ \cup \{\infty\}$, defined as:

$$OD_\tau^k(x^k, y^k) \equiv \inf\{\theta : y^k/\theta \in P_\tau^k(x^k)\}. \quad (4)$$

This function completely characterises the technology T_τ^k in the sense that:

$$OD_\tau^k(x^k, y^k) \leq 1 \Leftrightarrow (x^k, y^k) \in T_\tau^k. \quad (5)$$

We also use it to define the *output-oriented Farrell-type technical efficiency* (OTE) measure for DMU k in period τ as:

$$OTE_\tau^k(\cdot) \equiv OTE_\tau^k(x^k, y^k) \equiv 1/OD_\tau^k(x^k, y^k), \quad (x^k, y^k) \in \mathfrak{R}_+^{N+M}. \quad (6)$$

Likewise in input-orientation, the *input-oriented Shephard (1953) distance function* $ID_\tau^k : \mathfrak{R}_+^N \times \mathfrak{R}_+^M \rightarrow \mathfrak{R}_+ \cup \{\infty\}$, is defined as:

$$ID_\tau^k(y^k, x^k) \equiv \sup\{\lambda : x^k/\lambda \in L_\tau^k(y^k)\}. \quad (7)$$

Again, this function completely characterises T_τ^k in the sense that:

$$ID_\tau^k(y^k, x^k) \geq 1 \Leftrightarrow (x^k, y^k) \in T_\tau^k. \quad (8)$$

We likewise use it to define the *input-oriented Farrell-type technical efficiency* (ITE) measure for DMU k in period τ as:

$$ITE_\tau^k(\cdot) \equiv ITE_\tau^k(y^k, x^k) \equiv 1/ID_\tau^k(y^k, x^k), \quad (x^k, y^k) \in \mathfrak{R}_+^{N+M}. \quad (9)$$

These allow us to measure the efficiency of a DMU k , from either orientation, without needing to take prices into account.

We also consider the dual characterisation of $P_\tau^k(x^k)$ given by the *revenue function*:

$$R_\tau^k(x^k, p) \equiv \max_y \{py : y \in P_\tau^k(x^k)\}, \quad x^k \in \mathfrak{R}_+^N, \quad p \in \mathfrak{R}_{++}^M, \quad (10)$$

given an output price row-vector $p = (p_1, \dots, p_M) \in \mathfrak{R}_{++}^M$ corresponding to the M outputs. Note that p is assumed to be the same for all DMUs, a necessary assumption for deriving the aggregation results that follow, which we will discuss further in section 3. Given the revenue function, the *revenue efficiency* (RE) of a DMU k at period τ is:

$$RE_\tau^k(\cdot) \equiv RE_\tau^k(x^k, y^k, p) \equiv \frac{R_\tau^k(x^k, p)}{py^k}, \quad \text{for } py^k \neq 0, \quad (x^k, y^k) \in \mathfrak{R}_+^{N+M}, \quad p \in \mathfrak{R}_{++}^M. \quad (11)$$

It will always be the case, for $(x^k, y^k) \in T_\tau^k$, that $RE_\tau^k(x^k, y^k, p) \geq OTE_\tau^k(x^k, y^k)$. The multiplicative residual that closes the inequality, referred to as *output-oriented allocative efficiency* (OAE), is defined as:

$$OAE_\tau^k(\cdot) \equiv OAE_\tau^k(x^k, y^k, p) \equiv \frac{RE_\tau^k(x^k, y^k, p)}{OTE_\tau^k(x^k, y^k)}, \quad (x^k, y^k) \in \mathfrak{R}_+^{N+M}, \quad p \in \mathfrak{R}_{++}^M, \quad (12)$$

which immediately provides the following decomposition, which holds for any period τ and any DMU k ,

$$\begin{aligned} RE_\tau^k(x^k, y^k, p) &= OTE_\tau^k(x^k, y^k) \times OAE_\tau^k(x^k, y^k, p), \\ \forall (x^k, y^k) &\in \mathfrak{R}_+^{N+M}, \quad p \in \mathfrak{R}_{++}^M, \quad py^k \neq 0, \end{aligned} \quad (13)$$

which we will want an aggregate analogue of later.

Likewise for input orientation, we can consider the dual characterisation of $L_\tau^k(y^k)$, the *cost function*:

$$C_\tau^k(y^k, w) \equiv \min_x \{wx : x \in L_\tau^k(y^k)\}, \quad y^k \in \mathfrak{R}_+^M, \quad w \in \mathfrak{R}_{++}^N, \quad (14)$$

given an input price row-vector $w = (w_1, \dots, w_N) \in \mathfrak{R}_{++}^N$ corresponding to the N inputs. Again, this is assumed to be common for all DMUs, which is necessary for deriving the aggregation results that follow. The *cost efficiency* (CE) of a DMU k at period τ is:

$$CE_\tau^k(\cdot) \equiv CE_\tau^k(y^k, x^k, w) \equiv \frac{C_\tau^k(y^k, w)}{wx^k}, \quad \text{for } wx^k \neq 0, \quad (x^k, y^k) \in \mathfrak{R}_+^{N+M}, \quad w \in \mathfrak{R}_{++}^N. \quad (15)$$

We likewise always have the inequality $CE^k(y^k, x^k, w) \leq ITE^k(y^k, x^k)$, for $(x^k, y^k) \in T_\tau^k$, and the multiplicative residual that closes this, called *input-oriented allocative efficiency* (IAE), is defined as:

$$IAE_\tau^k(\cdot) \equiv IAE_\tau^k(y^k, x^k, w) \equiv \frac{CE_\tau^k(y^k, x^k, w)}{ITE_\tau^k(y^k, x^k)}, \quad (x^k, y^k) \in \mathfrak{R}_+^{N+M}, \quad w \in \mathfrak{R}_{++}^N, \quad (16)$$

which immediately provides the following decomposition, which holds for any period τ and any DMU k ,

$$CE_\tau^k(y^k, x^k, p) = ITE_\tau^k(y^k, x^k) \times IAE_\tau^k(y^k, x^k, w), \\ \forall (x^k, y^k) \in \mathfrak{R}_+^{N+M}, \quad w \in \mathfrak{R}_{++}^N, \quad wx^k \neq 0, \quad (17)$$

which we will also want an aggregate analogue of later.

With these measures, the *Hicks-Moorsteen productivity index* of Diewert (1992) and Bjurek (1996) can now be defined for the productivity change from periods s to t (for DMU k) as:

$$HM_{st}^k(\cdot) \equiv HM_{st}^k(y_s^k, y_t^k, x_s^k, x_t^k) \\ \equiv \left[\left(\frac{OTE_s^k(x_s^k, y_t^k)/OTE_s^k(x_s^k, y_s^k)}{ITE_s^k(y_s^k, x_t^k)/ITE_s^k(y_s^k, x_s^k)} \times \frac{OTE_t^k(x_t^k, y_t^k)/OTE_t^k(x_t^k, y_s^k)}{ITE_t^k(y_t^k, x_t^k)/ITE_t^k(y_t^k, x_s^k)} \right)^{-1} \right]^{1/2}. \quad (18)$$

Note that there is now a time subscript for inputs and outputs. For the output-oriented measures, inputs are held constant at the same period as the technology (while outputs are varied), and for the input-oriented measures, outputs are held constant at the same period as the technology (while inputs are varied).

Given the dual relationship between the technical efficiency measures and the revenue and cost efficiency measures, we can alternatively use the dual Hicks-Moorsteen productivity index, which takes the price information into account, defined in the same spirit as the original.

$$PHM_{st}^k(\cdot) \equiv PHM_{st}^k(y_s^k, y_t^k, x_s^k, x_t^k, p_s, p_t, w_s, w_t) \\ \equiv \left[\left(\frac{RE_s^k(x_s^k, y_t^k, p_t)/RE_s^k(x_s^k, y_s^k, p_s)}{CE_s^k(y_s^k, x_t^k, w_t)/CE_s^k(y_s^k, x_s^k, w_s)} \times \frac{RE_t^k(x_t^k, y_t^k, p_t)/RE_t^k(x_t^k, y_s^k, p_s)}{CE_t^k(y_t^k, x_t^k, w_t)/CE_t^k(y_t^k, x_s^k, w_s)} \right)^{-1} \right]^{1/2}. \quad (19)$$

We call it the *profitability Hicks-Moorsteen productivity index* from periods s to t for DMU k , because we can also represent (19) in terms of profitability components, i.e.:

$$\frac{RE_\tau^k(x_\tau^k, y_j^k, p_j)}{CE_\tau^k(y_\tau^k, x_j^k, w_j)} = \frac{R_\tau^k(x_\tau^k, p_j)/C_\tau^k(y_\tau^k, w_j)}{p_j y_j^k / w_j x_j^k}, \quad (20)$$

for $\tau, j = \{s, t\}$. Note that we again keep inputs (outputs) in the same period as the technology for output-oriented (input-oriented) measures, but allow output (input) prices to vary with the outputs (inputs) for the output-oriented (input-oriented) measures. Intuitively, by measuring revenue/cost with respect to a particular orientation, we are treating the DMU as making choices about that factor (outputs/inputs) given the prices of that factor in that period and a given amount of the other factor. Thus, it makes intuitive sense to allow the prices to vary with the factor of interest in that orientation, as decisions about that factor were made given those prices.

From (18) and (19) we immediately get

$$PHM_{st}^k(\cdot) = HM_{st}^k(\cdot) \times AHM_{st}^k(\cdot), \quad (21)$$

with this decomposition holding for any input-output-price combination, any two periods s and t , and for all k , where

$$\begin{aligned} AHM_{st}^k(\cdot) &\equiv AHM_{st}^k(y_s^k, y_t^k, x_s^k, x_t^k, p_s, p_t, w_s, w_t) \\ &\equiv \left[\left(\frac{OAE_s^k(x_s^k, y_t^k, p_t)/OAE_s^k(x_s^k, y_s^k, p_s)}{IAE_s^k(y_s^k, x_t^k, w_t)/IAE_s^k(y_s^k, x_s^k, w_s)} \right)^{-1} \right]^{1/2} \\ &\quad \times \left[\left(\frac{OAE_t^k(x_t^k, y_t^k, p_t)/OAE_t^k(x_t^k, y_s^k, p_s)}{IAE_t^k(y_t^k, x_t^k, w_t)/IAE_t^k(y_t^k, x_s^k, w_s)} \right)^{-1} \right]^{1/2}, \end{aligned} \quad (22)$$

is the *allocative Hicks-Moorsteen productivity index* from periods s to t for DMU k .

It is worth pausing here to reflect on the meaning of (21), as it appears to have been overlooked, but sometimes might be viewed as superior to the primal HMPI when prices are available, as it takes important economic information about prices into account. The primal HMPI measures the productivity change between periods s and t , with the first fraction measuring the change w.r.t. technology in period s and the second w.r.t. technology in period t , without considering prices. This is useful when price information is unavailable. However, the full decision of the DMU takes into account price information also, and this overall pro-

ductivity change is measured by the profitability HMPI, which considers not only whether the DMU is technically efficient but also whether it is maximising revenue and minimising cost. This provides an alternative to the profitability index proposed by O'Donnell (2012a), which, unlike that index, considers maximum revenue and minimum cost rather than revenue and cost at the technically efficient output and input levels. The decomposition (21) allows this to be decomposed into a consideration of changes with and without prices; the primal HMPI considers productivity changes due to changes in technology and efficiency, while the allocative HMPI considers productivity changes due to changes in the allocation of outputs/inputs, as well as changes in the prices between periods (which will itself change the revenue/cost efficiency of a given allocation). We will want an analogue of this decomposition to hold at the aggregate level, the topic we discuss next.

3 Aggregate Efficiency Measures

We now consider aggregate efficiency measures which measure the efficiency of a group of DMUs, which will be useful in constructing our aggregate HMPI measures in section 4. Here and in the following section we focus on aggregating all DMUs in a group, but it is possible to extend these results to aggregate separate subgroups of DMUs, and then consistently aggregate subgroups into larger groups (see Simar & Zelenyuk (2007)). This can be done at a cost of more complex notation and some additional derivations. We begin by presenting the results when there are some restrictions on the reallocation of inputs and outputs amongst DMUs in the group, and then relax these restrictions.

We denote the input and output allocations amongst DMUs within the group at a given period τ as $X_\tau = (x_\tau^1, \dots, x_\tau^K)$ and $Y_\tau = (y_\tau^1, \dots, y_\tau^K)$, and the sum of these over all DMUs in the group as $\bar{X}_\tau = \sum_{k=1}^K x_\tau^k$ and $\bar{Y}_\tau = \sum_{k=1}^K y_\tau^k$, respectively. Now consider a *group output set* for period τ , (first proposed by Färe & Zelenyuk (2003)) which is the Minkowski sum of the individual output sets for a given period τ :

$$\bar{P}_\tau(X) \equiv \sum_{k=1}^K P_\tau^k(x^k), \quad x^k \in \mathfrak{R}_+^N, \quad k = 1, \dots, K. \quad (23)$$

This group output set shows possible overall group output for a given input allocation amongst DMUs - that is, it does not allow inputs to be reallocated amongst DMUs in the group, a restriction we will relax below.

The group output set can be used to define a *group revenue function*, analogous to the

individual revenue function:

$$\overline{R}_\tau(X) \equiv \max_y \{py : y \in \overline{P}_\tau(X)\}, \quad x^k \in \mathfrak{R}_+^N, \quad k = 1, \dots, K, \quad p \in \mathfrak{R}_{++}^M, \quad (24)$$

and an accompanying *group revenue efficiency* measure

$$\overline{RE}_\tau(X, \overline{Y}, p) = \frac{\overline{R}_\tau(X, p)}{p\overline{Y}}, \quad \text{for } p\overline{Y} \neq 0. \quad (25)$$

It should be noted that this assumes all DMUs face common output prices, meaning that group revenue is maximised against the same prices as all individual revenue functions. In practice this could be the equilibrium market price, the industry average price, or the world price in the case of cross-country analysis, etc.

Likewise we can consider a *group input requirement set* for period τ (following Färe *et al.* (2004)) which is the Minkowski sum of the individual input requirement sets for a given period τ :

$$\overline{L}_\tau(Y) \equiv \sum_{k=1}^K L_\tau^k(y^k), \quad y^k \in \mathfrak{R}_+^M, \quad k = 1, \dots, K. \quad (26)$$

This shows possible overall group input levels that would allow a given set of output production by the DMUs (that is, output production cannot be reallocated amongst DMUs, a restriction we will again relax below). A *group cost function* can then be defined analogously to the individual cost function:

$$\overline{C}_\tau(Y, w) \equiv \max_x \{wx : x \in \overline{L}_\tau(Y)\}, \quad y^k \in \mathfrak{R}_+^M, \quad k = 1, \dots, K, \quad w \in \mathfrak{R}_{++}^N, \quad (27)$$

and the accompanying *group cost efficiency* measure is

$$\overline{CE}_\tau(Y, \overline{X}, w) \equiv \frac{\overline{C}_\tau(Y, w)}{w\overline{X}}, \quad \text{for } w\overline{X} \neq 0. \quad (28)$$

Again, it is assumed all DMUs face common (e.g. equilibrium) input prices, and hence group cost is minimised against the same prices as all individual cost functions.

In order to determine the aggregation functions, we now summarise a number of key aggregation results.

Lemma 1. *Given regularity axioms 1-6, and the above definitions, we have*

$$\overline{R}_\tau(X_\tau, p_j) = \sum_{k=1}^K R_\tau^k(x_\tau^k, p_j), \quad x_\tau^k \in \mathfrak{R}_+^N, \quad p_j \in \mathfrak{R}_{++}^M, \quad (29)$$

so group revenue efficiency is:

$$\overline{RE}_\tau(\cdot) \equiv \overline{RE}_\tau(X_\tau, \overline{Y}_j, p_j) = \sum_{k=1}^K RE_\tau(x_\tau^k, y_j^k, p_j) \cdot S_j^k, \quad (30)$$

with weights

$$S_j^k = \frac{p_j y_j^k}{p_j \overline{Y}_j}, \quad k = 1, \dots, K. \quad (31)$$

Moreover, $\overline{RE}_\tau(\cdot)$ can be decomposed into aggregate technical and allocative components, as

$$\overline{RE}_\tau(X_\tau, \overline{Y}_j, p_j) = \overline{OTE}_\tau(X_\tau, \overline{Y}_j) \times \overline{OAE}_\tau(X_\tau, \overline{Y}_j, p_j), \quad \forall \tau, j, \quad (32)$$

where output-oriented group technical efficiency is:

$$\overline{OTE}_\tau(\cdot) \equiv \overline{OTE}_\tau(X_\tau, \overline{Y}_j) = \sum_{k=1}^K OTE_\tau^k(x_\tau^k, y_j^k) \cdot S_j^k, \quad k = 1, \dots, K, \quad (33)$$

and output-oriented group allocative efficiency is:

$$\overline{OAE}_\tau(\cdot) \equiv \overline{OAE}_\tau(X_\tau, \overline{Y}_j, p_j) = \sum_{k=1}^K OAE_\tau^k(x_\tau^k, y_j^k, p_j) \cdot S_{ae,\tau,j}^k, \quad (34)$$

with

$$S_{ae,\tau,j}^k = \frac{p_j y_j^k OTE_\tau^k(x_\tau^k, y_j^k)}{\sum_{k=1}^K p_j y_j^k OTE_\tau^k(x_\tau^k, y_j^k)}, \quad k = 1, \dots, K. \quad (35)$$

Intuitively, this lemma says that the maximum overall revenue of the DMUs considered as a group is equal to the sum of their individual maximum revenues, given their individual input endowments and facing the same output prices. This allows group efficiencies to be expressed as a weighted sum of individual DMU efficiencies, and maintains a group level decomposition (32) analogous to the individual level decomposition (13). Note that these

weights are not *ad hoc* but are derived from revenue maximising behaviour w.r.t. the aggregation structure (23) and relative to common output prices. Also note that throughout, τ and j are two time periods (which can be the same).

Similarly, for the input orientation:

Lemma 2. *Given regularity axioms 1-5 and 7, and the above definitions, we have*

$$\overline{C}_\tau(Y_\tau, w_j) = \sum_{k=1}^K C_\tau^k(y_\tau^k, w_j), \quad y_\tau^k \in \mathfrak{R}_+^M, \quad w_j \in \mathfrak{R}_{++}^N, \quad (36)$$

so group cost efficiency is:

$$\overline{CE}_\tau(\cdot) \equiv \overline{CE}_\tau(Y_\tau, \overline{X}_j, w_j) = \sum_{k=1}^K CE_\tau(y_\tau^k, x_j^k, w_j) \cdot W_j^k, \quad (37)$$

with weights

$$W_j^k = \frac{w_j x_j^k}{w_j \overline{X}_j}, \quad k = 1, \dots, K. \quad (38)$$

Moreover, $\overline{CE}_\tau(\cdot)$ can be decomposed into aggregate technical and allocative components, as

$$\overline{CE}_\tau(Y_\tau, \overline{X}_j, w_j) = \overline{ITE}_\tau(Y_\tau, \overline{X}_j) \times \overline{IAE}_\tau(Y_\tau, \overline{X}_j, w_j), \quad \forall \tau, j, \quad (39)$$

where input-oriented group technical efficiency is:

$$\overline{ITE}_\tau(\cdot) \equiv \overline{ITE}_\tau(Y_\tau, \overline{X}_j) = \sum_{k=1}^K ITE_\tau^k(y_\tau^k, x_j^k) \cdot W_j^k, \quad k = 1, \dots, K, \quad (40)$$

and input-oriented group allocative efficiency is:

$$\overline{IAE}_\tau(\cdot) \equiv \overline{IAE}_\tau(Y_\tau, \overline{X}_j, w_j) = \sum_{k=1}^K IAE_\tau^k(y_\tau^k, x_j^k, w_j) \cdot W_{ae,\tau,j}^k, \quad (41)$$

with

$$W_{ae,\tau,j}^k = \frac{w_j x_j^k ITE_\tau^k(y_\tau^k, x_j^k)}{\sum_{k=1}^K w_j x_j^k ITE_\tau^k(y_\tau^k, x_j^k)}, \quad k = 1, \dots, K. \quad (42)$$

The intuition here is similar to that for the revenue case: the minimum overall cost for a group of DMUs is equal to the sum of their individual minimum costs, given their individual output production levels and facing the same input prices. Again, group efficiencies can then be expressed as a weighted sum of DMU efficiencies, and the group level decomposition (39) is directly analogous to the individual level decomposition (17). Likewise, these weights are not *ad hoc* but instead derived from economic theory principles.

These two lemmas present aggregate efficiency scores for the group, from either orientation, using the group output set, (23) and group input requirement set, (26), respectively. The proof of lemmas 1 and 2 is similar to that in Färe & Zelenyuk (2003) and Färe *et al.* (2004), and is therefore omitted. As was already noted, these lemmas take the allocation among DMUs of one factor as given, and considers the overall group level of the other factor. For example, the group output set takes each individual DMU's input endowment as given, and considers overall group output; but if DMUs are operating under non-constant returns to scale (or some have superior technologies), there may be unrealised output gains from reallocating inputs between DMUs (and likewise for reallocating output production when considering input orientation). These gains would be additional to those from all DMUs operating efficiently given their current endowments. To measure this, consider a *group potential technology*, which is the linear aggregation of individual DMU technologies in a given period τ :

$$T_{\tau}^g \equiv \sum_{k=1}^K T_{\tau}^k. \quad (43)$$

This technology aggregation structure was earlier used in Li & Ng (1995) and Nesterenko & Zelenyuk (2007) and we follow them here.¹ By aggregating technology sets instead of output and input requirement sets, this group potential technology allows full reallocation of inputs and outputs amongst DMUs in the group. T_{τ}^g can be equivalently characterised by the *group potential output set*:

$$P_{\tau}^g(\bar{X}) = \{y : (\bar{X}, y) \in T_{\tau}^g\}, \quad (44)$$

and likewise equivalently characterised by the *group potential input requirement set*:

$$L_{\tau}^g(\bar{Y}) = \{x : (x, \bar{Y}) \in T_{\tau}^g\}. \quad (45)$$

¹Blackorby & Russell (1999) also proposed this aggregation structure.

We now define the corresponding group efficiency measures. Specifically, let the *group potential output-oriented technical efficiency* be:

$$OTE_{\tau}^g(\cdot) \equiv OTE_{\tau}^g(\bar{X}_{\tau}, \bar{Y}_j) \equiv \max_{\theta} \{ \theta \bar{Y}_j : \theta \bar{Y}_j \in P_{\tau}^g(\bar{X}_{\tau}) \}. \quad (46)$$

Similarly, we can consider the dual characterisation of $P_{\tau}^g(\bar{X})$, the *group potential revenue function*:

$$R_{\tau}^g(\bar{X}_{\tau}, p_j) \equiv \max_y \{ p_j y : y \in P_{\tau}^g(\bar{X}_{\tau}) \}. \quad (47)$$

Given this, let the *group potential revenue efficiency* be:

$$RE_{\tau}^g(\cdot) \equiv RE_{\tau}^g(\bar{X}_{\tau}, \bar{Y}_j, p_j) \equiv \frac{R_{\tau}^g(\bar{X}_{\tau}, p_j)}{p_j \bar{Y}_j}. \quad (48)$$

It must also be clear that $RE_{\tau}^g(\cdot) \geq OTE_{\tau}^g(\cdot)$, and with the same logic as the individual level, we can close this inequality by defining the *group potential output-oriented allocative efficiency* as:

$$OAE_{\tau}^g(\cdot) \equiv OAE_{\tau}^g(\bar{X}_{\tau}, \bar{Y}_j, p_j) \equiv RE_{\tau}^g(\cdot) / OTE_{\tau}^g(\cdot). \quad (49)$$

It immediately follows that:

$$RE_{\tau}^g(\bar{X}_{\tau}, \bar{Y}_j, p_j) = OTE_{\tau}^g(\bar{X}_{\tau}, \bar{Y}_j) \times OAE_{\tau}^g(\bar{X}_{\tau}, \bar{Y}_j, p_j), \quad \forall \tau, j. \quad (50)$$

Intuitively, these measures are defined similarly to the individual efficiency measures, but now measure group efficiency w.r.t. the group potential output set (44). They thus allow for inputs to be reallocated amongst DMUs in the group each period before determining group output for that period (which already implicitly allows outputs to be reallocated within the group).

Likewise for the input orientation, let the *group potential input-oriented technical efficiency* be:

$$ITE_{\tau}^g(\cdot) \equiv ITE_{\tau}^g(\bar{Y}_{\tau}, \bar{X}_j) \equiv \min_{\lambda} \{ \lambda : \lambda \bar{X}_j \in L_{\tau}^g(\bar{Y}_{\tau}) \}. \quad (51)$$

Similarly, we can consider the dual characterisation of $L_{\tau}^g(\bar{Y})$, the *group potential cost func-*

tion:

$$C_\tau^g(\bar{Y}_\tau, w_j) \equiv \min_x \{w_j x : x \in L_\tau^g(\bar{Y}_\tau)\}. \quad (52)$$

Given this, let the *group potential cost efficiency* be:

$$RE_\tau^g(\cdot) \equiv RE_\tau^g(\bar{X}_\tau, \bar{Y}_j, p_j) \equiv \frac{R_\tau^g(\bar{X}_\tau, p_j)}{p_j \bar{Y}_j}. \quad (53)$$

It must also be clear that $CE_\tau^g(\cdot) \leq ITE_\tau^g(\cdot)$, and with the same logic as the individual level, we can close this inequality by defining the *group potential input-oriented allocative efficiency* as:

$$IAE_\tau^g(\cdot) \equiv IAE_\tau^g(\bar{Y}_\tau, \bar{X}_j, w_j) \equiv CE_\tau^g(\cdot)/ITE_\tau^g(\cdot). \quad (54)$$

It immediately follows that:

$$CE_\tau^g(\bar{Y}_\tau, \bar{X}_j, w_j) = ITE_\tau^g(\bar{Y}_\tau, \bar{X}_j) \times IAE_\tau^g(\bar{Y}_\tau, \bar{X}_j, w_j), \quad \forall \tau, j. \quad (55)$$

Intuitively, these measures are again defined analogously to the individual input-oriented efficiency measures, but now measure group efficiency w.r.t. the group potential input requirement set (45). This allows for output plans to be reallocated amongst DMUs in the group each period before determining the group input requirements for that period (which again already implicitly allows inputs to be reallocated within the group).

If we compare the group output and input requirement sets with and without allowing full reallocation, we have the following important results:

Lemma 3. *Given regularity axioms 1-7, and the above definitions, we have*

$$\bar{P}_\tau(X_\tau) \subseteq P_\tau^g(\bar{X}_\tau), \quad (56)$$

and

$$\bar{L}_\tau(Y_\tau) \subseteq L_\tau^g(\bar{Y}_\tau). \quad (57)$$

The proof of (56) is found in Nesterenko & Zelenyuk (2007), while the proof of (57) is analogous, and so is omitted.

In words, the linearly aggregated output and input requirement sets (not allowing full reallocation) will always be subsets of the group potential output and input requirement sets (allowing full reallocation). This implies that for any $(X_\tau, \bar{X}_\tau, p_j)$ we have $R_\tau^g(\bar{X}_\tau, p_j) \geq \bar{R}_\tau(X_\tau, p_j)$ which in turn implies $RE_\tau^g(\cdot) \geq \bar{RE}_\tau(\cdot)$. The multiplicative residual closing this inequality is responsible for reallocation, and so following the terminology in Nesterenko & Zelenyuk (2007) is called *group revenue reallocative efficiency*, and is defined as

$$RRE_\tau^g(\cdot) \equiv RRE_\tau^g(X_\tau, \bar{X}_\tau, p_j) \equiv R_\tau^g(\bar{X}_\tau, p_j) / \bar{R}_\tau(X_\tau, p_j), \quad (58)$$

i.e., we have the following decomposition:

$$RE_\tau^g(\cdot) = \bar{RE}_\tau(\cdot) \times RRE_\tau^g(\cdot), \quad \forall \tau. \quad (59)$$

We now state a useful decomposition for $RRE_\tau^g(\cdot)$.

Lemma 4. *Given regularity axioms 1-6, and the above definitions, $RRE_\tau^g(\cdot)$ can be decomposed as:*

$$RRE_\tau^g(\cdot) = OTRE_\tau^g(\cdot) \times OARE_\tau^g(\cdot), \quad \forall \tau, \quad (60)$$

where group output-oriented technical reallocative efficiency is:

$$OTRE_\tau^g(\cdot) \equiv OTE_\tau^g(\cdot) / \overline{OTE}_\tau(\cdot), \quad (61)$$

and group output-oriented allocative reallocative efficiency is:

$$OARE_\tau^g(\cdot) \equiv OAE_\tau^g(\cdot) / \overline{OAE}_\tau(\cdot). \quad (62)$$

Intuitively, this lemma identifies the residual group efficiency component due to allowing full reallocation of inputs in output orientation. This measures the efficiency gain for the group from allowing full reallocation, over and above the efficiency gain for the group of all DMUs being individually efficient given their current input endowments.

Given (57), similar comments apply in input orientation. Specifically, for any $(Y_\tau, \bar{Y}_\tau, w_j)$ we have $C_\tau^g(\bar{Y}_\tau, w_j) \leq \bar{C}_\tau(Y_\tau, w_j)$ which in turn implies $CE_\tau^g(\cdot) \leq \bar{CE}_\tau(\cdot)$. *Group cost realloca-*

tive efficiency is defined as:

$$CRE_{\tau}^g(\cdot) \equiv CRE_{\tau}^g(Y_{\tau}, \bar{Y}_{\tau}, w_j) \equiv C_{\tau}^g(\bar{Y}_{\tau}, w_j) / \bar{C}_{\tau}(Y_{\tau}, w_j), \quad (63)$$

i.e., we have the following decomposition:

$$CE_{\tau}^g(\cdot) = \bar{CE}_{\tau}(\cdot) \times CRE_{\tau}^g(\cdot), \quad \forall \tau. \quad (64)$$

We now state a useful decomposition for $CRE_{\tau}^g(\cdot)$.

Lemma 5. *Given regularity axioms 1-5 and 7, and the above definitions, $CRE_{\tau}^g(\cdot)$ can be decomposed as:*

$$CRE_{\tau}^g(\cdot) = ITRE_{\tau}^g(\cdot) \times IARE_{\tau}^g(\cdot), \quad \forall \tau, \quad (65)$$

where group input-oriented technical reallocative efficiency is:

$$ITRE_{\tau}^g(\cdot) \equiv ITE_{\tau}^g(\cdot) / \overline{ITE}_{\tau}(\cdot), \quad (66)$$

and group input-oriented allocative reallocative efficiency is:

$$IARE_{\tau}^g(\cdot) \equiv IAE_{\tau}^g(\cdot) / \overline{IAE}_{\tau}(\cdot). \quad (67)$$

Intuitively, each reallocative efficiency measure reveals the difference for the group between all DMUs being individually efficient in input orientation, given their individual output plans, and the group being collectively efficient in input orientation, allowing reallocation of outputs between DMUs.

Nesterenko & Zelenyuk (2007) also present reallocative measures for individual DMUs in output orientation, defined as

$$RRE_{\tau}^k(\cdot) \equiv RE_{\tau}^g(\cdot) / RE_{\tau}^k(\cdot), \quad (68)$$

$$OTRE_{\tau}^k(\cdot) \equiv OTE_{\tau}^g(\cdot) / OTE_{\tau}^k(\cdot), \quad (69)$$

$$OARE_{\tau}^k(\cdot) \equiv OAE_{\tau}^g(\cdot) / OAE_{\tau}^k(\cdot). \quad (70)$$

Likewise, for input orientation we can define:

$$CRE_{\tau}^k(\cdot) \equiv CE_{\tau}^g(\cdot)/CE_{\tau}^k(\cdot), \quad (71)$$

$$ITRE_{\tau}^k(\cdot) \equiv ITE_{\tau}^g(\cdot)/ITE_{\tau}^k(\cdot), \quad (72)$$

$$IARE_{\tau}^k(\cdot) \equiv IAE_{\tau}^g(\cdot)/IAE_{\tau}^k(\cdot). \quad (73)$$

These can then be aggregated into the group reallocative measures, which we summarise as lemma 6 (see Nesterenko & Zelenyuk (2007) for more details).

Lemma 6. *Given regularity axioms 1-7, and the above definitions:*

$$RRE_{\tau}^g = \left(\sum_{k=1}^K (RRE_{\tau}^k(x_{\tau}^k, y_j^k, p_j))^{-1} \cdot S_j^k \right)^{-1}, \quad (74)$$

$$OTRE_{\tau}^g = \left(\sum_{k=1}^K (OTRE_{\tau}^k(x_{\tau}^k, y_j^k))^{-1} \cdot S_j^k \right)^{-1}, \quad (75)$$

$$OARE_{\tau}^g = \left(\sum_{k=1}^K (OARE_{\tau}^k(x_{\tau}^k, y_j^k, p_j))^{-1} \cdot S_{ae,\tau,j}^k \right)^{-1}, \quad (76)$$

$$CRE_{\tau}^g = \left(\sum_{k=1}^K (CRE_{\tau}^k(y_{\tau}^k, x_j^k, w_j))^{-1} \cdot W_j^k \right)^{-1}, \quad (77)$$

$$ITRE_{\tau}^g = \left(\sum_{k=1}^K (ITRE_{\tau}^k(y_{\tau}^k, x_j^k))^{-1} \cdot W_j^k \right)^{-1}, \quad (78)$$

$$IARE_{\tau}^g = \left(\sum_{k=1}^K (IARE_{\tau}^k(y_{\tau}^k, x_j^k, w_j))^{-1} \cdot W_{ae,\tau,j}^k \right)^{-1}, \quad (79)$$

with the weights defined in (31), (35), (38) and (42).

Finally, decompositions (32), (59) and (60) imply this additional decomposition of group potential revenue efficiency:

$$RE_{\tau}^g(\cdot) = \overline{OTE}_{\tau}(\cdot) \times \overline{OAE}_{\tau}(\cdot) \times OTRE_{\tau}^g(\cdot) \times OARE_{\tau}^g(\cdot), \quad \forall \tau. \quad (80)$$

Likewise decompositions (39), (64) and (65) imply:

$$CE_{\tau}^g(\cdot) = \overline{ITE}_{\tau}(\cdot) \times \overline{IAE}_{\tau}(\cdot) \times ITRE_{\tau}^g(\cdot) \times IARE_{\tau}^g(\cdot), \quad \forall \tau. \quad (81)$$

With these aggregate efficiency measures, with and without full reallocation, we can now

construct our aggregate HMPIs, which we do in the following section. We aim to do so in a manner which maintains analogous decompositions to those expressed in (80) and (81).

4 Aggregate Hicks-Moorsteen Productivity Indexes

We begin by constructing a group potential profitability HMPI in terms of the group potential revenue and cost efficiencies (that is, an aggregate profitability HMPI allowing full reallocation), defined similarly to the individual profitability HMPI, and then decompose it into technical and allocative components.

Proposition 1. *Given regularity axioms 1-7, and the above definitions, let the group potential profitability Hicks-Moorsteen productivity index from periods s to t be given by*

$$\begin{aligned} PHM_{st}^g(\cdot) &\equiv PHM_{st}^g(\bar{Y}_s, \bar{Y}_t, \bar{X}_s, \bar{X}_t, p_s, p_t, w_s, w_t) \\ &= \left[\left(\frac{RE_s^g(\bar{X}_s, \bar{Y}_t, p_t)/RE_s^g(\bar{X}_s, \bar{Y}_s, p_s)}{CE_s^g(\bar{Y}_s, \bar{X}_t, w_t)/CE_s^g(\bar{Y}_s, \bar{X}_s, w_s)} \times \frac{RE_t^g(\bar{X}_t, \bar{Y}_t, p_t)/RE_t^g(\bar{X}_t, \bar{Y}_s, p_s)}{CE_t^g(\bar{Y}_t, \bar{X}_t, w_t)/CE_t^g(\bar{Y}_t, \bar{X}_s, w_s)} \right)^{-1} \right]^{1/2}, \end{aligned} \quad (82)$$

then for any two periods s and t it can be decomposed into technical and allocative components as

$$PHM_{st}^g(\cdot) = HM_{st}^g(\cdot) \times AHM_{st}^g(\cdot), \quad (83)$$

where

$$\begin{aligned} HM_{st}^g(\cdot) &\equiv HM_{st}^g(\bar{Y}_s, \bar{Y}_t, \bar{X}_s, \bar{X}_t) \\ &= \left[\left(\frac{OTE_s^g(\bar{X}_s, \bar{Y}_t)/OTE_s^g(\bar{X}_s, \bar{Y}_s)}{ITE_s^g(\bar{Y}_s, \bar{X}_t)/ITE_s^g(\bar{Y}_s, \bar{X}_s)} \times \frac{OTE_t^g(\bar{X}_t, \bar{Y}_t)/OTE_t^g(\bar{X}_t, \bar{Y}_s)}{ITE_t^g(\bar{Y}_t, \bar{X}_t)/ITE_t^g(\bar{Y}_t, \bar{X}_s)} \right)^{-1} \right]^{1/2}, \end{aligned} \quad (84)$$

is the group potential Hicks-Moorsteen productivity index from periods s to t , and

$$\begin{aligned} AHM_{st}^g(\cdot) &\equiv AHM_{st}^g(\bar{Y}_s, \bar{Y}_t, \bar{X}_s, \bar{X}_t, p_s, p_t, w_s, w_t) \\ &= \left[\left(\frac{OAE_s^g(\bar{X}_s, \bar{Y}_t, p_t)/OAE_s^g(\bar{X}_s, \bar{Y}_s, p_s)}{IAE_s^g(\bar{Y}_s, \bar{X}_t, w_t)/IAE_s^g(\bar{Y}_s, \bar{X}_s, w_s)} \right)^{-1} \right]^{1/2} \\ &\quad \times \left[\left(\frac{OAE_t^g(\bar{X}_t, \bar{Y}_t, p_t)/OAE_t^g(\bar{X}_t, \bar{Y}_s, p_s)}{IAE_t^g(\bar{Y}_t, \bar{X}_t, w_t)/IAE_t^g(\bar{Y}_t, \bar{X}_s, w_s)} \right)^{-1} \right]^{1/2}, \end{aligned} \quad (85)$$

is the group potential allocative Hicks-Moorsteen productivity index from periods s to t .

The proof of this follows from taking the group potential profitability HMPI, substituting in the decompositions of the group potential revenue and cost efficiency measures, (50) and (55), for each period, and then rearranging to separate out the group potential primal and allocative HMPI measures. Note that these measures are in the same form as the individual HMPIs, and have been derived from the other measures and relationship (83).

Intuitively, these group potential HMPIs capture the productivity change for the group between the two periods, allowing full reallocation of outputs and inputs amongst the DMUs. Improvements in this measure indicate that the group potential productivity (i.e., productivity when full reallocation is possible) has improved. This measure is particularly relevant in those cases where such reallocation is possible - for studying a firm with many branches, countries forming an economic union where such reallocation is relevant, etc. As the group potential efficiency results of Nesterenko & Zelenyuk (2007) decompose analogously to the individual measures, so these group potential HMPIs also decompose analogously to the individual measures.

Our main goal is to relate aggregate HMPI measures to the individual measures. To achieve this, we decompose these group potential HMPIs into the productivity change with and without allowing full reallocation, the latter of which can be related to the individual measures. We present these in the next two propositions, then show their relationship to the group potential HMPIs as corollaries.

Proposition 2. *Given regularity axioms 1-7, and the above definitions, let the group profitability Hicks-Moorsteen productivity index from periods s to t be given by*

$$\begin{aligned} \overline{PHM}_{st}(\cdot) &\equiv \overline{PHM}_{st}(Y_s, Y_t, X_s, X_t, p_s, p_t, w_s, w_t) \\ &= \left[\left(\frac{\sum_{k=1}^K RE_s(x_s^k, y_t^k, p_t) \cdot S_t^k / \sum_{k=1}^K RE_s(x_s^k, y_s^k, p_s) \cdot S_s^k}{\sum_{k=1}^K CE_s(y_s^k, x_t^k, w_t) \cdot W_t^k / \sum_{k=1}^K CE_s(y_s^k, x_s^k, w_s) \cdot W_s^k} \right)^{-1} \right]^{1/2} \\ &\quad \times \left[\left(\frac{\sum_{k=1}^K RE_t(x_t^k, y_t^k, p_t) \cdot S_t^k / \sum_{k=1}^K RE_t(x_t^k, y_s^k, p_s) \cdot S_s^k}{\sum_{k=1}^K CE_t(y_t^k, x_t^k, w_t) \cdot W_t^k / \sum_{k=1}^K CE_t(y_t^k, x_s^k, w_s) \cdot W_s^k} \right)^{-1} \right]^{1/2}, \quad (86) \end{aligned}$$

then for any two periods s and t it can be decomposed into technical and allocative components

as

$$\overline{PHM}_{st}(\cdot) = \overline{HM}_{st}(\cdot) \times \overline{AHM}_{st}(\cdot), \quad (87)$$

where

$$\begin{aligned} \overline{HM}_{st}(\cdot) &\equiv \overline{HM}_{st}(Y_s, Y_t, X_s, X_t) \\ &= \left[\left(\frac{\sum_{k=1}^K OTE_s(x_s^k, y_t^k) \cdot S_t^k / \sum_{k=1}^K OTE_s(x_s^k, y_s^k) \cdot S_s^k}{\sum_{k=1}^K ITE_s(y_s^k, x_t^k) \cdot W_t^k / \sum_{k=1}^K ITE_s(y_s^k, x_s^k) \cdot W_s^k} \right)^{-1} \right]^{1/2} \\ &\quad \times \left[\left(\frac{\sum_{k=1}^K OTE_t(x_t^k, y_t^k) \cdot S_t^k / \sum_{k=1}^K OTE_t(x_t^k, y_s^k) \cdot S_s^k}{\sum_{k=1}^K ITE_t(y_t^k, x_t^k) \cdot W_t^k / \sum_{k=1}^K ITE_t(y_t^k, x_s^k) \cdot W_s^k} \right)^{-1} \right]^{1/2}, \end{aligned} \quad (88)$$

is the group Hicks-Moorsteen productivity index from periods s to t , and

$$\begin{aligned} \overline{AHM}_{st}(\cdot) &\equiv \overline{AHM}_{st}(Y_s, Y_t, X_s, X_t, p_s, p_t, w_s, w_t) \\ &= \left[\left(\frac{\sum_{k=1}^K OAE_s(x_s^k, y_t^k, p_t) \cdot S_{ae,s,t}^k / \sum_{k=1}^K OAE_s(x_s^k, y_s^k, p_s) \cdot S_{ae,s,s}^k}{\sum_{k=1}^K IAE_s(y_s^k, x_t^k, w_t) \cdot W_{ae,s,t}^k / \sum_{k=1}^K IAE_s(y_s^k, x_s^k, w_s) \cdot W_{ae,s,s}^k} \right)^{-1} \right]^{1/2} \\ &\quad \times \left[\left(\frac{\sum_{k=1}^K OAE_t(x_t^k, y_t^k, p_t) \cdot S_{ae,t,t}^k / \sum_{k=1}^K OAE_t(x_t^k, y_s^k, p_s) \cdot S_{ae,t,s}^k}{\sum_{k=1}^K IAE_t(y_t^k, x_t^k, w_t) \cdot W_{ae,t,t}^k / \sum_{k=1}^K IAE_t(y_t^k, x_s^k, w_s) \cdot W_{ae,t,s}^k} \right)^{-1} \right]^{1/2}, \end{aligned} \quad (89)$$

is the group allocative Hicks-Moorsteen productivity index from periods s to t .

Again, the proof follows from taking the group profitability HMPI, substituting in the decompositions of the group revenue and cost efficiency measures, (32) and (39), for each period, and then rearranging to separate out the group primal and allocative HMPI measures. Note that for the group profitability and primal HMPIs, the weights for the output-oriented measures are those of the same period as the output-price combination, and the weights for the input-oriented measures are those of the same period as the input-price combination, rather (in both cases) than the period of the technology. For the group allocative HMPI, the weights depend on both the period of the technology and the period of the variable factors (revenue shares for output orientation, cost shares for input orientation).

Intuitively, each of the group HMPIs measure the productivity change of the overall group

taking current input endowments (for output-oriented measures) and output production (for input-oriented measures) as given; that is, without allowing full reallocation. Improvements in these measures indicate that the group productivity (taking the input/output allocation amongst DMUs as given) has improved. They have each been constructed in terms of the group efficiency measures, which in turn are constructed from the individual efficiency measures (with appropriate weights). It is the latter that are usually estimated in practice, and this result shows how these individual measures can be consistently aggregated into a group productivity index. Moreover, the aggregation results of Färe & Zelenyuk (2003) for the group efficiency measures decompose analogously to the individual measures (following (32) and (39)), and so our group productivity indexes also decompose analogously to the individual productivity indexes, (21). Thus the group profitability HMPI can be decomposed following (87) into the group productivity change due to changes in group efficiency or technology (the group primal HMPI) and group productivity change due to changes in the allocation of factors within each DMU in the group or changes in the prices faced by the group (the group allocative HMPI).

We also obtain similar results for the group reallocative HMPIs, as we summarise next.

Proposition 3. *Given regularity axioms 1-7, and the above definitions, let the group profitability reallocative Hicks-Moorsteen productivity index from periods s to t be given by*

$$\begin{aligned}
PRHM_{st}^g(\cdot) &\equiv PRHM_{st}^g(\bar{Y}_s, \bar{Y}_t, \bar{X}_s, \bar{X}_t, Y_s, Y_t, X_s, X_t, p_s, p_t, w_s, w_t) \\
&= \left[\left(\frac{RRE_s^g(\bar{X}_s, X_s, \bar{Y}_t, Y_t, p_t)}{RRE_s^g(\bar{X}_s, X_s, \bar{Y}_s, Y_s, p_s)} \right)^{-1} \right]^{1/2} \\
&\quad \times \left[\left(\frac{CRE_s^g(\bar{Y}_s, Y_s, \bar{X}_t, X_t, w_t)}{CRE_s^g(\bar{Y}_s, Y_s, \bar{X}_s, X_s, w_s)} \right)^{-1} \right]^{1/2}, \\
&= \left[\frac{\sum_{k=1}^K (RRE_s^k(x_s^k, y_t^k, p_t))^{-1} \cdot S_t^k / \sum_{k=1}^K (RRE_s^k(x_s^k, y_s^k, p_s))^{-1} \cdot S_s^k}{\sum_{k=1}^K (CRE_s^k(y_s^k, x_t^k, w_t))^{-1} \cdot W_t^k / \sum_{k=1}^K (CRE_s^k(y_s^k, x_s^k, w_s))^{-1} \cdot W_s^k} \right]^{1/2} \\
&\quad \times \left[\frac{\sum_{k=1}^K (RRE_t^k(x_t^k, y_t^k, p_t))^{-1} \cdot S_t^k / \sum_{k=1}^K (RRE_t^k(x_t^k, y_s^k, p_s))^{-1} \cdot S_s^k}{\sum_{k=1}^K (CRE_t^k(y_t^k, x_t^k, w_t))^{-1} \cdot W_t^k / \sum_{k=1}^K (CRE_t^k(y_t^k, x_s^k, w_s))^{-1} \cdot W_s^k} \right]^{1/2}, \quad (90)
\end{aligned}$$

then for any two periods s and t , it can be decomposed as

$$PRHM_{st}^g(\cdot) = RHM_{st}^g(\cdot) \times ARHM_{st}^g(\cdot), \quad (91)$$

where

$$\begin{aligned}
RHM_{st}^g(\cdot) &\equiv RHM_{st}^g(\bar{Y}_s, \bar{Y}_t, \bar{X}_s, \bar{X}_t, Y_s, Y_t, X_s, X_t) \\
&= \left[\left(\frac{OTRE_s^g(\bar{X}_s, X_s, \bar{Y}_t, Y_t)/OTRE_s^g(\bar{X}_s, X_s, \bar{Y}_s, Y_s)}{ITRE_s^g(\bar{Y}_s, Y_s, \bar{X}_t, X_t)/ITRE_s^g(\bar{Y}_s, Y_s, \bar{X}_s, X_s)} \right)^{-1} \right]^{1/2} \\
&\quad \times \left[\left(\frac{OTRE_t^g(\bar{X}_t, X_t, \bar{Y}_t, Y_t)/OTRE_t^g(\bar{X}_t, X_t, \bar{Y}_s, Y_s)}{ITRE_t^g(\bar{Y}_t, Y_t, \bar{X}_t, X_t)/ITRE_t^g(\bar{Y}_t, Y_t, \bar{X}_s, X_s)} \right)^{-1} \right]^{1/2}, \\
&= \left[\frac{\sum_{k=1}^K (OTRE_s^k(x_s^k, y_t^k))^{-1} \cdot S_t^k / \sum_{k=1}^K (OTRE_s^k(x_s^k, y_s^k))^{-1} \cdot S_s^k}{\sum_{k=1}^K (ITRE_s^k(y_s^k, x_t^k))^{-1} \cdot W_t^k / \sum_{k=1}^K (ITRE_s^k(y_s^k, x_s^k))^{-1} \cdot W_s^k} \right]^{1/2} \\
&\quad \times \left[\frac{\sum_{k=1}^K (OTRE_t^k(x_t^k, y_t^k))^{-1} \cdot S_t^k / \sum_{k=1}^K (OTRE_t^k(x_t^k, y_s^k))^{-1} \cdot S_s^k}{\sum_{k=1}^K (ITRE_t^k(y_t^k, x_t^k))^{-1} \cdot W_t^k / \sum_{k=1}^K (ITRE_t^k(y_t^k, x_s^k))^{-1} \cdot W_s^k} \right]^{1/2}, \quad (92)
\end{aligned}$$

is the group reallocative Hicks-Moorsteen productivity index from periods s to t , and

$$\begin{aligned}
ARHM_{st}^g(\cdot) &\equiv ARHM_{st}^g(\bar{Y}_s, \bar{Y}_t, \bar{X}_s, \bar{X}_t, Y_s, Y_t, X_s, X_t, p_s, p_t, w_s, w_t) \\
&= \left[\left(\frac{OARE_s^g(\bar{X}_s, X_s, \bar{Y}_t, Y_t, p_t)/OARE_s^g(\bar{X}_s, X_s, \bar{Y}_s, Y_s, p_s)}{IARE_s^g(\bar{Y}_s, Y_s, \bar{X}_t, X_t, w_t)/IARE_s^g(\bar{Y}_s, Y_s, \bar{X}_s, X_s, w_s)} \right)^{-1} \right]^{1/2} \\
&\quad \times \left[\left(\frac{OARE_t^g(\bar{X}_t, X_t, \bar{Y}_t, Y_t, p_t)/OARE_t^g(\bar{X}_t, X_t, \bar{Y}_s, Y_s, p_s)}{IARE_t^g(\bar{Y}_t, Y_t, \bar{X}_t, X_t, w_t)/IARE_t^g(\bar{Y}_t, Y_t, \bar{X}_s, X_s, w_s)} \right)^{-1} \right]^{1/2}, \\
&= \left[\frac{\sum_{k=1}^K (OARE_s^k(x_s^k, y_t^k, p_t))^{-1} \cdot S_{ae,s,t}^k / \sum_{k=1}^K (OARE_s^k(x_s^k, y_s^k, p_s))^{-1} \cdot S_{ae,s,s}^k}{\sum_{k=1}^K (IARE_s^k(y_s^k, x_t^k, w_t))^{-1} \cdot W_{ae,s,t}^k / \sum_{k=1}^K (IARE_s^k(y_s^k, x_s^k, w_s))^{-1} \cdot W_{ae,s,s}^k} \right]^{1/2} \\
&\quad \times \left[\frac{\sum_{k=1}^K (OARE_t^k(x_t^k, y_t^k, p_t))^{-1} \cdot S_{ae,t,t}^k / \sum_{k=1}^K (OARE_t^k(x_t^k, y_s^k, p_s))^{-1} \cdot S_{ae,t,s}^k}{\sum_{k=1}^K (IARE_t^k(y_t^k, x_t^k, w_t))^{-1} \cdot W_{ae,t,t}^k / \sum_{k=1}^K (IARE_t^k(y_t^k, x_s^k, w_s))^{-1} \cdot W_{ae,t,s}^k} \right]^{1/2}, \quad (93)
\end{aligned}$$

is the group allocative reallocative Hicks-Moorsteen productivity index from periods s to t .

Again, the proof follows from taking the group profitability reallocative HMPI, substituting into it the decompositions of the group revenue and cost reallocative efficiency measures, (60) and (65), for each period, and then rearranging to separate out the group primal and allocative reallocative HMPI measures. For each measure, the last equality (expressing it in terms of individual reallocative efficiency measures) follows from lemma 6.

In words, the group reallocative HMPIs capture the productivity change component due

to allowing full reallocation of inputs and outputs between DMUs in the group, beyond that of all DMUs in the group operating efficiently.

Moreover, the group potential profitability HMPI can decompose into the group profitability HMPI and the group profitability reallocative HMPI, as we present in the next corollary.

Corollary 1. *Given regularity axioms 1-7, and the above definitions, we have*

$$PHM_{st}^g(\cdot) = \overline{PHM}_{st}(\cdot) \times PRHM_{st}^g(\cdot), \quad (94)$$

for any two periods s and t .

The proof follows from taking the group potential profitability HMPI, substituting in the decompositions of group potential revenue and cost efficiency, (59) and (64), and rearranging out the group profitability and group profitability reallocative HMPIs.

The value of this decomposition (94) is that it can reveal the source of group potential productivity changes. For example, if group potential productivity improves, this could be in a way which also improves group productivity proportionally, e.g. technological improvement (then group reallocative productivity would be close to unity). Alternatively, it could be in a way which is neutral for group productivity, e.g. shifts along the frontier (then group reallocative productivity would increase proportionally). It could also be in a way which lowers group productivity, e.g. shifts towards the optimal group allocation at the cost of individual efficiency (then group reallocative productivity would increase even more than group potential productivity), or with some combination of improvement of group productivity and group reallocative productivity.

Similar decompositions hold for the group potential primal and allocative HMPIs, as we summarise in the next corollary.

Corollary 2. *Given regularity axioms 1-7, and the above definitions, we have*

$$HM_{st}^g(\cdot) = \overline{HM}_{st}(\cdot) \times RHM_{st}^g(\cdot), \quad (95)$$

and

$$AHM_{st}^g(\cdot) = \overline{AHM}_{st}(\cdot) \times ARHM_{st}^g(\cdot), \quad (96)$$

both for any two periods s and t .

The proof follows from taking the group potential (allocative) HMPI, substituting in the decompositions of group potential technical (allocative) efficiency, (61) and (66) ((62) and (67)), and rearranging out the group (allocative) and group (allocative) reallocative HMPIs.

The primal decomposition of group potential HMPI, (95), is particularly important, because it does not require as much information as its dual, the group potential profitability HMPI. The dual information (input and output prices) is not always available, which motivates the need for a decomposition that does not require such information, i.e. this primal decomposition (see below for how price independent weights can be calculated for the group HMPI). Again, this decomposition reveals the source of changes in group potential productivity - from changes in group productivity without allowing full reallocation (88), and/or from changes due to allowing full reallocation (92), similarly to the group potential profitability HMPI.

Finally we can determine a full decomposition of the group potential profitability HMPI, analogous to the efficiency decompositions (80) and (81).

Corollary 3. *Given regularity axioms 1-7, and the above definitions, we have*

$$PHM_{st}^g(\cdot) = \overline{HM}_{st}(\cdot) \times \overline{AHM}_{st}(\cdot) RHM_{st}^g(\cdot) \times ARHM_{st}^g(\cdot), \quad (97)$$

for any two periods s and t .

The proof follows from substituting the decompositions of the group and group reallocative profitability HMPIs, (87) and (91), into the decomposition of the group potential profitability HMPI, (94).

This decomposition is valuable as it provides a fuller decomposition of the group potential profitability HMPI, identifying the change due to the group technical HMPI ($\overline{HM}_{st}(\cdot)$), the group allocative HMPI ($\overline{AHM}_{st}(\cdot)$), the group technical reallocative HMPI ($RHM_{st}^g(\cdot)$) and the group allocative reallocative HMPI ($ARHM_{st}^g(\cdot)$). This gives researchers a clearer indication of the sources of productivity change in the group.

Note again that all the group HMPI measures can be calculated from the individual efficiency scores, after appropriate aggregation. It should be stressed that the aggregation

weights used here for aggregating individual efficiency scores are not *ad hoc* but are derived from economic theory and consistent with other aggregation results. Incidentally, note that the weights derived are intuitive measures of the ‘economic importance’ of each firm in each orientation - observed revenue shares in output orientation, and observed cost shares in input orientation. We certainly do not claim that these are the best possible weights, but the value in using them here is that we know how they are derived and from what assumptions. Moreover, they maintain group decompositions analogous to the individual level, and in this productivity context the derivation scheme reveals which weights to use for each period.

In general, these group HMPI measures may yield different results from the case when the simple (equally-weighted) sample mean is used. For example, consider an industry that is dominated by a few large firms and also contains a number of much smaller firms. If the large firms decline in productivity (over a given period) and the small firms improve, a simple mean would suggest that overall industry productivity has improved, but weighting by the economic importance of the firms will reveal that overall industry productivity has actually declined. This is just one example where the two measures (the (weighted) group HMPI measures derived above, and the (equally-weighted) simple mean) will yield quite different results. This is not to say that the simple arithmetic mean is useless - rather, it should be used as a complementary statistic to estimate the first moment of the distribution of HMPI. We would argue, however, that it is important to also compare it to an average that accounts for the economic weight of each observation, i.e. the group measures we have derived here.

5 Practical Matters

Here we discuss three matters related to the practical estimation of these measures: estimating the group potential measures, calculating price independent weights, and comparing harmonic and geometric averages for the HMPI.

Estimation of Group Potential Measures

Note that the group potential HMPI measures are not calculated from the individual efficiency scores, but calculated directly from the group potential technology. At present, general methods to estimate the group potential technology allowing different firms to have different technologies (as in the theory outlined above) have not been developed. However, we can still estimate the group potential efficiency measures if we make two additional assumptions, both of which are usual for many methods, including data envelopment analysis, a common estimation method for HMPIs. These assumptions are that T_τ^k is convex and identical across

DMUs for each period. With these assumptions the results of Li & Ng (1995) can be used to obtain

$$T_\tau^g = KT_\tau, \text{ where } T_\tau = T_\tau^k \text{ is convex, } \forall k = 1, \dots, K, \forall \tau, \quad (98)$$

and so for any period τ we have:

$$P_\tau^g(\bar{X}_\tau) = KP_\tau(\tilde{x}_\tau), \quad (99)$$

where $\tilde{x}_\tau \equiv K^{-1} \sum_{k=1}^K x_\tau^k$, so $P_\tau(\tilde{x}_\tau)$ is the output set of the average DMU in the group in period τ . Using these results, the output-oriented group potential efficiencies are equal to the efficiency measures of the average DMU in the group, that is, we have:

$$OTE_\tau^g(\bar{X}_\tau, \bar{Y}_j) = OTE_\tau(\tilde{x}_\tau, \tilde{y}_j), \quad (100)$$

$$RE_\tau^g(\bar{X}_\tau, \bar{Y}_j, p_j) = RE_\tau(\tilde{x}_\tau, \tilde{y}_j, p_j), \quad (101)$$

$$OAE_\tau^g(\bar{X}_\tau, \bar{Y}_j, p_j) = OAE_\tau(\tilde{x}_\tau, \tilde{y}_j, p_j) = RE_\tau(\tilde{x}_\tau, \tilde{y}_j, p_j) / OTE_\tau(\tilde{x}_\tau, \tilde{y}_j), \quad (102)$$

where $\tilde{y}_j \equiv K^{-1} \sum_{k=1}^K y_j^k$, for any period j , and where OTE, RE and OAE are as defined in (6), (11) and (12) respectively, with superscript k dropped. The proof of (100)-(102) can be found in Nesterenko & Zelenyuk (2007).

We likewise present the equivalent results for the input-oriented measures, where for any period τ we have:

$$L_\tau^g(\bar{Y}_\tau) = KL_\tau(\tilde{y}_\tau), \quad (103)$$

so $L_\tau(\tilde{y}_\tau)$ is the input requirement set of the average DMU in the group in period τ . Using these results, the input-oriented group potential efficiencies are equal to the efficiency measures of the average DMU in the group, that is, we have:

$$ITE_\tau^g(\bar{Y}_\tau, \bar{X}_j) = ITE_\tau(\tilde{y}_\tau, \tilde{x}_j), \quad (104)$$

$$CE_\tau^g(\bar{Y}_\tau, \bar{X}_j, w_j) = CE_\tau(\tilde{y}_\tau, \tilde{x}_j, w_j), \quad (105)$$

$$IAE_\tau^g(\bar{Y}_\tau, \bar{X}_j, w_j) = IAE_\tau(\tilde{y}_\tau, \tilde{x}_j, w_j) = CE_\tau(\tilde{y}_\tau, \tilde{x}_j, w_j) / ITE_\tau(\tilde{y}_\tau, \tilde{x}_j), \quad (106)$$

where ITE, CE and IAE are as defined in (9), (15) and (16) respectively, with superscript k dropped. (The proof is analogous to that found in Nesterenko & Zelenyuk (2007).) Note that

the measures (100) and (104) are the aggregate efficiency measures suggested by Førsund & Hjalmarsson (1979). Calculating the group potential measures as the average DMU (when all DMUs have the same convex technology) enables further intuition. If all DMUs were individually efficient but spread across different points of the frontier, the average DMU would be inefficient relative to that frontier (that is, the group potential measure would be inefficient). This is because, though the DMUs are individually efficient, if they pooled their resources and technology, they could do better - and the gap between their individually efficient and collectively efficient level is the group reallocative efficiency.

Price Independent Weights

In practice, price information is not always available to a researcher (whether input prices, output prices or both). The aggregation scheme can still be used, for example with shadow prices (as in Li & Ng (1995)). Alternatively, price independent weights can be used, which were originally developed by Färe & Zelenyuk (2003) and extended by Färe & Zelenyuk (2007) and Simar & Zelenyuk (2007), all for the output orientation. Here we present these results and extend them ourselves to the input orientation.

For the output orientation, an additional assumption is made that the share of industry revenue from each output is a known constant across DMUs (though it can differ over time); the resulting output revenue shares can then be used to calculate price independent weights. Specifically, for a given time period τ , assume:

$$\frac{p_{m,\tau} \bar{Y}_{m,\tau}}{\sum_{m=1}^M p_{m,\tau} \bar{Y}_{m,\tau}} = a_{m,\tau}, \quad m = 1, \dots, M, \quad (107)$$

where $\bar{Y}_{m,\tau} = \sum_{k=1}^K y_{m,\tau}^k$, and $a_{m,\tau} \in [0, 1]$ ($m = 1, \dots, M$) are constants (estimated or assumed) such that $\sum_{m=1}^M a_{m,\tau} = 1$. After obtaining such constants, let

$$\varpi_{m,\tau}^k = y_{m,\tau}^k / \bar{Y}_{m,\tau}, \quad (108)$$

be the industry share of DMU k in producing the m th output in period τ . The output-oriented price independent weights in period τ for each DMU are then:

$$S_\tau^k = \sum_{m=1}^M a_{m,\tau} \varpi_{m,\tau}^k, \quad k = 1, \dots, K, \quad (109)$$

which are the weighted sum of a DMU's share of industry output for each output, weighted by the industry revenue share of each output in period τ . A special case, applicable if the $a_{m,\tau}$

were unavailable, is to assume them to be identical for all outputs, yielding an unweighted arithmetic average of output shares, as in Färe & Zelenyuk (2003). For extension to the subgroup context to derive price independent weights for aggregating within and between subgroups in output orientation, see Simar & Zelenyuk (2007).

Considering then the input orientation, make an analogous additional assumption that the share of industry cost from each input is a known constant across DMUs (which can vary across time); we then use these input cost shares to derive price independent weights. That is, for a given period τ assume:

$$\frac{w_{n,\tau} \bar{X}_{n,\tau}}{\sum_{n=1}^N w_{n,\tau} \bar{X}_{n,\tau}} = b_{n,\tau}, \quad n = 1, \dots, N, \quad (110)$$

where $\bar{X}_{n,\tau} = \sum_{k=1}^K x_{n,\tau}^k$, and $b_{n,\tau} \in [0, 1]$ ($n = 1, \dots, N$) are constants (estimated or assumed) such that $\sum_{n=1}^N b_{n,\tau} = 1$. After obtaining such constants, let

$$\omega_{n,\tau}^k = x_{n,\tau}^k / \bar{X}_{n,\tau}, \quad (111)$$

be the industry share of DMU k in using the n th input. The input-oriented price independent weights for each DMU are then:

$$W_{\tau}^k = \sum_{n=1}^N b_{n,\tau} \omega_{n,\tau}^k, \quad k = 1, \dots, K, \quad (112)$$

which are the weighted sum of a DMU's share of industry input for each input, weighted by the industry cost share of each input, in period τ . Again, if the $b_{n,\tau}$ were unavailable, a special case would be to assume them to be identical for all inputs, yielding an unweighted arithmetic average of input shares. Likewise these results could be extended to the subgroup case, adapting the results of Simar & Zelenyuk (2007).

Geometric vs. Harmonic Averaging

The aggregation results derived here led to a harmonic averaging scheme for the individual distance functions used to calculate the Hicks-Moorsteen productivity index, but we noted that in practice simple (equally weighted) averages of individual Hicks-Moorsteen productivity indexes had been used. The weights and weighting scheme used here are derived from economic theory; the question is: Is there any relationship between this weighting scheme (for the group HMPI measures) and some geometric weighting scheme with appropriate

weights?² To consider this, let us rewrite the group Hicks-Moorsteen productivity index in terms of distance functions instead of technical efficiency measures, using (6) and (9).

$$\begin{aligned} \overline{HM}_{st}(\cdot) &= \left[\frac{(\sum_{k=1}^K [OD_s^k(x_s^k, y_t^k)]^{-1} \cdot S_t^k)^{-1} / (\sum_{k=1}^K [OD_s^k(x_s^k, y_s^k)]^{-1} \cdot S_s^k)^{-1}}{(\sum_{k=1}^K [ID_s^k(y_s^k, x_t^k)]^{-1} \cdot W_t^k)^{-1} / (\sum_{k=1}^K [ID_s^k(y_s^k, x_s^k)]^{-1} \cdot W_s^k)^{-1}} \right]^{1/2} \\ &\times \left[\frac{(\sum_{k=1}^K [OD_t^k(x_t^k, y_t^k)]^{-1} \cdot S_t^k)^{-1} / (\sum_{k=1}^K [OD_t^k(x_t^k, y_s^k)]^{-1} \cdot S_s^k)^{-1}}{(\sum_{k=1}^K [ID_t^k(y_t^k, x_t^k)]^{-1} \cdot W_t^k)^{-1} / (\sum_{k=1}^K [ID_t^k(y_t^k, x_s^k)]^{-1} \cdot W_s^k)^{-1}} \right]^{1/2}, \quad (113) \end{aligned}$$

Similarly, we can define the geometric analogue of (113) as

$$\begin{aligned} \overline{HM}_{st}^G(\cdot) &= \left[\frac{\prod_{k=1}^K OD_s^k(x_s^k, y_t^k)^{U_t^k} / \prod_{k=1}^K OD_s^k(x_s^k, y_s^k)^{U_s^k}}{\prod_{k=1}^K ID_s^k(y_s^k, x_t^k)^{V_t^k} / \prod_{k=1}^K ID_s^k(y_s^k, x_s^k)^{V_s^k}} \right]^{1/2} \\ &\times \left[\frac{\prod_{k=1}^K OD_t^k(x_t^k, y_t^k)^{U_t^k} / \prod_{k=1}^K OD_t^k(x_t^k, y_s^k)^{U_s^k}}{\prod_{k=1}^K ID_t^k(y_t^k, x_t^k)^{V_t^k} / \prod_{k=1}^K ID_t^k(y_t^k, x_s^k)^{V_s^k}} \right]^{1/2}, \quad (114) \end{aligned}$$

where U_τ^k and V_τ^k ($\tau = s, t$) are some weights - an equally weighted geometric average would assume the same weights for all k .

Clearly (113) and (114) will be different in general. However, when they use the same weights, then $(\sum_{k=1}^K [OD_\tau^k(x_\tau^k, y_j^k)]^{-1} \cdot S_j^k)^{-1}$ and $\prod_{k=1}^K OD_\tau^k(x_\tau^k, y_j^k)^{S_j^k}$ have the same first-order approximation around unity (a natural point to approximate productivity indexes), which is $\sum_{k=1}^K OD_\tau^k(x_\tau^k, y_j^k) \cdot S_j^k$, a result which will hold for $\tau, j = s, t$ and also for the input-oriented distance function. This result then means that:

$$\overline{HM}_{st}(\cdot) \cong \overline{HM}_{st}^G(\cdot), \quad \text{for } (U_s^k, U_t^k) = (S_s^k, S_t^k), \quad (V_s^k, V_t^k) = (W_s^k, W_t^k), \quad (115)$$

that is, (113) and (114) have the same first-order approximation around unity, given both are using the same weights. This means that if researchers prefer to use geometric averaging rather than harmonic, they can use the weights derived from economic theory, and their results will have the same first-order approximation around unity as the harmonic weighting scheme derived from economic theory. Similar results will hold for the group profitability and allocative HMPIs.

²The geometric mean is natural in this context, given the multiplicative way in which the index is constructed.

6 Real Data Example

We here present an empirical example which illustrates the aggregate primal Hicks-Moorsteen productivity indexes derived here and contrasts them with the arithmetic and geometric averages, showing that they can lead to quite different conclusions in practice.

The aggregate indexes constructed in this paper could be calculated using a variety of estimation methods (such as Data Envelopment Analysis (DEA), Stochastic Frontier Analysis (SFA) etc.) and make no assumptions specific to a particular technique. They could also be applied at a variety of levels (such as firm-level, industry-level, country-level etc.). For this illustration we use the DEA estimator (Farrell (1957), Charnes *et al.* (1978)), which is a popular estimator in practice, especially for the HMPI and other productivity indexes. This estimator has a number of desirable statistical properties, including consistency (see Korostelev *et al.* (1995a,b)), a known limiting distribution (see Kneip *et al.* (2008), Park *et al.* (2010)) and consistent bootstrapping techniques (see Kneip *et al.* (2008)). Intuitively, data envelopment analysis estimates the technology by finding the smallest convex free-disposal set that fits the observed data, and then measures efficiency relative to this estimated technology. Specifically, the *estimated output-oriented technical efficiency* with technology (and inputs) in period τ and outputs in period j is:

$$\begin{aligned} \widehat{OTE}_\tau(x_\tau, y_j) = \max_{\theta, z_1, \dots, z_K} \{ \theta > 0 : & \sum_{k=1}^K z_k y_{\tau, m}^k \geq \theta y_{j, m}, \quad m = 1, \dots, M, \\ & \sum_{k=1}^K z_k x_{\tau, n}^k \leq x_{\tau, n}, \quad n = 1, \dots, N, \quad z_k \geq 0, \quad k = 1, \dots, K, \quad \sum_{k=1}^K z_k = 1 \}, \end{aligned} \quad (116)$$

where $\tau, j = \{s, t\}$, to allow us to estimate the four combinations of technologies and outputs we need for our HMPI. Likewise, the *estimated input-oriented technical efficiency* with technology (and outputs) in period τ and inputs in period j is:

$$\begin{aligned} \widehat{ITE}_\tau(y_\tau, x_j) = \min_{\lambda, z_1, \dots, z_K} \{ \lambda > 0 : & \sum_{k=1}^K z_k y_{\tau, m}^k \geq y_{\tau, m}, \quad m = 1, \dots, M, \\ & \sum_{k=1}^K z_k x_{\tau, n}^k \leq \lambda x_{j, n}, \quad n = 1, \dots, N, \quad z_k \geq 0, \quad k = 1, \dots, K, \quad \sum_{k=1}^K z_k = 1 \}. \end{aligned} \quad (117)$$

These estimated technical efficiency scores are then used to calculate the theoretical measures defined above, just substituting $\widehat{OTE}_\tau(\cdot)$ and $\widehat{ITE}_\tau(\cdot)$ in place of $OTE_\tau(\cdot)$ and $ITE_\tau(\cdot)$ respectively. Note that the superscript k is now dropped as the DEA approach assumes that all DMUs have access to the same technology.

We also use a macroeconomic data set, as it might be of interest to a wider audience (application to other contexts would be similar). In particular we use the data from the frequently cited study of Kumar & Russell (2002), which observes capital, labour and GDP for 57 countries from 1965 to 1990 (originally from the Penn World Tables). We focus our attention on the years 1965 and 1990, estimating the productivity change between them. As in the original study, we do not use price data and so only estimate technical efficiency and the primal HMPIs (individual, group, group potential, and group reallocative versions). It is a necessary assumption for deriving the aggregation results that all countries face the same input and output prices. For output, GDP has already been adjusted for purchasing power parity. For the inputs (labour and capital), common prices are a reasonable assumption if there is similar capital and labour mobility between countries. We therefore restrict our attention to the 21 countries in the sample who were OECD members in 1965,³ where such labour and capital mobility is a more relevant assumption. We calculate price independent weights for aggregation in input orientation as discussed in the previous section. For the shares of each input for each period, we estimate them using OLS with a Cobb-Douglas production function, with shares around 0.4 for labour and 0.6 for capital.⁴

Table 1 presents the group HMPI (88), group reallocative HMPI (92), and group potential HMPI (84) respectively, alongside the simple arithmetic, simple geometric, and weighted geometric (using the same weights as the group HMPI) means of the individual HMPIs. We present these results under the assumption of variable returns to scale, which allows the group potential HMPI to include potential productivity gains due to firms operating at non-constant returns to scale. Considering the theoretical aggregate measures, the group potential HMPI indicates a decline in productivity of about 6%. On the other hand, the group HMPI has declined by 11.2% over the time period. This indicates that group potential productivity (allowing full reallocation) has declined slower than group productivity (taking input endowments/output production as given). The difference is due to improvements in

³Austria, Belgium, Canada, Denmark, France, West Germany, Greece, Iceland, Ireland, Italy, Japan, Luxembourg, Netherlands, Norway, Portugal, Spain, Sweden, Switzerland, Turkey, the United Kingdom, and the United States of America.

⁴We tried other shares for calculating the price independent weights, including the common practice of assuming 1/3 shares for capital, 2/3 for labour, and found that choice of shares did not substantially change the results nor change any qualitative conclusions.

Table 1: Estimates of Aggregate Productivity Change using HMPI

Aggregation Method	$\overline{HM}_{65,90}$	$RHM_{65,90}^g$	$HM_{65,90}^g$
Theoretical Aggregate	0.8995	1.0489	0.9435
Simple Arithmetic mean	1.0472		
Simple Geometric mean	1.0172		
Weighted Geometric mean	0.8610		

Table 1: *Notes:* Estimation is via DEA, assuming variable returns to scale, and using price independent weights for aggregation. Calculated with Matlab by authors, using data from Kumar & Russell (2002).

productivity due to potential reallocation of inputs and outputs among countries between 1965 and 1990.

It is important to note that both $\overline{HM}_{st}(\cdot)$ and $HM_{st}^g(\cdot)$ are theoretical benchmarks, defining hypothetical measures under given theoretical assumptions. The researcher will need to decide which underlying assumptions better reflect the area of practical application. If reallocation of inputs and outputs between the DMUs is possible, then the group potential measure $HM_{st}^g(\cdot)$ might be more appropriate to focus on, using the decomposition (95) for understanding the size of the productivity change due to different sources. On the other hand, if reallocation of inputs and outputs between DMUs is difficult, the group measure $\overline{HM}_{st}(\cdot)$ might be more appropriate to focus on, while the other measures provide an indication of the potential gains were such reallocation possible. Where reality is somewhere in the middle, as in this case - there is capital and labour mobility between countries but they are not all in a monetary or customs union directly attempting to reallocate resources - then both measures are important and the group reallocative measure $RHM_{st}^g(\cdot)$ is useful to reveal the difference between the other two measures.

Next, comparing the group HMPI (which is constructed from the original efficiency measures) with the simple (equally weighted) means of individual HMPIs, we see that there is a large difference. In particular, the simple means would suggest that overall productivity had improved by 1.7% or 4.7%, but when we account for the economic importance of each country using the group HMPI, we see that overall productivity has actually *declined* by

11.2% over the period⁵. This shows that we can get not only quantitatively different but also qualitatively different conclusions from the group HMPI and the simple means. Note that the simple means are also larger than even the group potential HMPI.

For this example, the simple means indicate an increase in overall productivity, while the group HMPI (and group potential HMPI) indicates a decrease. This suggests that individually there was more improvement than decline in productivity across countries, but some of those which decline have larger economic weight. While the weights differ between periods and for input and output orientation, we find that there are some countries which are always weighted much higher than others. Under equal weighting, all countries in this sample would be weighted 4.76%. Using the price independent weights derived above, (109) and (112), we find that (for both orientations and both periods, and under variable returns to scale) many countries have weights below 1%, including two with weights below 0.1% (Iceland and Luxembourg) and only six countries are weighted above 5% (France, Germany, Italy, Japan, the United Kingdom and the United States; with Japan (weighted around 15%) and the United States (weighted around 38%) taking the lion's share). Full results for individual HMPIs, their components, and their weights, are shown in the appendix.

We also include, for comparison, the geometric mean which uses the same weights as those for the group HMPI. As we discussed above, these weights are justified by economic theory for the group HMPI but not for the weighted geometric mean; however, both have the same first order approximation around unity. In this case we find that the results of these two measures are closer than those of the aggregate HMPI and the simple geometric mean. In particular, the weighted geometric mean has a lower estimate than the group HMPI by 4.3%, whereas the the simple geometric mean has a higher estimate by 13.1%.

Overall this empirical example has illustrated that accounting for the weight of each DMU in determining aggregate Hicks-Moorsteen productivity indexes can lead to both qualitatively and quantitatively different conclusions relative to using a simple mean. In turn, such differences can lead to very different policy implications. Again, this is not to say that the simple arithmetic mean is useless, but to say that it should be interpreted in connection with the group HMPI, as a measure related to the first moment of individual productivity indexes rather than as an industry productivity index. By contrast, the group HMPI properly accounts for the economic importance of each DMU, using weights that are not *ad hoc* but

⁵Note that the productivity measured here is multi-factor productivity, accounting for both labour and capital. By contrast, the sample mean of the labour productivity index was 1.880 for this data, indicating on average an 88% improvement in labour productivity between 1965 and 1990 for these countries.

derived from economic theory. Likewise, where full reallocation amongst DMUs is possible, it is worth focussing on the group potential HMPI, and its decomposition into the group and group reallocative HMPIs.

7 Conclusion

In this paper we have presented Hicks-Moorsteen productivity indexes that take price information into account, and have derived an aggregation scheme for these indexes, with and without allowing full reallocation of inputs and outputs amongst DMUs in the group. The aggregation scheme is justified by economic theory, consistent with previous aggregation results, and maintains aggregate decompositions that are analogous to the decompositions at the individual level. The group potential HMPIs allow us to determine the overall productivity change allowing full reallocation, and decompose this into components with and without full reallocation. Unlike the simple mean, the latter (group HMPI) indexes take into account the economic importance of each DMU, in an intuitive and theoretically grounded way, and so provide an important alternative to the simple mean, as demonstrated in our empirical example.

While we have derived theoretical measures for aggregate Hicks-Moorsteen productivity indexes, in practice (as in our empirical illustration) we have only the estimated measures. An important extension to this work would be to develop a bootstrapping methodology for the aggregate HMPIs as well as for individual HMPIs (which to our knowledge has not yet been done). In the DEA context, this could be done by merging the ideas of Simar & Wilson (1999) and Daskovska *et al.* (2010) with Simar & Zelenyuk (2007) and Simar & Wilson (2011). Another natural extension would be to determine a method for estimating the group potential technology without assuming that technology is convex and identical across DMUs.

8 Appendix

Table 2: Estimates of Individual Components of HMPI

Country	OTE				ITE				HMPI
	65(90)	65(65)	90(90)	90(65)	65(90)	65(65)	90(90)	90(65)	
Austria	0.522	1.146	1.366	3.000	0.443	0.872	0.734	2.719	0.8130
Belgium	0.684	1.415	1.160	2.400	0.555	0.707	0.862	1.803	1.2676
Canada	0.469	1.254	1.070	2.858	0.462	0.798	0.935	2.534	1.2340
Denmark	0.740	1.318	1.421	2.532	0.512	0.759	0.704	1.673	0.9493
France	0.564	1.249	1.211	2.684	0.490	0.801	0.825	2.431	1.0101
Germany, W.	0.740	1.446	1.246	2.437	0.540	0.692	0.802	1.831	1.1434
Greece	0.665	1.738	1.602	4.186	0.250	0.575	0.625	2.183	0.9225
Iceland	0.354	1.000	1.000	2.826	0.589	1.000	1.000	4.788	0.9911
Ireland	0.478	1.351	1.118	3.156	0.328	0.741	0.896	3.033	1.0214
Italy	0.608	1.480	1.120	2.726	0.453	0.676	0.893	2.262	1.2518
Japan	0.323	1.289	1.625	6.479	0.203	0.741	0.615	4.969	0.7343
Luexembourg	0.459	1.000	1.000	2.179	0.819	1.000	1.000	1.568	1.5749
Netherlands	0.554	1.187	1.120	2.400	0.526	0.843	0.893	2.293	1.0557
Norway	0.651	1.591	1.260	3.079	0.442	0.637	0.793	1.142	1.6972
Portugal	0.342	1.148	1.000	3.358	0.240	0.871	1.000	4.359	0.8446
Spain	0.393	1.000	1.186	3.016	0.372	1.000	0.843	4.191	0.6960
Sweden	0.712	1.238	1.297	2.253	0.590	0.808	0.771	1.732	0.9911
Switzerland	0.690	1.170	1.122	1.904	0.700	0.856	0.891	1.190	1.3270
Turkey	0.356	1.322	1.138	4.231	0.155	0.757	0.874	4.550	0.7387
U.K.	0.550	1.000	1.000	1.817	0.527	1.000	1.000	2.739	0.7968
U.S.A.	0.500	1.000	1.000	2.000	0.656	1.000	1.000	3.026	0.9316
Aggregate	0.508	1.150	1.169	2.643	0.472	0.858	0.855	2.975	0.8995
Re-allocative	1.057	1.057	1.065	1.065	0.967	0.959	0.940	0.862	1.0489
Potential	0.537	1.215	1.244	2.814	0.457	0.823	0.804	2.563	0.9435

Table 2: *Notes:* Individual components of HMPI, presented by country. For each column, the first number in the heading is the period of the reference technology, the second (in brackets) is the period of the variable factor (output in output orientation, inputs in input orientation). Calculated with Matlab by authors, using data from Kumar & Russell (2002).

Table 3: Estimated Price Independent Weights of Individual Countries

Country	OTE:65	OTE:90	ITE:65	ITE:90
Austria	0.0088	0.0085	0.0088	0.0094
Belgium	0.0126	0.0115	0.0142	0.0110
Canada	0.0336	0.0397	0.0346	0.0383
Denmark	0.0079	0.0062	0.0084	0.0071
France	0.0701	0.0687	0.0717	0.0675
Germany, W.	0.0913	0.0790	0.1057	0.0983
Greece	0.0052	0.0060	0.0082	0.0081
Iceland	0.0002	0.0003	0.0002	0.0003
Ireland	0.0023	0.0028	0.0028	0.0027
Italy	0.0583	0.0627	0.0708	0.0569
Japan	0.0875	0.1542	0.1324	0.2061
Luxembourg	0.0006	0.0005	0.0008	0.0005
Netherlands	0.0179	0.0170	0.0171	0.0154
Norway	0.0051	0.0055	0.0119	0.0067
Portugal	0.0043	0.0064	0.0066	0.0071
Spain	0.0289	0.0325	0.0271	0.0319
Sweden	0.0143	0.0110	0.0141	0.0123
Switzerland	0.0129	0.0096	0.0185	0.0141
Turkey	0.0111	0.0183	0.0250	0.0349
U.K.	0.0822	0.0661	0.0703	0.0567
U.S.A.	0.4450	0.3936	0.3509	0.3147
Total	1	1	1	1

Table 3: *Notes:* Price independent weights for each orientation and period, calculated with Matlab by authors, with input shares for each period estimated via OLS with a Cobb-Douglas production function, using data from Kumar & Russell (2002).

9 References

References

- Bjurek, Hans. 1996. The Malmquist Total Factor Productivity Index. *The Scandinavian Journal of Economics*, **98**(2), 303–313.
- Blackorby, Charles, & Russell, R. Robert. 1999. Aggregation of Efficiency Indices. *Journal of Productivity Analysis*, **12**(1), 5–20.
- Caves, Douglas W., Christensen, Laurits R., & Diewert, W. Erwin. 1982. The Economic Theory of Index Numbers and the Measurement of Input, Output, and Productivity. *Econometrica*, **50**(6), 1393–1414.
- Charnes, Abraham, Cooper, William, & Rhodes, Edwardo. 1978. Measuring the efficiency of decision making units. *European Journal of Operational Research*, **2**(6), 429–444.
- Daskovska, Alexandra, Simar, Léopold, & Van Belleghem, Sbastien. 2010. Forecasting the Malmquist productivity index. *Journal of Productivity Analysis*, **33**(2), 97–107.
- Diewert, W. Erwin. 1992. Fisher ideal output, input, and productivity indexes revisited. *Journal of Productivity Analysis*, **3**(3), 211–248.
- Färe, Rolf, & Primont, Daniel. 1995. *Multi-output production and duality: theory and applications*. New York: Kluwer Academic Publishers.
- Färe, Rolf, & Zelenyuk, Valentin. 2003. On aggregate Farrell efficiencies. *European Journal of Operational Research*, **146**(3), 615–620.
- Färe, Rolf, & Zelenyuk, Valentin. 2007. Extending Färe and Zelenyuk (2003). *European Journal of Operational Research*, **179**(2), 594–595.
- Färe, Rolf, Grosskopf, Shawna, & Zelenyuk, Valentin. 2004. Aggregation of cost efficiency: indicators and indexes across firms. *Academia Economic Papers*, **32**, 395–411.
- Farrell, M. J. 1957. The Measurement of Productive Efficiency. *Journal of the Royal Statistical Society. Series A (General)*, **120**(3), 253–290.
- Førsund, Finn R, & Hjalmarsson, Lennart. 1979. Generalized Farrell Measures of Efficiency: An Application to Milk Processing in Swedish Dairy Plants. *The Economic Journal*, **89**(354), 294.

- Kneip, Alois, Simar, Léopold, & Wilson, Paul W. 2008. Asymptotics and Consistent Bootstraps for DEA Estimators in Nonparametric Frontier Models. *Econometric Theory*, **24**(6), 1663–1697.
- Korostelev, Alexander, Simar, Léopold, & Tsybakov, Alexandre B. 1995a. Efficient Estimation of Monotone Boundaries. *The Annals of Statistics*, **23**(2), 476–489.
- Korostelev, Alexander, Simar, Léopold, & Tsybakov, Alexandre B. 1995b. On estimation of monotone and convex boundaries. *Publications de l'Institut de Statistique des Universités de Paris*, **39**(1), 3–18.
- Kumar, Subodh, & Russell, R. R. 2002. Technological change, technological catch-up, and capital deepening: relative contributions to growth and convergence. *The American Economic Review*, **92**(3), 527–548.
- Li, Sung-Ko, & Ng, Ying C. 1995. Measuring the productive efficiency of a group of firms. *International Advances in Economic Research*, **1**(4), 377–390.
- Mayer, Andreas, & Zelenyuk, Valentin. 2013. *Aggregation of Malmquist productivity Indexes allowing for Reallocation of Resources*. CEPA Working Papers Series WP06/2013. School of Economics, University of Queensland, Australia.
- Nesterenko, Vladimir, & Zelenyuk, Valentin. 2007. Measuring potential gains from reallocation of resources. *Journal of Productivity Analysis*, **28**(1/2), 107–116.
- O'Donnell, Chris J. 2012a. An aggregate quantity framework for measuring and decomposing productivity change. *Journal of Productivity Analysis*, **38**(3), 255.
- O'Donnell, Christopher J. 2012b. Nonparametric Estimates of the Components of Productivity and Profitability Change in U.S. Agriculture. *American Journal of Agricultural Economics*, **94**(4), 873.
- Park, Byeong U, Jeong, Seok-Oh, & Simar, Léopold. 2010. Asymptotic Distribution of Conical-Hull Estimators of Directional Edges. *Annals of Statistics*, **38**(3), 1320–1340.
- Shephard, Ronald W. 1953. *Cost and production functions*. Princeton: Princeton University Press.
- Shephard, Ronald W. 1970. *Theory of cost and production functions*. Princeton, N.J: Princeton University Press.

- Simar, Léopold, & Wilson, Paul W. 1999. Estimating and bootstrapping Malmquist indices. *European Journal of Operational Research*, **115**(3), 459–471.
- Simar, Léopold, & Wilson, Paul W. 2011. Inference by the m out of n bootstrap in nonparametric frontier models. *Journal of Productivity Analysis*, **36**(1), 33–53.
- Simar, Léopold, & Zelenyuk, Valentin. 2007. Statistical Inference for Aggregates of Farrell-Type Efficiencies. *Journal of Applied Econometrics*, **22**(7), 1367–1394.
- Zelenyuk, Valentin. 2006. Aggregation of Malmquist productivity indexes. *European Journal of Operational Research*, **174**(2), 1076–1086.