Centre for Efficiency and Productivity Analysis

Working Paper Series
No. WP05/2016

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Date: May 2016

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ISSN No. 1932 - 4398

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Abstract

Measuring profit efficiency is a challenging task. This paper synthesizes existing approaches to form a general Farrell-type model of profit efficiency. Our derivations help us unveil new and interesting relationship between existing profit efficiency measures and the Farrell-type profit efficiency measures. In turn, this helps us establishing a complete framework of studying efficiency behavior of firms, where the profit efficiency measure satisfies some desirable properties and contains Farrell output-oriented or input-oriented measures of technical efficiency and allocative efficiency as multiplicative elements. The new component, revenue efficient allocative efficiency, introduced in this paper can help firms to make decision and has not been studied in the literature before.

Subject classification	:	Organizational studies: productivity. Profit efficiency. Data envelopment analysis
Area of review	:	Optimization

JEL Codes: C44, D24

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1. Introduction

In measuring technical efficiency of production two types of scaling inputs and outputs are frequently used: (i) scaling each component of a vector; and (ii) scaling the vector by itself. These two approaches are often referred to as the Russell measure and the Farrell measure, respectively. In this paper we apply the above scaling methods to the profit function and introduce some Farrell-type efficiency measures in this setting.

The Farrell technical efficiency measure was proposed by Farrell (1957) and then generalized and popularized by Charnes, Cooper and Rhodes (1978), Färe and Lovell (1978), and Banker, Charnes and Cooper (1984), just to mention a few. To strengthen the ability of measures of technical efficiency to differentiate firm performance, researchers impose conditions to the empirical model. For examples, "assurance region" (Thompson et al. 1986), "factor weights" (Ruggiero 2000), "performance standards" (Cook and Zhu 2006) and "input allocation" (Cherchye et al. 2013) are added to the model of measuring technical efficiency. When both price and quantity data are available, investigating the profit efficiency would be a natural way to study the performance of a firm, and this is the approach we take in this paper.

Farrell (1957) also introduced the concepts of overall (cost or revenue) efficiency and its decomposition into technical and allocative efficiencies. These efficiency measures can be characterized by the scaling method (ii) stated above. Here we refer to them as Farrell-type efficiency measures. The dominating popularity of Farrell-type efficiency measures in practice is reflected by thousands of empirical works in the past several decades. It is desirable to include them as components when the performance of a firm is analyzed. However, all Farrell output-oriented and input-oriented measures have not been related (at least explicitly) to any profit efficiency measure as multiplicative components and filling this gap is one of the main contributions of this paper.

It is worth noting that measuring profit efficiency is a particularly challenging task. For example, the maximal profit level can be zero, positive, or even undefined. Meanwhile, the actual profit level, besides possibility to be positive, can also be zero or even negative. This complicates the problem of defining a suitable measure of profit efficiency so that it is well defined mathematically as well as has convenient (e.g., percentage) interpretation. Various measures of profit efficiency were suggested to overcome these difficulties.

Here we will synthesize existing ideas in the literature (some of which were proposed in a different context) into a Farrell-type framework of studying efficiency behavior of firms. Importantly, we will explicitly relate to profit maximization principle and derive several Farrell-type profit efficiency measures that have useful interpretations and some advantages over existing measures. Related ideas of these profit-efficiency measures go back to the works of Banker and Maindiratta (1988), Chavas and Cox (1994), Färe and Primont (1995), Chambers et al. (1998), and Ray (2004). The general model introduced in this paper helps us unveil new and interesting relationship between existing profit efficiency measures and the Farrell-type profit efficiency measures. Further, the output-oriented profit efficiency measure contains Farrell output-oriented measures of technical efficiency and allocative efficiency as multiplicative elements.

We illustrate our approach with a help of data on US banks (from Ray (2004)). By employing Data Envelopment Analysis (DEA) , a new component, revenue efficient allocative efficiency, is identified to help the firm to determine her production scale and input mix. Our approach can also be adapted to other variants, such as the IDEA (Cooper, Park and Yu (2001) and Zhu (2004)), CAR-DEA (Cook and Zhu (2008)), DEA with non-homogeneous firms (Cook et al. (2013)), and based on DEA nonparametric models of optimizing behavior (Cherchye et al (2008)), to mention just a few.

In the next section, we introduce a general framework of profit efficiency measure. Then section 3 discusses the Farrell output-oriented profit efficiency measure specifically. In section 4, some interpretations of this measure are provided. In section 5, we show that the relationship among different measures in Farrell-type framework, technical efficiency, revenue efficiency and cost efficiency are just some special case of profit efficiency under different constraints. Section 6 expresses multiplicative decomposition of the profit efficiency measure. Section 7 links this Farrell-type profit efficiency measure to some existing profit efficiency measures. Section 8 illustrates the

Farrell-type profit efficiency measure and its decomposition. Finally, we conclude our main results in section 9.

2. A General Framework for Farrell-type Measures of Profit Efficiency

Let $x \in \mathbb{R}_{+}^{N}$ and $q \in \mathbb{R}_{+}^{M}$ be input and output vectors. The corresponding prices are denoted with row-vectors $w \in \mathbb{R}_{+}^{N}$ and $p \in \mathbb{R}_{+}^{M}$, respectively. The production technology is given by $\mathfrak{I} = \{(x, q): x\ can\ produce\ q\}$ which we assume satisfying standard regularity conditions (see Färe and Primont (1995), Table 2.1, p. 27). Let (x^0, q^0) be an observed input-output vector and let the observed total revenue (or sales) at output prices p^0 and observed total costs at input prices w^0 be denoted by $R^0 = p^0 q^0$ and $C^0 = w^0 x^0$ respectively. Further, let π^0 be the observed profit, i.e., $\pi^0 = p^0 q^0 - w^0 x^0$. The maximal profit a firm can attain facing prices (p^0, w^0) and using technology $\mathfrak{I}$ is given by

$$\pi(p^0, w^0) = \sup_{x,q} \{p^0 q - w^0 x: (x, q) \in \mathfrak{I}\} \tag{1}$$

In the following discussions, a firm will be referred to as *profit efficient* if an only if it achieves maximal profit, i.e., when $\pi(p^0, w^0) = \pi^0$.

Before going further with a definition of profit efficiency, it is important to note that there are quite many cases where using $\pi(p, w)$ is problematic. This includes the case of global increasing returns to scale (IRS), where $\pi(p, w) = +\infty$. Similar situation is with the constant returns to scale (CRS), where $\pi(p, w) = +\infty$ or $\pi(p, w) = 0$, depending on the prices (p, w) relative to the shape of frontier of technology set $\mathfrak{I}$. One potential remedy to such cases is to allow for additional constraints because in practice, firms indeed often face various constraints: e.g., fixed inputs or endowments at a given period, minimal employment requirement, maximal amount of inputs available, budget limits, etc. We will denote such constraints with a constraint set Z and, so the constrained profit will be denoted with $\pi(p, w|Z)$, where $Z = \{(x, q): R_{lb} \leqq pq \leqq R_{ub}, q_{lb} \leqq q \leqq q_{ub}, C_{lb} \leqq wx \leqq C_{ub}, and\ x_{lb} \leqq x \leqq x_{ub}\}$. One could also add many other types of constraints. Such constraint set has been adopted extensively in the measurement of technical efficiency. For example, when there are some nondiscretionary input variables (Kopp 1981, and Banker and Morey 1986), the lower and upper bounds for some inputs are equal. If each input n and output m is associated with a "weight" w_n or p_m, the assurance region method introduced by Thompson et al. (1986) imposes constraints like $\alpha \leqq w_i/w_j \leqq \beta$ where α and β are two positive numbers. In the special case when each lower bound is zero and each upper bound is positive infinity, the constraint set becomes $Z = \mathbb{R}_+^{M+N}$ and $\pi(w,p|Z)$ will be the usual profit function. We will refer to $\pi(w,p)$ and $\pi(p,w|Z)$ as *unconstrained and constrained maximal profit function*, respectively. An example of such generalized profit function is given by

$$\pi(p,w|Z) = \sup_{x,q}\{pq - wx : (x,q) \in \mathfrak{I} \cap Z\} \tag{2}$$

Furthermore, let $\mathfrak{D} = \{(x,q;w,p) \in \mathbb{R}_+^N \times \mathbb{R}_+^M \times \mathbb{R}_+^N \times \mathbb{R}_+^M : (x,q) \in \mathfrak{I}, w \in \mathbb{R}_{++}^N, p \in \mathbb{R}_{++}^M\}$. A general measure of profit efficiency for the observed quantity vector (x^0,q^0) and price vector (w^0,p^0) is a function $\phi: \mathfrak{D} \to \mathbb{R}_+ \cup \{+\infty\}$, and subject to Z, defined as

$$\phi(x^0,q^0;w^0,p^0|\,Z) = \sup_{\theta,\lambda,x,q}\{f(\lambda_1,\dots,\lambda_M;\theta_1,\dots,\theta_N) :$$

$$\sum_{m=1}^{M} p_m^0(\theta_m q_m^0) - \sum_{n=1}^{N} w_n^0(\lambda_n x_n^0) \leqq p^0 q - w^0 x,$$

$$(x,q) \in \mathfrak{I} \cap Z\} \tag{3}$$

for all $(x^0,q^0) \in \mathfrak{I}$ and $f(.)$ is bounded from above under the constraints, while $\pi(p^0,w^0|Z) = +\infty$ when there is no upper bound for the the value of $f(.)$ under the restrictions. Different forms of $f(\lambda_1,\dots,\lambda_N;\theta_1,\dots,\theta_M)$ can be chosen. In this paper we will focus on the case where $\theta_m = \theta$ for $m = 1,\dots,M$, and $\lambda_n = \lambda$ for $n = 1,\dots,N$. Thus, (3) becomes

$$\phi(x^0,q^0;w^0,p^0|\,Z)$$
$$= \sup_{\theta,\lambda,x,q}\{f(\lambda,\theta): p^0(\theta q^0) - w^0(\lambda x^0) \leqq p^0 q - w^0 x, (x,q) \in \mathfrak{I} \cap Z\}. \tag{4}$$

Incidentally, if in addition we have $\lambda = 1/\theta$ and $f(.) = \theta$, then (4) becomes

$$\phi(x^0,q^0;w^0,p^0|\,Z)$$

$$= \sup_{\theta,x,q}\{\theta: p^0(\theta q^0) - w^0(x^0/\theta) \leqq p^0 q - w^0 x, (x,q) \in \Im \cap Z\}.$$

This is the reciprocal of the overall graph efficiency measure introduced by Färe, Grosskopf and Lovell (1985, p. 119). On the other hand, if we let $\theta = \lambda = \beta$ and $f(.) = \beta$ instead, then (4) becomes

$$\phi(x^0, q^0; w^0, p^0 | Z)$$
$$= \sup_{\theta,x,q}\{\beta: \beta(p^0 q^0 - w^0 x^0) \leqq p^0 q - w^0 x, (x,q) \in \Im \cap Z\}$$
$$= \frac{\pi(x^0, q^0; p, w|Z)}{\pi^0}.$$

This is the reciprocal of the profit efficiency measure defined in Banker and Maindiratta (1988).

The attention of this paper is on $f(\theta,\lambda) = \theta - \lambda + 1$. The measure of profit efficiency (4) becomes

$$\phi(x^0, q^0; w^0, p^0 | Z)$$
$$= \sup_{\theta,\lambda,x,q}\{\theta - \lambda + 1: p(\theta q^0) - w(\lambda x^0) \leqq p^0 q - w^0 x, (x,q) \in \Im \cap Z\}. \quad (5)$$

This is a formulation that contains many ways. Specifically, let $\lambda = 1$ and use subscript "o" to denote the measure in (5), then we have

$$\phi_o(x^0, q^0; w^0, p^0 | Z) = \sup_{\theta,x,q}\{\theta: p^0(\theta q^0) - w^0(x^0) \leqq p^0 q - w^0 x, (x,q) \in \Im \cap Z\}.$$

We call it the *output-oriented Farrell profit efficiency measure*. Note that

$$\phi_o(x^0, q^0; w^0, p^0 | Z)$$
$$= \sup_{\theta,x,q}\{\theta: \theta(p^0 q^0) - w^0 x^0 \leqq p^0 q - w^0 x, (x,q) \in \Im \cap Z\}$$
$$= \sup_{\theta,x,q}\{\theta: \theta R^0 - C^0 \leqq \pi(p^0, w^0|Z)\}$$
$$= \frac{\pi(p^0, w^0|Z) + C^0}{R^0} = 1 + \frac{\pi(p^0, w^0|Z) - \pi^0}{R^0}. \quad (6)$$

Since $\pi(p^0, w^0|Z) - \pi^0 \geqq 0$, we have $\phi_o(x^0, q^0; w^0, p^0 | Z) \geqq 1$, and the measure tells us that the firm can raise profit level by $(\phi_o - 1) \times 100\%$ of the observed total revenue.

Similarly, if $\theta = 1$ and use subscript "i" to denote the measure in (5), only input side is considered and we call the expression *input-oriented Farrell profit efficiency measure,* then

$$\phi_i(x^0, q^0; w^0, p^0 | Z)$$

$$= \sup_{\lambda,x,q}\{2-\lambda : p^0(q^0) - w^0(\lambda x^0) \leqq p^0 q - w^0 x, (x,q) \in \Im \cap Z\}$$

$$= 2 - \inf_{\lambda,x,q}\{\lambda : R^0 - \lambda C^0 \leqq \pi(p^0, w^0|Z)\}$$

$$= 2 - \frac{R^0 - \pi(p^0, w^0|Z)}{C^0}$$

$$= 1 + \frac{\pi(p^0, w^0|Z) - \pi^0}{C^0}. \qquad (7)$$

Again, $\phi_i(x^0, q^0; w^0, p^0|\, Z) \geqq 1$. This measure has a meaning of raising the profit level by $(\phi_i - 1)\times$ 100% of the observed total costs.

Incorporating both output and input sides simultaneously is possible. One example is to let $\lambda = 2 - \theta$ in (5). We call this the *jointly oriented Farrell profit efficiency* measure and use a subscript “io” to denote it. Then

$$\phi_{io}(x^0, q^0; w^0, p^0|\, Z)$$

$$= \sup_{\theta,x,q}\left\{\theta - (2-\theta) + 1 : p^0(\theta q^0) - w^0\left((2-\theta)x^0\right) \leqq p^0 q - w^0 x, (x,q) \in \Im \cap Z\right\}.$$

$$= \sup_{\theta,x,q}\{2\theta - 1 : \theta(R^0 + C^o) - 2C^0 \leqq \pi(p^0, w^0|Z)\}$$

$$= \sup_{\theta,x,q}\left\{2\theta - 1 : \theta \leqq \frac{\pi(p^0, w^0|Z) + 2C^o}{R^0 + C^o} = \frac{\pi(p^0, w^0|Z) - \pi^0}{R^0 + C^o} + 1\right\}$$

$$= 1 + \frac{\pi(p^0, w^0|Z) - \pi^0}{(R^0 + C^o)/2}. \qquad (8)$$

If $(R^0 + C^o)/2$ is treated as an estimate of the firm size, then the profit efficiency measure ϕ_{io} can be interpreted as the firm can increase profit level by $(\phi_{io} - 1)\times 100\%$ of the firm size.

In summary, we have introduced a *Farrell measure of profit efficiency* (5) that includes radial changes in outputs in (6) or inputs in (7) as special cases. Further, it also allows for simultaneous radial changes of inputs and outputs in (8). Each of these three measures in (6), (7), and (8) are greater than or equal to one. If total revenue, total costs and the average of revenue and costs are regarded as three different indicators of firm size, then $\phi_i, \phi_O, and\ \phi_{io}$ all have the interpretation that the potential increase in profit level as a percentage of firm size.

Clearly, it is beyond the space permitted by any journal to consider all these (and other) cases in reasonable details in one paper and so we will focus only on one of them—the output-oriented Farrell Profit Efficiency measure. We develop a framework of profit efficiency measure that encompasses existing Farrell output-oriented efficiency measures as special cases of the model and as multiplicative components of the measure. All discussions of the output-oriented Farrell profit efficiency measure apply, with some modifications, to the other two measures: the *input-oriented Farrell efficiency measure* and the *jointly oriented Farrell efficiency measure.*

3. The Farrell Output-oriented Measure of Profit Efficiency and its Properties

Formally, we define the *Farrell output-oriented measure of profit efficiency,* $\phi_o\colon \mathfrak{D} \to \mathbb{R}_+ \cup \{+\infty\}$, as

$$\phi_o(x^0, q^0; w^0, p^0 | Z) = \sup_{\theta,x,q}\{\theta > 0 : p^0(\theta q^0) - w^0(x^0) \leqq p^0 q - w^0 x, (x,q) \in \mathfrak{I} \cap Z\} \tag{9}$$

for all $(x^0, q^0) \in \mathfrak{I} \cap Z$ and $pq^0 \neq 0$, while $\phi_o(x^0, q^0 | p, w) = +\infty$ for $pq^0 = 0$.[1]

This measure can be restated in terms of the maxi-max (or sup-sup) principle—as a two-stage optimization strategy, first optimizing for profit and then optimizing for measuring the distance between the profit efficient allocation and the actual allocation, i.e.,

$$\begin{aligned}
&\phi_o(x^0, q^0; w^0, p^0 | Z) \\
&= \sup_{\theta}\left\{\sup_{x,q}\{\theta > 0 :\ p^0(\theta q^0) - w^0(x^0) \leqq p^0 q - w^0 x, (x,q) \in \mathfrak{I} \cap Z\}\right\} \\
&= \sup_{\theta}\left\{\sup_{x,q}\left\{\frac{p^0 q - w^0 x + w^0 x^0}{p^0 q^0} : (x,q) \in \mathfrak{I} \cap Z\right\} \geqq \theta > 0\right\}
\end{aligned} \tag{10}$$

for all $(x^0, q^0) \in \mathfrak{I} \cap Z$ and $p^0 q^0 \neq 0$, while $\phi_o(x^0, q^0; w^0, p^0 | Z) = +\infty$ for $p^0 q^0 = 0$.

Intuitively, this measure expands observed outputs equi-proportionally or radially, in the spirit of the Farrell measure, to raise the profit level to the maximal possible profit level under the reference technology. The constraints Z (e.g., on revenue, cost, outputs and inputs) are included for the sake of generality, to encompass various practical cases where a firm is required to satisfy some conditions. In the following discussions, let θ^* be a solution in (9). Note that $(x^0, \theta^* q^0)$ may be an infeasible input-output vector.

A geometric illustration of the Farrell profit efficiency measure is in order. Consider Figure 1, where "**abcde"** denotes the production frontier and *f1* is the observed input-output vector. The output-oriented technical efficiency is measured by $q_1{}'/q_1$. pp' is the iso-profit line. We can hypothetically expand the output to raise profit until it coincides the optimal iso-profit line pp'. Then $q_1"/q_1$ can be regarded as a measure of profit efficiency, while the gap between technical efficiency and the profit efficiency can be regarded as the (profit-oriented) allocative efficiency, which is measured via $q_1"/q_1$ in Figure 1.

It is also worth reminding again that without some upper bounds on x, q or wx, pq, there will be no maximum profit level if technology exhibits IRS and so, as can be shown, $\phi_o(x^0, q^0; w^0, p^0 | Z) = +\infty$. Similar situation may occur under CRS. In a sense, the upper constraints we have in the definition help regularizing the profit function and, in turn, the related Farrell profit efficiency measure for such cases, besides helping to tailor these concepts closer to the reality.

[Insert **Figure 1** here]

Many researchers have stated, explicitly or implicitly, some desirable properties of the profit efficiency measure. To adapt them to our context, let Φ be a profit efficiency measure. Here we summarize

P.1 Φ is a well-defined function for all maximal and observed profit levels, and for all $(x^0, q^0) \in \mathfrak{I}$.

P.2 There exists a value *c* such that a firm is identified as profit efficient if and only if $\Phi = c$.

P.3 Φ is homogeneous of degree zero in input and output prices.

P.4 Φ is independent of units of measurement, i.e., commensurable.

These properties have been discussed by many authors before, see Nahm and Vu (2013, p. 46), Asmild et al. (2007, p. 318), among others. The intuition behind them and their importance are obvious. It is natural to accept them as axioms that one would expect any profit efficiency measure to satisfy.

Our measure does satisfy all the above four properties. In particular, suppose the observed total revenue is positive and the maximum profit level is a finite number. I.e., $R^0 > 0$ and $\pi(p^0, w^0|Z) > 0$. In (6),

$$\phi_o(x^0, q^0; w^0, p^0 | Z) = 1 + \frac{\pi(p^0, w^0|Z) - \pi^0}{R^0}. \tag{11}$$

Since the denominator is positive and the difference between the maximum and observed profits is finite, the Farrell output-oriented profit efficiency measure is well-defined for all observed input and output quantities and prices as long as the total revenue is positive. Further, in view of (11), it is clear that $\phi_o(x^0, q^0; w^0, p^0 | Z) = 1$ if and only if $\pi(p^0, w^0|Z) - \pi^0 = 0$ which means profit efficient. Property P.2 is satisfied, too. Also one expression in (6) is $\phi_o(x^0, q^0; w^0, p^0 | Z) = [\pi(p^0, w^0|Z) + w^0 x^0]/p^0 q^0$. Since the profit function is homogeneous of degree one in input and output prices, we have, for t > 0, $\phi_o(x^0, q^0; tw^0, tp^0 | Z) = [\pi(tp^0, tw^0|Z) + tw^0 x^0]/tp^0 q^0 = [\pi(p^0, w^0|Z) + w^0 x^0]/p^0 q^0 = \phi_o(x^0, q^0; w^0, p^0 | Z)$. The Farrell output-oriented profit efficiency measure is homogeneous of degree zero in input and output prices. Finally, let (x^*, q^*) be a profit-maximizing input-output vector. Then $\phi_o(x^0, q^0; w^0, p^0 | Z) = [(p^0 q^* - w^0 x^*) + w^0 x^0]/p^0 q^0$. Since after changes of units of measurement, prices and quantities will adjust so that $p_m q_m$ and $w_n x_n$ remain unchanged for all m and n, the commensurability of $\phi_o(x^0, q^0; w^0, p^0 | Z)$ follows. Hence the Farrell output-oriented profit efficiency measure satisfies all the four properties P.1 to P.4.

In addition to the above properties for any profit efficiency measure, the output-oriented profit efficiency measure is homogeneous of degree *-1* in the observed output vector q^0. It is also continuous in input and output prices as well as in inputs and outputs (for $pq^0 > 0$).

4. Intuition of the Farrell Output-oriented Profit Efficiency Measure

To simplify further discussions and notations, we assume that the maximum profit level exists at the input and output prices under consideration and $p^0 q^0 > 0$. Many interpretations of the Farrell output-oriented profit efficiency measure can be found by rearranging its formula.

Firstly, from (6), $\phi_o(x_0, q^0; p^0, w^0|Z) = (\pi(p^0, w^0|Z) + C^o)/R^o$. It follows that

$$\phi_o(x^o, q^o; w^0, p^0|Z) = \frac{C^o}{R^o} + \frac{\pi(p^0, w^0|Z)}{R^o}. \tag{12}$$

The interpretation of (12) is very useful because $\phi_o(\cdot)$ is expressed in terms of two factors: (i) realized cost-benefit ratio, and (ii) the best possible profit margin for the firm. Both of these factors are well understood by practitioners in business and investment community. Indeed, the cost-benefit ratio is among the most important key performance indicators (KPI) in business practice, as is the profit margin indicator.

Secondly, subtracting and adding R^0 to the numerator of (6), we get

$$\phi_o(x^o, q^o; w^0, p^0|Z) = \frac{\pi(p^0, w^0|Z) - \pi^o}{R^o} + 1. \tag{13}$$

The Farrell output-oriented profit efficiency measure $\phi_o(\cdot)$ in the form of (13) tells about unrealized or lost profit in percentage to actual total revenue or sales and so can be understood by practitioners as the 'lost profit margin' (additively normalized by *1*).

Thirdly, let R^* and C^* be the profit-maximizing total revenue and total costs respectively and noting that $\pi(p^0, w^0|Z) = R^* - C^*$, one can rewrite (6) as follows:

$$\phi_o(x^o, q^o; w^0, p^0|Z) = \frac{R^*}{R^0} + \frac{C^0 - C^*}{R^0} \tag{14}$$

This interpretation is very useful because it shows decomposition into (i) ratio of best possible revenue to actual total revenue (sales) and (ii) excessive cost (beyond the possible minimum) relative to the actual sales. Note that R^*/R^0 might be larger, equal or small than unity, which can be interpreted as undersell, efficient sale and oversell, all relative to the profit maximizing efficiency criterion. Similarly, $(C^0 - C^*)/R^0$ might be larger, equal or small than unity, which can be interpreted as overspend, efficient cost and underspend, all with respect to the profit maximizing efficiency criterion.

5. Some Important Special Cases

Many special cases can be derived from the proposed Farrell output-oriented profit efficiency measure in (9) by setting various restrictions to cater to particular situations a firm may be facing in practice. We will present just a few here, which are perhaps the most interesting to general audience.

When the maximum profit level is zero (which can happen in long-run competitive equilibrium), the profit efficiency measure will coincide with the very intuitive and frequently used in practice cost-benefit ratio (reciprocal to the 'return to dollar' efficiency measure), i.e., $\phi_0 = C^o/R^o$.

When we impose restrictions on quantities and prices, we find that the Farrell profit efficiency measure is closely related to other Farrell measures. First, when $x_{lb} = x_{ub} = x^0$, i.e., the firm is required (e.g., in the short-run) to utilize the already available resources and there are no bounds on pq and q. Let $Z(x = x^0) = \{(x,q): (x,q) \in Z, x = x^0\}$. Then the profit efficiency measure coincides with the standard revenue efficiency measure, *i.e.*, from (10),

$$\begin{aligned}
&\phi_o\big(x^o, q^o; w^0, p^0 | Z(x = x^0)\big) \\
&= \sup_{\theta}\left\{\sup_{x,q}\left\{\frac{p^0 q - w^0 x^0 + w^0 x^0}{p^0 q^0} : (x,q) \in \Im \cap Z(x = x^0)\right\} \geqq \theta > 0\right\} \\
&= \sup_{\theta}\left\{\sup_{x,q}\left\{\frac{p^0 q}{p^0 q^0} : (x,q) \in \Im \cap Z(x = x^0)\right\} \geqq \theta > 0\right\} \\
&= \frac{R(x^0, p^0)}{p^0 q^0} = RE(x^0, q^0, p^0). \qquad (15)
\end{aligned}$$

for all $(x^0, q^0) \in \Im$ and $p^0 q^0 > 0$, where $R(x^0, p^0)$ is the maximum revenue at (x^0, p^0). Second, in addition to $x_{lb} = x_{ub} = x_0$ consider the supremum when the output prices can take any positive values, and the constraints in *Z*, other than the input vector, are non-binding. Then, (10) becomes

$$\begin{aligned}
&\phi_o(x^o, q^o; w^0, p | Z(x = x^0), p > 0) \\
&= \sup_{\theta}\left\{\sup_{x,q}\left\{\frac{pq - w^0 x^0 + w^0 x^0}{p^0 q^0} : (x,q) \in \Im \cap Z(x = x^0)\right\} \geqq \theta > 0 : p > 0\right\} \\
&= \sup_{\theta,q}\left\{\theta : \theta \leqq \frac{pq}{pq^0}, (x,q) \in \Im \cap Z(x = x^0), p > 0\right\}
\end{aligned}$$

Let the solution to this optimization problem be θ^* and q^*. For the case where the output set is convex, due to the Supporting Hyperplane theorem in mathematics, there exists some nonzero price vector $p^s \geq 0$ such that $p^s q^* \geqq p^s q$ for $(x^0, q) \in \mathfrak{I}$. As proved similarly in Debreu (1951),

$$\sup_{\theta,q}\left\{\theta : \theta \leqq \frac{pq}{pq^0}, (x,q) \in \mathfrak{I} \cap Z(x = x^0), p > 0\right\}$$

$$= \inf_{p>0}\sup_{q\in\mathfrak{I}}\left\{\frac{pq}{pq^0}\right\} = \frac{p^s q^*}{p^s q^0}$$

$$= \sup_{\theta}\{\theta : (x^0, \theta q^0) \in \mathfrak{I}\} = F_0(x^0, q^0).$$

where $F_0(x^0, q^0)$ is the *Farrell output-oriented technical efficiency measure*. Thus we have

$$\phi_o\big(x^o, q^o; p^s, w | Z(x = x^0)\big) = F_0(x^0, q^0).^2 \tag{16}$$

Therefore the Farrell output-oriented profit efficiency measure encompasses the Farrell output-oriented measures of revenue efficiency and technical efficiency as special cases. Recall that the profit efficiency measure can be expressed in the form of equation (13) and the value of $F_0(x^0, q^0)$ is independent of output prices, all the profit efficiency, revenue efficiency and technical efficiency can be interpreted as the percentage of inefficiency losses in profit to the total revenue.

6. Decomposition of the Farrell Profit Efficiency Measure

The relationships in (15) and (16) provide a unified framework that unites the profit efficiency with revenue efficiency and Farrell technical efficiency, as well as help establishing various decompositions of the former. In view of the increasing restrictions imposed on the Farrell output-oriented profit efficiency measure, we have

$$\phi_o(x^o, q^o; w^0, p^0 | Z) \geqq \phi_o\big(x^o, q^o; w^0, p^0 | Z(x = x^0)\big) \geqq \phi_o\big(x^o, q^o; w^0, p^s | Z(x = x^0)\big). \tag{17}$$

Applying (15) and (16) to (17), it follows that

$$\phi_o(x^o, q^o; w^0, p^0 | Z) \geqq RE(x^o, q^o, p^0) \geqq F_o(x^o, q^o).^3 \tag{18}$$

We can define the *profit allocative efficiency* as

$$AE_\pi(x^o, q^o; w^0, p^0 | Z) = \frac{\phi_o(x^o, q^o; w^0, p^0 | Z)}{F_0(x^o, q^o)} \geqq \frac{RE(x^o, q^o, p^0)}{F_0(x^o, q^o)} \geqq 1. \tag{19}$$

Rewrite (19), the output-oriented profit efficiency measure can be decomposed into two components: profit allocative efficiency and Farrell output-oriented technical efficiency:

$$\phi_o(x^o, q^o; w^0, p^0|Z) = AE_\pi(x^o, q^o; w^0, p^0|Z) \cdot F_0(x^o, q^o). \tag{20}$$

Denote the output-oriented allocative efficiency by $AE_o(x^o, q^o, p^0)$. Recall that $AE_o(x^o, q^o, p^0) = RE(x^o, q^o, p^0)/F_0(x^o, q^o)$. It is clear from (19) that $AE_\pi \geqq AE_0$. Thus $\phi_o = AE_\pi \times F_0 \geqq AE_0 \times F_0$. Define

$$AE_{re}(x^o, q^o; w^0, p^0|Z) = \frac{AE_\pi(x^o, q^o; w^0, p^0|Z)}{AE_o(x^o, q^o, p^0)} = \frac{\phi_o(x^o, q^o; w^0, p^0|Z)}{RE(x^o, q^o, p^0)}. \tag{21}$$

AE_{re} shows the improvement of profits from the maximum revenue for a given input vector to maximum profits when all inputs and outputs are variable. We call AE_{re} the *revenue efficient allocative efficiency*. So (20) can be further written as

$$\phi_o(x^o, q^o; w^0, p^0|Z) = AE_{re}(x^o, q^o; w^0, p^0|Z) \times AE_o(x^o, q^o, p^0) \times F_0(x^o, q^o). \tag{22}$$

Decomposition of profit efficiency is in general additive in the literature of efficiency analysis. A decomposition that included multiplicative components was suggested by Aparicio, Pastor and Ray (2013). However, their decomposition of profit efficiency consists of both additive and multiplicative elements. None of those decompositions included explicitly Farrell technical and allocative efficiency measures as components. Our decomposition in (22) fills in the gap of the literature. It provides decomposition of the profit efficiency that explicitly includes the Farrell technical and allocative efficiency measures as multiplicative components with meaningful interpretation in each item.

7. Relationship to other measures

As we mentioned in the introduction, some roots to the framework of profit efficiency discussed here, can be found in various works in different contexts. The most prominent and perhaps most closely related formula is found in Chavax and Cox (1994). Specifically, in their insightful paper, Chavax and Cox (1994) used the following formula

$$D_o(x^0, q^0) = \min_\theta \left\{ \theta : \; p^0 \left(\frac{q^0}{\theta} \right) - w_0 x_0 \;\leqq\; \pi(p^0, w^0) \right\}. \tag{23}$$

After fixing some details (e.g., changing *min* to *inf*, re-defining the domain and range of the function to circumvent peculiar cases, etc.), one can see that the $D_o(x^0, q^0)$ in (23) is the reciprocal of the Farrell output-oriented measure of profit efficiency in the unrestricted case. It is also important to note that the attention of Chavax and Cox (1994) was not the profit efficiency measurements but the outer bounds of the input and output distance functions and they (and to the best of our knowledge others), have not explored the possibility of using this formula as a profit efficiency measure. In a related work, Asmild et al. (2007, Theorem 4, p. 316) showed that the same formula is the limit of a linked cone model.

Another very important and perhaps the most general model of profit efficiency measure is the directional technology measure of profit efficiency, introduced by Chambers, Chung and Färe (1998).

$$\phi_T^d(x^o, q^o; w^0, p^0; g_x, g_q) = \frac{\pi(p^0, w^0) - \pi^0}{pg_q + wg_x}. \tag{24}$$

The measure ϕ_T^d actually contains many existing measures as special cases. It is closely related to the measures proposed in this paper. Indeed, note that if we let $g_q = q^0$ and $g_x = 0$ in (24), then their measure is equivalent to the unrestricted version of the output oriented Farrell profit efficiency measure. Specifically, we have

$$\phi_T^d(x^o, q^o; w^0, p^0; g_x, g_q) = \frac{\pi(p^0, w^0) - \pi^0}{p^0 q^0} = \left(\frac{\pi(p^0, w^0) - \pi^0}{R^0} + 1\right) - 1 = \phi_o - 1.$$

The vitality of this result for the Farrell profit efficiency measure is that one does not need to derive a new duality theory for this measure as well as other technical properties—it is already provided in Chambers, Chung and Färe (1998), for the unrestricted case and can be adapted accordingly when some restrictions are necessary.

Furthermore, Ray (2004, pp. 233 – 234) proposed the following profit efficiency measure

$$\phi_{Ray}(x^o, q^o; w^0, p^0) = (\pi(p^0, w^0) \, - \, \pi^0) \, / \, R^0.$$

Clearly, this measure is equivalent to the Farrell profit efficiency in the unrestricted case (i.e., $\phi_o = \phi_{Ray} + 1$). It is also worth noting that both Chambers, Chung and Färe (1998) and Ray (2004) treated the denominator in their formulation as an arbitrary normalization. In this paper, we explicitly show that such normalization can be justified by a certain orientation as in (6). The normalization can be used to relate the profit efficiency measure to the output-oriented Farrell measure of technical efficiency for decomposing the profit efficiency measure into various sources, as we do in the previous section.

It is also important to note that Ray (2004) also mentioned normalizing the difference by total costs, which has been applied by Färe, Grosskopf and Weber (2004) and Das and Ghoshb (2009).[4] It can be shown by similar reasoning and derivations as we have done above that normalization by total cost can be justified if one instead use the input oriented Farrell profit efficiency measure.

8 Numerical Illustration

The theoretical developments we presented above are general rather than specifically related to a particular estimation approach. Here, in this section we present numerical illustration using one of the most popular methods of estimating efficiency measures in practice—the data envelopment analysis (DEA) approach, while other popular approaches (e.g., stochastic frontier analysis approach) can also be used.

Specifically, suppose there are *K* firms. It is observed that firm *k* uses *N* inputs to produce *M* outputs. The observed input and output vectors are $x^k = (x_{k1}, \cdots, x_{kN})$ and $q^k = (q_{k1}, \cdots, q_{kM})$. For illustration, no explicit bounds are imposed. Let empirical technology set used in the computation be

$$T^{vrs} = \Big\{(x,q): \sum_{k=1}^{K} z_k q_{km} \geqq q_m, \sum_{k=1}^{K} z_k x_{kn} \leqq x_n, \sum_{k=1}^{K} z_k = 1\,,$$

$$z_j \geqq 0, j = 1, \cdots, K, m = 1, \cdots M, n = 1, \cdots, N\Big\}.$$

The various efficiency measures mentioned in previous sections are computed as follows:

Farrell output-oriented profit efficiency

$\phi_o(x^0, q^0; w^0, p^0)$

$$= \max_{\theta,z,x,q}\{\theta: p^0(\theta q^0) - w^0x^0 \leqq p^0q - w^0x, (x,q) \in T^{vrs}, x \geqq 0, q \geqq 0\}$$

Revenue efficiency

$$RE(x^0, q^0, p^0) = \max_{\theta,z,q}\{\theta: p^0(\theta q^0) \leqq p^0q, (x^0, q) \in T^{vrs}, q \geqq 0\}.$$

Farrell output oriented technical efficiency

$$F_0(x^0, q^0) = \max_{\theta,z}\{\theta: (x^0, \theta q) \in T^{vrs}\}.$$

Output oriented allocative efficiency

$$AE_o(x^0, q^0, p^0) = RE(x^0, q^0, p^0)/\ F_0(x^0, q^0)$$

Revenue efficient allocative efficiency

$$AE_{re}(x^0, q^0; p^0, w^0) = \phi_o(x^0, q^0; p^0, w^0)/RE(x^0, q^0, p^0)$$

We compute the values of these measures using the data set in Ray (2004). There are price and quantity data of five outputs and four inputs. Each observation is a bank in the US in 1996. For illustration, although all 50 observations are used for computation, only the results of the first 20 banks are reported in the following Table.

[Insert Table 1 here]

In Table 1, the geometric mean of the Farrell output-oriented profit efficiency measure (ϕ_o) is 2.85. This means that, on average, the banks could increase their profit levels by about 185% of the observed total revenue. On average, the value of the output-oriented technical efficiency measure (F_o) is 1.02 which means many banks are operating on or close to the production frontier. Although the banks are fairly allocatively inefficient (AE_o = *1.15*), the value of the revenue efficient allocative efficiency is the biggest (AE_{re} = *2.42*). On average, the sources of profit inefficiency come from technical inefficiency (2.2%), output-oriented allocative inefficiency (15.1%), and revenue efficient allocative efficiency (142.4%). Thus most profit inefficiency comes from revenue efficient allocative inefficiency. When firms in this sample are revenue efficient, they can still, on average, raise the profit level by 142% by changing input mix and input quantities. This component of allocative inefficiency after the firm is revenue efficient has not been discussed in the literature before.

9. Conclusion

By synthesizing various approaches in the literature, this paper presents a cohesive framework of profit efficiency measure. The Farrell output-oriented measure of profit efficiency derived from this model satisfies some nice properties discussed by other researchers in the literature. This measure has intuitive interpretation and includes Farrell output-oriented measure of technical efficiency, revenue efficiency as special cases. This is the first time in the literature that a profit efficiency measure is expressed as multiplicative elements containing Farrell technical efficiency and revenue efficiency.

Notes

1. The framework for the input oriented version is analogous and so is omitted for the sake of space.

2. Färe and Primont (1995, p. 129) has a similar expression that $1/D_o(x^o, q^o) = \sup_\theta\{\theta: p(\theta q^o) - wx^0 \leqq \pi(p, w), (p, w) \geq 0\}$. Their derivation requires convex technology set whereas the technology set in our formulation is not assumed but inputs are fixed. When the technology set is not convex, it is possible that $\sup_\theta\{\theta: p(\theta q^o) - wx^0 \leqq \pi(p, w), (p, w) \geq 0\} > 1/D_o(x^o, q^o)$.

3. The left- and right-hand side of (18) can be rewritten as $(\pi(p^o, w^o) + w^o x^0)/p^o q^0 \geqq 1/D_o(x^0, q^0)$ where $D_o(x^0, q^0)$ is the output distance function. Färe and Primont (1995, p. 131) have mentioned this inequality from duality relation. Färe and Grosskopf (2004, p. 39) have stated the same through the relation between the profit function and the directional output distance function.

4. Aparicio, Pastor, and Ray (2013) also proposed another normalization--using the average total costs of all firms.

References

Asmild, M., J. C. Paradi, D. N. Reese, F. Tam. 2007. Measuring Overall Efficiency and Effectiveness using DEA. *European Journal of Operational Research* 178: 305 – 321.

Aparicio, J., J. T. Pastor, S. C. Ray. 2013. An overall measure of technical inefficiency at the firm and at the industry level: the 'lost profit on outlay. *European Journal of Operational Research* 226: 154 – 162.

Banker, R. D., A. Charnes, W. W. Cooper. 1984. Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Management science 30*: 1078 – 1092.

Banker, R. D., A. Maindiratta. 1988. Nonparametric Analysis of Technical and Allocative Efficiencies in Production. *Econometrica* 56: 1315 – 1332.

Banker, R., R. C. Morey. 1986. Efficiency Analysis for Exogeneously Fixed Inputs and Outputs. *Operations Research* 34: 513 – 521.

Chambers, R. G., Y. Chung, R. Färe. 1998. Profit, Directional Distance Functions, and Nerlovian Efficiency. *Journal of Optimization Theory and Applications* 98: 351 - 364.

Charnes, A., W. W. Cooper, E. Rhodes. 1978. Measuring the efficiency of decision making units. *European journal of operational research 2*: 429-444.

Chavas, J. P., T. L. Cox. 1994. A Primal-Dual Approach to Nonparametric Productivity Analysis: The Case of U.S. Agriculture. *Journal of Productivity Analysis* 4: 359 – 373.

Cherchye, L., B. De Rock, B. Dierynck, F. Roodhooft, J. Sabbe. 2013. Opening the "Black Box" of Efficiency Measurement: Input Allocation in Multioutput Settings. *Operations Research* 61: 1148 – 1165.

Cherchye, L. B. De Rock, F. Vermeulen (2008) "Analyzing Cost-Efficient Production Behavior Under Economies of Scope: A Nonparametric Methodology," *Operations Research* 56(1):204-221.

Cook, W.D., J. Harrison, R. Imanirad, P. Rouse, and J. Zhu (2013) "Data Envelopment Analysis with Nonhomogeneous DMUs" *Operations Research* 61(3):666-676.

Cook, W. D., J. Zhu. 2006. Incorporating Multiprocess Performance Standards into the DEA Framework. *Operations Research* 54: 656 – 665.

Cook, W.D. and J. Zhu (2008) “CAR-DEA: Context-Dependent Assurance Regions in DEA,” *Operations Research* 56(1):69-78.

Cooper, W.W., K. S. Park and Gang Yu (2001) “An Illustrative Application of Idea (Imprecise Data Envelopment Analysis) to a Korean Mobile Telecommunication Company,” *Operations Research* 49(6):807-820.

Das, A., S. Ghosh. 2009. Financial deregulation and profit efficiency: A nonparametric analysis of Indian banks. *Journal of Economics and Business* 61: 509-528.

Debreu, G. 1951. The coefficient of resource utilization. *Econometrica* 19: 273-292.

Färe, R., S. Grosskopf. 2004. Efficiency Indicators and Indexes. In R. Färe and S. Grosskopf eds, *New Directions : Efficiency and Productivity*. Boston : Kluwer Academic Publishers.

Färe, R., S. Grosskopf,C. A. K. Lovell. 1985. *The Measurement of Efficiency of Production*. Boston: Kluwer-Nijhoff Publishing.

Färe, R., S. Grosskopf, W. L. Weber. 2004. The effect of risk-based capital requirements on profit efficiency in banking. *Applied Economics* 36: 1731-1743.

Färe, R., C. A. K. Lovell. 1978. Measuring the technical efficiency of production. *Journal of Economic theory 19*: 150-162.

Färe, R., D. Primont. 1995. *Multi-Output Production and Duality: Theory and Applications*. Boston: Kluwer Academic Publishers.

Farrell, M. J., 1957. The measurement of productive efficiency. *Journal of the Royal Statistical Society. Series A (General)*, *120:*253-290.

Kopp, R. J. 1981. The Measurement of Productive Efficiency: A Reconsideration. *Quarterly Journal of Economics* 96: 477 – 503.

Nahm, D., H. Vu. 2013. Profit Efficiency and Productivity of Vietnamese Banks: A New Index Approach. *Journal of Applied Finance and Banking* 3: 45 – 65.

Ray, S. C. 2004. *Data Envelopment Analysis: Theory and Techniques for Economics and Operations Research*. Cambridge University Press.

Ruggiero, J. 2000. Measuring Technical Efficiency. *European Journal of Operational Research* 121: 138 – 150.

Thompson, R. G., F. D. Singleton, Jr., R. M. Thrall, B. A. Smith. 1986. Comparative Site Evaluations for Locating a High-Energy Physics Lab in Texas. *Interfaces* 16: 35 – 49.

Zhu, J. (2004) "Imprecise DEA via Standard Linear DEA Models with a Revisit to a Korean Mobile Telecommunication Company," Operations Research 52(2):323-329.

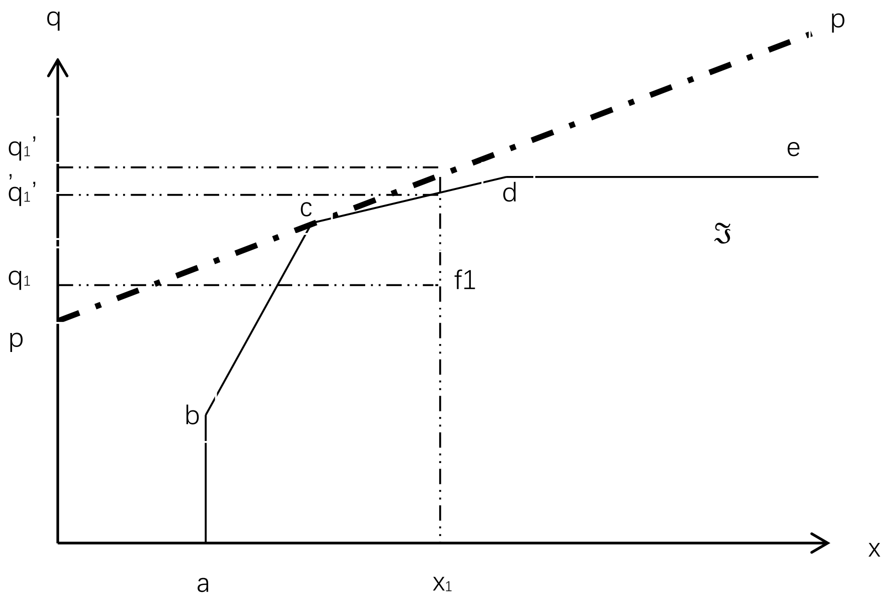

Figure 1. The output-oriented Farrell profit efficiency measure

Table 1: Farrell output-oriented profit efficiency and its components

Firm	Technical efficiency	Allocative efficiency	Revenue efficient allocative efficiency	Profit efficiency
f1	1.00	1.09	1.48	1.61
f2	1.00	1.60	2.62	4.18
f3	1.00	1.00	2.79	2.79
f4	1.12	1.12	2.49	3.14
f5	1.00	1.00	2.99	2.99
f6	1.06	1.13	2.44	2.93
f7	1.00	1.17	2.24	2.62
f8	1.10	1.17	2.30	2.96
f9	1.00	1.06	2.39	2.52
f10	1.00	1.46	1.91	2.80
f11	1.00	1.00	2.89	2.89
f12	1.00	1.00	3.03	3.03
f13	1.00	1.00	2.86	2.86
f14	1.09	1.09	2.34	2.78
f15	1.00	1.23	2.31	2.85
f16	1.06	1.43	2.14	3.23
f17	1.00	1.27	2.11	2.68
f18	1.00	1.14	2.42	2.76
f19	1.00	1.28	3.00	3.83
f20	1.00	1.00	2.43	2.43
Geometric Mean	1.02	1.15	2.42	2.85