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Abstract

This study tackles the problem of building nonparametric production technologies that satisfy homotheticity and (Hicks) neutrality. Since no current method is available, the paper proposes a method for constructing a homothetic metatechnology and proves that this metatechnology can be obtained as the union of homothetic group technologies that satisfy (Hicks) neutrality. It is shown that all these technologies are the minimal technologies that will satisfy the given set of axioms, therefore providing an axiomatic foundation for the method. There is a non-negligible computational aspect in the fact that all technologies are not only LP computable but are in fact computable as a linear function of the size of the dataset (in fact in the non-convex case enumeration algorithms can be used). An empirical illustration is provided to show the strength and range of applicability of the method.

Key Words: Data Envelopment Analysis, Free Disposal Hull, Input Homotheticity, Hicks Neutrality, Efficiency

1 Introduction

This paper proposes a method for building a data generated technology that satisfies homotheticity and (Hicks) neutrality. According to Blackorby et al. (1976) the notion of neutrality of technical change dates back at least to Hicks (1932) with his definition of neutral technical change as technical change that leaves the marginal product of different production factors constant. Although the notion of neutrality of technical change has been around for some time, its implementation through nonparametric production analysis, such as data envelopment analysis (DEA) or free disposal hull(FDH), has never been attempted. This notion of neutrality clearly extends to the metafrontier setting proposed by O'Donnell et al. (2008). In such a setting there are multiple reference technologies that define different groups of firms or decision making units (DMU). The distance between group frontiers has been called a technology gap and neutrality would imply that this technology gap takes a very special form.

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In a recent paper Peyrache (2024) builds on previous work of Olesen (2014) to show that it is possible to build a homothetic reference technology. This method can be applied to the group technologies, but the union of these group homothetic technologies will not give rise to a homothetic metatechnology. The reason for this somewhat counter intuitive result is that the homothetic group technologies do not satisfy neutrality. As it will be shown in this study, it is possible to build group technologies that satisfy neutrality and homotheticity and give rise to a metafrontier that is homothetic. A complete axiomatic characterization of this result will be provided and an empirical application will show how to use the method. I will derive the method both for the metafrontier defined as a union of sets and for the metafrontier defined as a pooled technology. The two definitions only differ in the case of convexity and the strategy of defining the metafrontier as the convexification of the union of the group technologies will be addressed as well.

The ability to construct group technologies that satisfy homotheticity and neutrality also solves another open problem that has not been addressed in the literature. This relates to the ability to test for neutrality in the tradition of Hanoch and Rothschild (1972). Since the neutral homothetic group technologies will be obtained as an enlargement of the variable returns to scale (VRS) group technologies, testing for the neutrality assumption will reduce to compare these two technologies. As pointed out in Peyrache (2024), since the homothetic technologies can be computed fast (computational time is linear in the number of observations), this is a non-negligible computational gain compared to using, for example, an extension of the approach presented in Primont and Primont (1994).

One last point should be made about computational complexity. The method proposed has roughly the same computational complexity of a DEA or FDH efficiency evaluation program. Therefore it will be a series of linear programs (LP) for DEA and an enumeration algorithm for FDH. By reducing the dimensionality of the problem, there is a significant gain in terms of discrimination power. Peyrache (2024) documented that this can be substantial, with the homothetic FDH having basically the same discrimination power as a standard DEA (a similar results is obtained in this paper in the empirical application). This is a significant result since one can retain discrimination power by using a homothetic FDH instead of a standard DEA, while at the same time having very significant computational gains over large datasets. This is even more relevant for the neutral homothetic technology proposed in this paper, since the neutrality assumption allows to use observations from the whole dataset to build the reference set, therefore maintaining a good degree of discrimination under non-convexity. This has important consequences in analyzing the impact of returns to scale in the structure of the industry under evaluation. This can be accomplished by studying the scaling function directly in the graph of the technology (see Olesen and Ruggiero (2014, 2022) for an example).

2 Preliminary Definitions

Consider a production process that uses P inputs $x \in \mathbb{R}_+^P$ to produce a single output $y \in \mathbb{R}_+$. The production possibilities set for group $t = 1, \ldots, N$ is defined as

$$T_t = \{x \text{ can produce } y \text{ in group } t\}$$

It should be immediately noted that the N groups can also be interpreted as N time periods, giving rise to period specific technologies. The input requirement set $L_t(y)$ and the input isoquant $IL_t(y)$ can be defined as:

$$L_t(y) = \left\{x \in \mathbb{R}_+^P \mid (x, y) \in T_t\right\}$$
$$IL_t(y) = \left\{x \in \mathbb{R}_+^P : x \in L_t(y),\, \lambda x \notin L_t(y),\, \forall\, 0 \leq \lambda < 1\right\}$$

The technology set can be described functionally via the input distance function:

$$D_I(x, y, t) = \sup\{\theta > 0 : (x/\theta, y) \in T_t\}$$

If the grouping variable t is interpreted as time periods, then the difference between any two technology sets T_t and T_s is interpreted as technical change. If the grouping is interpreted in the metafrontier tradition, then the difference is interpreted as a technology gap. The grouping variable t may also represent a mix of groups and time periods. In other words all the results of this paper will hold in the case in which one considers metafrontiers over several time periods. Notice that here the only relevant difference is if the grouping variable t is ordinal or not[1] and since none of the results of this paper rely on the variable to be ordinal, this means that the difference is only interpretational.

Consider two group technologies T_s, T_t, with $T_s \neq T_t$. If $T_s \subset T_t$, then T_t is a more productive technology (if t is a time index and $t > s$, this means there is no technical regress). On the contrary if $T_s \supset T_t$, then T_s is a more productive technology (if t is a time index and $t > s$, this

[1] I will not consider the case of a continuous grouping variable (although this may be an interesting development left for future research). In this section, for the sake of tradition, I will describe all the necessary functional restrictions imposed on these technology sets in terms of the time period interpretation. This is mostly because the economic literature has developed this material in terms of time and the associated notion of technical change.

means there is no technical progress). Finally it is said that the two technology sets intersects if $T_t \neq T_s$, but $T_s \not\subset T_t$ and $T_s \not\supset T_t$. Input neutrality is defined as[2]:

$$D_I(x, y, t) = D_I(x, y, 1)/B(y, t) \tag{1}$$

and I will refer to this functional restriction simply as *neutrality*. I will refer to two technologies that satisfy neutrality as being the homothethy of each other or being homothetic to each other. For example if the definition above is satisfied by two technologies T_t, T_s, then I will simply say that T_t and T_s are homothetic to each other, or that T_t is the homothety of T_s (and viceversa). I will also refer to these two technologies as being neutral, in the sense that they satisfy the functional definition of neutrality as stated in equation (1). It is important to stress that neutrality is a property that is interesting only if there are at least two group technologies, since if there is only one group technology, the definition of neutrality is trivially satisfied! Therefore neutrality is a property that pertains to a given group technology in its relationship to another group technology. It should also be stressed that neutrality does not imply that the group technologies are nested or, to say it differently, neutrality is not sufficient to avoid that two technologies intersect.

A group technology T_t is said to be homothetic if the input distance function satisfies the following functional restriction:

$$D_I(x, y, t) = D_I(x, 1, t)/H(y, t) = X(x, t)/H(y, t) \tag{2}$$

where the input aggregator function $X(x, t)$ is linearly homogeneous in the input vector (as stated in the equation, this function can be chosen to be $X(x, t) = D_I(x, 1, t)$ where the unit isoquant is chosen as a reference). The function $H(y, t)$ is an aggregate input requirement function and represents the minimal amount of aggregate input that is needed to produce a given output. Peyrache (2013) shows that a technology is both neutral and homothetic (which means each group technology T_t is homothetic and any two group technologies T_t, T_s are the homothety of each other) if and only if the input distance function can be written as:

$$D_I(x, y, t) = D_I(x, 1, 1)/B(y, t) = X(x)/B(y, t) \tag{3}$$

where the input aggregator function $X(x)$ is now independent of t (this function can be

[2]Blackorby et al. (1976) identified three alternative definitions of neutrality of technical change: Hicks neutrality, implicit Hicks neutrality and extended Hicks neutrality. This paper is using the notion of implicit Hicks neutrality as defined in Blackorby et al. (1976).

chosen to be $X(x) = D_I(x, 1, 1)$). This definition implies that every single group technology T_t is homothetic and that any two given group technologies T_t, T_s are homothetic to each other. I will refer to the functional restriction of equation (3) as *neutral homotheticity.*

If the technology is homothetic, then it can be represented using its graph and base isoquant (see Peyrache (2024) for a full discussion). In the case in which the technology is neutral homothetic, then a time independent input aggregator $X(x)$ can be defined. This means that it is possible to use a common metric to aggregate inputs across the different technologies T_t. The graphs of the group technologies are then given by:

$$G_t = \{(X, y) : (x, y) \in T_t,\, X = X(x)\} \tag{4}$$

As noted earlier, the graphs of the group technologies may intersect with each other, since neutral homotheticity is not sufficient for the technology sets to be nested.

3 Data Generated Technologies

Consider a set of data points, decision making units (DMUs), or observations (x_{kt}, y_{kt}) ($\forall k = 1, \ldots, K$ and $\forall t = 1, \ldots, N$) collected for each group t in a dataset in the form of matrix $D_t = [x_{kt}, y_{kt}]$. All the technologies discussed in this paper will satisfy the following two properties.

A1. Feasibility of observed DMUs: the observed data points belong to the production set, $(x_{kt}, y_{kt}) \in T_t$, $\forall k = 1, \ldots, K$ and $\forall t = 1, \ldots, N$.

A2. Free Disposability: $\forall (x, y) \in T_t$, if $x_1 \geq x$, $y_1 \leq y$, then $(x_1, y_1) \in T_t$, $\forall t = 1, \ldots, N$.

3.1 Group Technologies

The group free disposal hull (FDH) technology is defined as:

$$T_t^V = \left\{ (x, y) : \sum_{k=1}^{K} \lambda_{kt} x_{kt} \leq x, \sum_{k=1}^{K} \lambda_{kt} y_{kt} \geq y, \sum_{k=1}^{K} \lambda_{kt} = 1, \lambda_{kt} \in \{0, 1\} \right\} \tag{5}$$

and it is the smallest technology that satisfies properties $A1, A2$. It is also said that dataset D_t *generates* or is a *generator* for technology T_t^V, since the definition of the set is based on the given generator (see Dulá and Thrall (2001)).

The free disposal conical extension of a production set T_t is defined as:

$$C(T_t) = \{(x, y) : x \geq \delta\overline{x},\ y \leq \delta\overline{y},\ (\overline{x}, \overline{y}) \in T_t,\ \delta \geq 0\} \tag{6}$$

The technology satisfies CRS if and only if $T_t = C(T_t)$. The cone technology $C(T_t)$ is said to be *generated* by T_t, or it is said that T_t is a *generator* of $C(T_t)$. The cone technology can in most cases be generated directly from the dataset D_t by using the axiom of scalability. In fact, the conical extension $C(T_t)$ will always satisfy scalability, defined as:

P1. Scalability: $\forall (x, y) \in T_t$ and $\delta \geq 0$ then $(\delta x, \delta y) \in T_t$.

The reverse is also true: if a free disposable technology T_t satisfies scalability, then it is a cone in the sense of definition (6)[3]. Using the conical extension operator defined in equation (6), one can generate the following CRS non-convex technology (FDH-CRS) from the FDH technology (5)[4]:

$$T_t^C = C(T_t^V) = \left\{(x, y) : \delta \sum_{k=1}^{K} \lambda_{kt} x_{kt} \leq x, \delta \sum_{k=1}^{K} \lambda_{kt} y_{kt} \geq y, \sum_{k=1}^{K} \lambda_{kt} = 1, \lambda_{kt} \in \{0, 1\}, \delta \geq 0\right\} \tag{7}$$

The cone extension so defined will always return the minimal technology that will satisfy A1, A2 and P1. It should be noted that the conical extension of the FDH set is a non-convex cone. This non-convex cone technology has been studied in Kerstens and Eeckaut (1999), Podinovski (2004). Briec and Kerstens (2006) show that computation of distance functions using the non-convex cone technology just defined can be accomplished by using an enumeration algorithm. This makes computation over large datasets feasible and fast.

A production technology is said to satisfy convexity if it satisfies the following property[5]:

P2. Convexity: $(\mathbf{x}_1, y_1) \in T$ and $(\mathbf{x}_2, y_2) \in T \implies [\gamma\mathbf{x}_1 + (1 - \gamma)\mathbf{x}_2, \gamma y_1 + (1 - \gamma) y_2] \in T$, $0 \leq \gamma \leq 1$.

[3]To see this, consider a technology T that satisfies free disposability (A2) and scalability (P1). Then for any point $(\overline{\mathbf{x}}, \overline{y}) \in T$, scalability implies $(\delta\overline{\mathbf{x}}, \delta\overline{y}) \in T$ for $\delta \geq 0$, and free disposability implies $(\mathbf{x}, y) \in T$, for all $(\mathbf{x} \geq \delta\overline{\mathbf{x}}, y \leq \delta\overline{y})$, which is equivalent to definition (6).

[4]This will imply a restriction on the data in order for $C(T_t)$ to be a strict subset of the positive orthant. What is required is that there are no observations that produce strictly positive output using a zero input vector. This is also known as the no free lunch assumption: $\nexists k : \mathbf{x}_k = \mathbf{0}_P$ and $y_k > 0$.

[5]Associated with the convexity property it is possible to define a convex operator which will extend any given set to make it convex. Kerstens et al. (2019) define this operator as follows:

$$Conv(T) = \left\{(x, y) \in \mathbb{R}_+^{P+1} : x = \sum_i \alpha_i \overline{x}_i, x = \sum_i \alpha_i \overline{x}_i, (\overline{x}_i, \overline{y}_i) \in T, \sum_i \alpha_i = 1, \alpha_i \geq 0\right\}$$

Enlarging the FDH hull by convexity will return the following VRS hull (also known as the BCC or DEA):

$$\Psi_t^V = Conv(T_t^V) = \left\{(x, y) : \sum_{k=1}^{K} \lambda_{kt} x_{kt} \leq x, \sum_{k=1}^{K} \lambda_{kt} y_{kt} \geq y, \sum_{k=1}^{K} \lambda_{kt} = 1, \lambda_{kt} \geq 0\right\} \quad (8)$$

This is the minimal set satisfying A1, A2, P2. The convex CRS technology can be obtained as the conical extension of (8) or as the convex closure of (7):

$$\Psi_t^C = C(\Psi_t^V) = Conv(T_t^C) = \left\{(x, y) : \sum_{k=1}^{K} \lambda_{kt} x_{kt} \leq x, \sum_{k=1}^{K} \lambda_{kt} y_{kt} \geq y, \lambda_{kt} \geq 0\right\} \quad (9)$$

This set will be the minimal set that satisfies A1, A2, P1 and P2. The last property and operator to be discussed relates to homotheticity. Homotheticity is defined as follows

P3. Input Homotheticity. $\exists H_{kt}, H_{jt} : H_{jt} IL_t(y_{kt}) = H_{kt} IL_t(y_{jt}), \forall k, j, t.$

Defining a homothetic operator that will enlarge a given technology in order to return a technology that satisfies homotheticity is not a trivial task. Peyrache (2024) shows that it is possible to build a minimal homothetic technology as an enlargement of the FDH. A minimal requirement for such an enlarged set is that it is contained in the CRS set. Thus, in order to define such a homothetic technology, the following requirement needs to be added:

R1. $T_t \subseteq T_t^C, \forall t = 1, \ldots, T.$

This requirement states that the homothetic free disposal hull (HFDH) technology has to be contained in the FDH-CRS technology (this extends to the convex case). Based on assumptions A1, A2, P3 and R1, Peyrache (2024) shows how to build a minimal homothetic technology T_t^H. The conical extension of this group homothetic technology will return the group CRS technology: $C(T_t^H) = T_t^C$. This also means that the homothetic technology is a generator of the CRS technology; in fact a minimal generator. In other words, for any given CRS technology T_t^C it is possible to build a homothetic technology T_t^H that will generate the CRS technology as its conical extension. In a nutshell, the homothetic technology is obtained by computing the input aggregates as[6] $X_{jt} = D_I^C(x_{jt}, 1, t)$ and then use these aggregates to build an enlarged

[6]Compute the input aggregates as follows:

$$\begin{array}{lll} \dfrac{1}{X_{jt}} = & \min\limits_{\theta,\lambda_k} & \theta \\ & s.t & \sum_k \lambda_k \frac{x_{kt}}{y_{kt}} \leq \theta x_{jt} \\ & & \sum_k \lambda_k = 1 \\ & & \lambda_k \in \{0, 1\} \end{array} \quad (10)$$

dataset that will be a generator for T_t^H. This enlarged dataset can be built by noting that the definition of input homotheticity implies that if $(x_{kt}, y_{kt}) \in T_t$, then $(x_{kt}\frac{X_{jt}}{X_{kt}}, y_{jt}) \in T_t$. The following enlarged dataset of K^2 observations will then provide a generator for the homothetic set T_t^H:

$$D_t^H = \left[x_{kt}\frac{X_{jt}}{X_{kt}}, y_{jt} \right] \tag{11}$$

If parsimony is necessary, Peyrache (2024) shows how to obtain the frame from this enlarged dataset. Alternatively, one can use the procedure described in Dulá and Thrall (2001) to extract the frame from this generator set. D_t^H will generate a homothetic set. I will refer to the procedure for obtaining the generator of the homothetic group technology T_t^H as the *homothetic operator*, or the *homothetic extension* of the VRS technology T_t^V, or, possibly more appropriate in this context, the minimal *homothetic generator* of the CRS technology T_t^C. This last sentence is justified in the sense that the homothetic set so obtained is the smallest homothetic set that can generate the CRS technology as its conical extension and contains all the data points. The basic notion is that the base isoquant is taken from the CRS technology and then the graph is obtained using the FDH definition. The technology so obtained will be deemed T_t^H.

3.2 Pooled Technologies and Metatechnologies

The definition of a metatechnology has been done in two main alternative ways: one can define the metatechnology as the union of the group technologies; alternatively, one can define the metatechnology as the pooled technology. The two notions basically differ in the way they treat the convexity assumption, as it should be clarified shortly. O'Donnell et al. (2008) seem to define the metatechnology as the union of sets, although while implementing it using DEA they use the pooled definition. Pastor and Lovell (2005) and Walheer (2018, 2023) use the pooled definition for the metatechnology (under convexity). Kerstens et al. (2019), Jin et al. (2020) and Jin et al. (2024) support the idea that the metatechnology should be a union of sets. The difference lies in the fact that some authors take $\bigcup_t T_t$ and some others $Conv\left(\bigcup_t T_t\right)$ as the definition of the metatechnology (irrespective of the fact if the group technologies satisfy convexity). Notice that this difference only arises for convex technologies, since in the case of the FDH the pooled technology is the union of the group FDH technologies. To maintain the language simple and parsimonious I will refer to the metafrontier as the union of sets and to the pooled technology in the standard way of using the pooled dataset as a generator for the technology[7]. Since the purpose of the current paper is to mainly provide a method to build group technologies that satisfy neutrality and homotheticity, I will illustrate the method using both the union of sets and the pooled definitions of the metafrontier. To avoid confusion in the language I will refer to the metafrontier as the union of sets, which will avoid ambiguity. This way the metafrontier will always be the union of the group technologies and the pooled technology the one defined on the pooled dataset.

Starting with the FDH technology T_t^V it is easy to show that the pooled technology is also the union of the group technologies. The pooled FDH is:

$$T^V = \left\{ (x, y) : \sum_{t=1}^{N} \sum_{k=1}^{K} \lambda_{kt} x_{kt} \leq x, \sum_{t=1}^{N} \sum_{k=1}^{K} \lambda_{kt} y_{kt} \geq y, \sum_{t=1}^{N} \sum_{k=1}^{K} \lambda_{kt} = 1, \lambda_{kt} \in \{0, 1\} \right\} \tag{12}$$

Interestingly this derives from the fact that the FDH technology is already defined as a union of sets. In fact an alternative way of defining the FDH technology is as the union of the sets $T_{kt}^V = \{(x, y) : x \geq x_{kt}, y \leq y_{kt}\}$, i.e. $T_t^V = \bigcup_k T_{kt}^V$ and therefore $T^V = \bigcup_t T_t^V = \bigcup_t \bigcup_k T_{kt}^V$. Similarly, the pooled FDH-CRS is:

$$T^C = \left\{ (x, y) : \delta \sum_{t=1}^{N} \sum_{k=1}^{K} \lambda_{kt} x_{kt} \leq x, \delta \sum_{t=1}^{N} \sum_{k=1}^{K} \lambda_{kt} y_{kt} \geq y, \sum_{t=1}^{N} \sum_{k=1}^{K} \lambda_{kt} = 1, \lambda_{kt} \in \{0, 1\}, \delta \geq 0 \right\} \tag{13}$$

It is easy to show that the metatechnologies are the pooled technologies: $T^V = \bigcup_t T_t^V$; $T^C = \bigcup_t T_t^C$.

The following lemma clarifies the relationship between the conical extension operator and the union operator (Kerstens et al. (2019) provide a similar lemma; I provide a proof for completeness, since I am using a slightly different definition of conical extension).

Lemma 3.1. *The conical extension of the union of sets is the union of their conical extensions:* $C\left(\bigcup_t T_t\right) = \bigcup_t C\left(T_t\right)$.

Proof. To show this consider a point $(x, y) \in \bigcup_t C(T_t)$. Using the definition of conical extension, this means that for some t, $\exists (\overline{x}, \overline{y}) \in T_t$ such that $x \geq \delta \overline{x}, y \leq \delta \overline{y}$ for some $\delta \geq 0$. If $\delta > 0$, then $(x/\delta, y/\delta) \in T_t$ which implies $(x/\delta, y/\delta) \in \bigcup_t T_t$; then $(x, y) \in C\left(\bigcup_t T_t\right)$. If $\delta = 0$, then condition $x \geq \delta \overline{x}, y \leq \delta \overline{y}$ simplifies to $x \geq 0, y \leq 0$, which implies $(x, y) \in C\left(\bigcup_t T_t\right)$.

On the contrary, suppose that $(x, y) \in C(\bigcup_t T_t)$, then using the definition of conical extension $x \geq \delta \overline{x}, y \leq \delta \overline{y}$, for $\delta \geq 0$ and $(\overline{x}, \overline{y}) \in T_t$, for some t. This means that $(x, y) \in C(T_t)$, therefore

[7] The alternative of using FDH group technologies and then the convexification of their union as the metatechnology has not been suggested, thus I will not discuss this case.

$(x, y) \in \bigcup_t C(T_t)$. QED

It is also clear that since $T_t^C = C(T_t^V)$, then $T^C = C(T^V)$, due to the Lemma just introduced. This defines the VRS and CRS metatechnologies by using their pooled definition in the non-convex case. For the convex case things are complicated by the fact that the union of sets is not the pooled technology. The pooled convex VRS technology is:

$$\Psi_{pooled}^V = \left\{(x, y) : \sum_{t=1}^N \sum_{k=1}^K \lambda_{kt} x_{kt} \leq x, \sum_{t=1}^N \sum_{k=1}^K \lambda_{kt} y_{kt} \geq y, \sum_{t=1}^N \sum_{k=1}^K \lambda_{kt} = 1, \lambda_{kt} \geq 0\right\} \tag{14}$$

The pooled CRS is:

$$\Psi_{pooled}^C = \left\{(x, y) : \sum_{t=1}^N \sum_{k=1}^K \lambda_{kt} x_{kt} \leq x, \sum_{t=1}^N \sum_{k=1}^K \lambda_{kt} y_{kt} \geq y, \lambda_{kt} \geq 0\right\} \tag{15}$$

The convex metafrontiers defined as union of sets are:

$$\begin{aligned} \Psi^V = \textstyle\bigcup_t \Psi_t^V = \Bigg\{(x, y) : \quad & \sum_{t=1}^N \sum_{k=1}^K \lambda_{kt} x_{kt} \leq x, \sum_{t=1}^N \sum_{k=1}^K \lambda_{kt} y_{kt} \geq y, \\ & \sum_{k=1}^K \lambda_{kt} = b_t, \sum_t b_t = 1, b_t \in \{0, 1\}, \lambda_{kt} \geq 0\Bigg\} \end{aligned} \tag{16}$$

and

$$\begin{aligned} \Psi^C = \textstyle\bigcup_t \Psi_t^C = \Bigg\{(x, y) : \quad & \sum_{t=1}^N \sum_{k=1}^K \lambda_{kt} x_{kt} \leq x, \sum_{t=1}^N \sum_{k=1}^K \lambda_{kt} y_{kt} \geq y, \\ & \sum_{k=1}^K \lambda_{kt} = \delta b_t, \sum_t b_t = 1, b_t \in \{0, 1\}, \delta \geq 0, \lambda_{kt} \geq 0\Bigg\} \end{aligned} \tag{17}$$

The next question that should be raised is if it is possible to build a homothetic metafrontier and pooled homothetic technology starting with the group homothetic technologies T_t^H and Ψ_t^H (these two homothetic technologies can be built using the homothetic operator on T_t^C and Ψ_t^C). It is tempting to apply the same definition of metatechnology to the group-specific homothetic technologies to obtain the homothetic metatechnology: $\bigcup_t T_t^H$. This definition will satisfy the basic requirement that $T_t^V \subseteq T_t^H \subseteq T_t^C$, which implies (using Lemma 3.1) $T^V \subseteq \bigcup_t T_t^H \subseteq T^C$. This also means that the conical extension of this metatechnology coincides with the CRS metatechnology $C(\bigcup_t T_t^H) = T^C$. Unfortunately, this technology is not homothetic, i.e. homotheticity of the group technologies is not sufficient for homotheticity of their union. This stems from the fact that homotheticity of the group technologies does not imply neutrality, therefore the homothetic group technologies are not homothetic to each other. Thus homotheticity of the group technologies is not a sufficient condition for the homotheticity of the metatechnology, since these group homothetic technologies will fail to be homothetic to each other, i.e. they will fail neutrality. Therefore $\bigcup_t T_t^H$ must be disregarded as a candidate for a homothetic metatechnology, i.e. this will provide a metatechnology but this metatechnology will not be homothetic.

Things are different when it comes to the pooled homothetic technology, since in this case the homothetic operator can be applied directly to the pooled CRS technology T^C (Ψ^C_{pooled} and Ψ^C can be used as well). In fact, it is possible to find a minimal pooled homothetic technology that satisfies:

R2. $T \subseteq T^C$

Requirement $R2$ is the pooled version of requirement $R1$, since $\bigcup_t T_t \subseteq \bigcup_t T_t^C = T^C$. $R2$ is weaker than requirement $R1$, in fact it is implied by $R1$, although it does not imply $R1$. Call the pooled homothetic technology so obtained T^H. Since this pooled homothetic technology is the minimal technology that satisfies $R2$, I will regard this technology as being the best candidate and requirement $R2$ as being a very minimal requirement that should be maintained throughout the analysis. A similar argument can be put forward for the convex case and the use of Ψ^C_{pooled} and Ψ^C. Notice however that in spite of the fact that $L^C(1) \subseteq L^C_{\Psi}(1) \subseteq L^C_{\Psi_{pooled}}(1)$, these technolgies are not-nested (they intersect): $\Psi^H_{pooled} \neq \Psi^H \neq T^H$.

In order to obtain a generator dataset for the pooled homothetic technology, the input aggregates can be computed by solving the program associated with the computation of the following input distance function $X_{js} = X(x_{js}) = D^C_I(x_{js}, 1)$ for all observations in the dataset:

$$
\begin{aligned}
\frac{1}{X_{js}} = \quad & \min \quad \theta \\
& st \quad \delta \sum_{t=1}^{N} \sum_{k=1}^{K} \lambda_{kt} \mathbf{x}_{kt} \leq \theta \mathbf{x}_{js} \\
& \qquad \delta \sum_{t=1}^{N} \sum_{k=1}^{K} \lambda_{kt} y_{kt} \geq 1 \\
& \qquad \sum_{t=1}^{N} \sum_{k=1}^{K} \lambda_{kt} = 1 \\
& \qquad \lambda_{kt} \in \{0,1\}, \forall k, t \\
& \qquad \delta \geq 0
\end{aligned}
\tag{18}
$$

An enumeration algorithm exists for the computation of this program (see Briec and Kerstens (2006)). To obtain the program for the convex case, one eliminates the binary variables constraints, transforming it into a linear program. The following dataset will be a generator for T^H:

$$D^H = \left[x_{kt} \frac{X_{js}}{X_{kt}}, y_{js} \right] \tag{19}$$

Notice that this dataset will be of dimension $(NK)^2$, contrary to the dimensionality of the generators D_T^H of T_t^H that return a pooled dataset of dimension NK^2. Given the dimensionality of $(NK)^2$, this generator is not a very parsimonious representation of the technology set. The most parsimonious representation of T^H can be obtained by extracting the frame from this set of observations using the method presented in Dulá and Thrall (2001), although this could be computationally intensive, since it basically involves solving $(NK)^2$ programs. The alternative provided in Peyrache (2024) can be adapted here by considering the set of extreme efficient points for the pooled CRS technology as defined in (this has been defined in Dulá and Thrall (2001)). Notice that the approach of Dulá and Thrall (2001) can be improved computationally by starting with the definition of the pooled FDH dominance set (assuming that no observations are repeated and no observations are scalings of each other) which corresponds to the frame of the pooled FDH:

$$S^{FDH} = \{(x_{js}, y_{js}) : (x_{js} \leq x_{kt}) \cap (y_{js} \geq y_{kt}), \forall k, t\} \tag{20}$$

This set is likely to be much smaller than the original pooled dataset. From this set it is possible to extract the frame of the FDH-CRS (or the convex CRS) as follows:

$$S^C = \left\{ (x_{js}, y_{js}) \in S^{FDH} : D_I^C(x_{js}, y_{js}) = 1 \right\} \tag{21}$$

Since the pooled FDH set can be obtained by enumeration, the frame of the CRS technology can be obtained faster than in the method proposed by Dulá and Thrall (2001), with this latest method providing an upper bound to the computational time (in the case all observations are extreme scale efficient; very unlikely in any empirical application). Call the dimensionality of this set $M < NK$. Now, given this set it is possible to define the base unit isoquant as the list: $a_{js} = x_{js}/y_{js}$, with $(x_{js}, y_{js}) \in S^C$. Then the frame of the homothetic technology is given by (assuming no output is duplicated):

$$D^H = [a_{js} X_{kt}, y_{kt}] \tag{22}$$

thus providing a generator with MNK observations, much lower than the $(NK)^2$ required before (by a factor of $M < NK$).

The next obvious question is if it is possible to find group technologies that satisfy homotheticity, are homothetic to each other and give rise to the pooled homothetic technology just defined as their metatechnology (i.e. as a union of sets). One very intuitive way of building such technologies would be to take the input aggregates just computed and define the following generator sets of dimension NK^2:

$$D_t^{NH} = \left[x_{js}\frac{X_{kt}}{X_{js}}, y_{kt} \right] \tag{23}$$

If computational parsimony is necessary, then it is possible to work directly with the frame:

$$D^H = [a_{js}X_{kt}, y_{kt}] \tag{24}$$

which is of dimension MK for each group technology. This will generate group technologies T_t^{NH} that are neutral homothetic. Moreover, since $\bigcup_t D_t^{NH} = D^H$, this means that $\bigcup_t T_t^{NH} = T^H$, satisfying the basic requirement that the metatechnology is generated as union of sets. Although this approach provides a way of building group technologies that satisfy homotheticity and neutrality, it is not clear what the properties of these group technologies are. In particular, it is not clear if these technologies can be given axiomatic foundation. The next section should address this issue and also provide the frame (a parsimonious generator) for these neutral homothetic sets.

I will refer to the metafrontiers as simply T^V and T^C and their associated distance functions as $D_I^V(x, y)$ and $D_I^C(x, y)$. All the results obtained for T^V and T^C will extend to $\Psi^V, \Psi^V_{pooled}, \Psi^C, \Psi^C_{pooled}$. I will indicate in the discussion and the proofs when results may diverge.

4 The Neutral Homothetic Group Technologies

In this section I will consider two technologies that satisfy both neutrality and homotheticity, one under the CRS assumption and the other one under the VRS assumption (i.e. one will satisfy scalability P1 and the other will not). Before presenting the methods used to build such technologies it is useful to introduce the property of *neutral homotheticity*, the joint satisfaction of neutrality and homotheticity:

A3. Neutral Homotheticity: $\exists B_{kt}, B_{js} : B_{kt}IL(y_{kt}, t) = B_{js}IL(y_{js}, s), \forall k, j, s, t.$

This property is stating that all input sets have to be scaling of each other even if they belong to different group technologies. The definition of homotheticity P3 on the contrary makes the same statement for each technology separately. Therefore A3 is stronger than P3.

4.1 The Neutral CRS Group Technologies

It is useful to start the discussion with the group CRS technologies T_t^C. These technologies are homothetic (they always satisfy P3) but, in general, they are not homothetic to each other (they will violate A3), therefore they are not neutral. On the contrary, the metafrontier (or pooled) CRS technology is homothetic. Strangely enough, such technologies are not neutral, yet they return a metatechnology that is homothetic, showing that neither neutrality nor homotheticity are necessary for the metafrontier to be homothetic (remember that the union of the homothetic group technologies T_t^H is not homothetic). In what follows it is assumed that the group CRS technologies are not neutral, i.e. they fail property A3. If this were not the case, then as it will be shown below, the group homothetic technologies T_t^H will in fact satisfy axiom A3. It is therefore assumed that T_t^C do not satisfy condition A3 and it is therefore necessary to find appropriate enlargements of these technologies that will satisfy such condition. Assumption A3 can be easily checked using the procedure outlined in this subsection, since the resulting neutral CRS group technologies (call them T_t^{NC}) will in this case be the same as T_t^C, the CRS group technologies. In the case of CRS technologies, scalability (P1) and neutral homotheticity (A3) will imply a very neat property that is presented in the next lemma.

Lemma 4.1. *If the technology satisfies A1, A2, A3 and P1, then there exist constants A_t such that ($\forall t, s = 1, \ldots, N$):*

$$A_s \, IL_s(1) = A_t \, IL_t(1)$$

Proof. Using the definition of neutral homotheticity (A3) and scalability (P1), it is possible to write:

$$B_{kt} y_{kt} IL_t(y_{js}) = B_{js} y_{js} IL_s(y_{js})$$

Define now $B'_{kt} = B_{kt} y_{kt}$. Using scalability this implies:

$$B'_{kt} IL_t(1) = B'_{js} IL_s(1)$$

Consider now $x \in IL_t(1)$, which means $\frac{B'_{kt}}{B'_{js}} x \in IL_s(1)$. Since $x \in IL_t(1)$ if and only if

$D_I(x, 1, t) = 1$, it is possible to write:

$$\frac{B'_{kt}}{B'_{js}} = \frac{D_I(x,1,t)}{D_I(x,1,s)} = \frac{D_I(1,1,t)}{D_I(1,1,s)}$$

where the last passage follows from the neutral homotheticity assumption. Since the right hand side only depends on s, t, so does the left hand side. Therefore it must be: $B'_{kt} = B_{kt}y_{kt} = A_t$. In other words scalability and neutral homotheticity imply that:

$$B_{kt} = \frac{A_t}{y_{kt}}$$

This also means that $B_{kt}y_{kt} = B_{jt}y_{jt}$. QED

The importance of this lemma should not be underestimated, since it implies a stronger notion of neutrality[8], where the time shift function is reduced to a set of constants independent of the output level. I will call this set of constants the *neutral deflators*, since they will "deflate" the unit isoquant of the group CRS technologies to their base level.

The first step in building a technology that satisfies A1, A2, A3, P1 and R2 is to use scalability and express all the group technologies in terms of their unit input set: $x \in L_t(y)$ implies $x/y \in L_t(1)$ (any other output level can be chosen, this is innocuous). Then any difference in the technologies can be seen as a difference in this unit isoquant (since all the others are linear scalings). In other words, all these CRS group technologies will only differ in the shape of the unit isoquant, since given this unit isoquant the rest of the technology is obtained as a cone by scaling. Since the neutral deflators A_t are not known, a method needs to be found in order to compute them. Requirement $R2$ can be restated as requiring that $L_t(1) \subseteq L^C(1)$, which means any neutral deflator will need to satisfy a condition that the enlarged CRS technology still satisfy this condition. Consider the following set of programs for all k, t:

$$a_{kt} = \min_{\theta} \left\{ \theta : \theta \frac{x_{kt}}{y_{kt}} \in L^C(1) \right\} \tag{25}$$

These coefficients are projecting each observation onto the unit CRS metafrontier (pooled) isoquant. The neutral deflators can then be derived as ($\forall t = 1, \ldots, N$):

$$A_t = \max_k \{a_{kt}\} \tag{26}$$

[8] Blackorby et al. (1976) call this concept extended Hicks neutrality. There is also a theorem in Blackorby et al. (1976) that proves a result similar to the one presented in the lemma.

where clearly $A_t \leq 1$ where equality only happens for those group technologies that have one observation on the metafrontier (pooled) unit isoquant. Given these deflators, the neutral CRS group technologies are then given by the unit input set $L_t^{NC}(1) = L^C(1)/A_t = L^C(1/A_t)$. The technology is defined as a scaling of this unit input set:

$$T_t^{NC} = \big\{(x, y) : x/y \in L_t^{NC}(1)\big\} = \big\{(x, y) : A_t x \in L^C(y)\big\} \tag{27}$$

The following proposition should clarify that these deflators are optimal in the sense of providing the smallest or tightest technology that satisfies the given requirements.

Theorem 4.1. *T_t^{NC} are the smallest group technologies satisfying axioms $A1, A2, A3, P1$ and requirement R2.*

Proof. Requirement R2 states that $\bigcup_t T_t \subseteq T^C$. Suppose now that one were to choose $a_t < A_t$, where A_t is the optimal coefficient determined above. Then $\exists k : a_t x_{kt} \notin L^C(1)$, violating R2. It must then be that $a_t \geq A_t$.

Consider now the possibility that $a_t > A_t$. In such a case $\forall k$, $\frac{A_t}{a_t} x_{kt} \in L_t(1)$, but $\frac{A_t}{a_t} x_{kt} \notin IL_t(1)$, i.e. none of the observations of group t will be part of the unit isoquant of the group technology. This means that the associated technology is not minimal.

Therefore T_t^{NC} will be the smallest technologies compatible with the stated axioms. QED

Corollary 4.1. *The neutral CRS technologies T_t^{NC} are nested, with the neutral deflators A_t determining this nesting.*

Proof. This is a consequence of scalability, neutrality and homotheticity. QED

Interestingly enough, even if the neutral CRS group technologies T_t^{NC} are enlargements of the CRS group technologies T_t^C, they will return the same metatechnology!

Corollary 4.2. $\bigcup_t T_t^C = \bigcup_t T_t^{NC} = T^C$.

Proof. Requirement R2 states that $\bigcup_t T_t^{NC} \subseteq T^C$. Since $T_t^C \subseteq T_t^{NC}$, then $\bigcup_t T_t^C \subseteq \bigcup_t T_t^{NC}$. These two conditions can be stated as $\bigcup_t T_t^C \subseteq \bigcup_t T_t^{NC} \subseteq T^C$, and given that $\bigcup_t T_t^C = T^C$, the statement follows. QED

4.2 The Neutral Homothetic Group Technologies

The previous section provided CRS group technologies that satisfy neutrality. In this section I am trying to understand if a similar procedure can be delivered for the VRS technologies. The class of technologies that satisfy A1, A2 and P3 (and P2 if convexity is assumed) is $\mathcal{T}_t$; if requirement R1 is added then the class becomes $\mathcal{H}_t \subseteq \mathcal{T}_t$. Call $\overline{\mathcal{H}}_t$ the complement to $\mathcal{H}_t$ in $\mathcal{T}_t$, so that $\overline{\mathcal{H}}_t \bigcup \mathcal{H}_t = \mathcal{T}_t$. Define the class of group technologies that satisfy A1, A2, A3 (and P2 if convexity is assumed) as $\mathcal{Z}_t$, and indicate a technology in this class as $T_t \in \mathcal{Z}_t$, $\forall t = 1, \ldots, N$. It is clear that $\mathcal{Z}_t \subseteq \mathcal{T}_t$, since A3 is stronger than P3.

Lemma 4.2. *$\forall T_t \in \mathcal{Z}_t$, $T_t^V \subseteq T_t$.*

Proof. Trivial.

Ok, I'll show you...

Consider $(x, y) \in T_t^V$. Using the definition of T_t^V, $\exists k : x \geq x_{kt}, y \leq y_{kt}$. Feasibility of observations (A1), implies that $(x_{kt}, y_{kt}) \in T_t$ and free disposability (A2) implies that $(x, y) \in T_t$, proving $T_t^V \subseteq T_t$. The proof is similar in the convex case. QED

Lemma 4.3. *The neutral CRS group technology is a subset of the conical extension of the neutral homothetic group technology: $\forall T_t \in \mathcal{Z}_t$, $T_t^{NC} \subseteq C(T_t)$*

Proof. Consider a point $(x, y) \in T_t^{NC}$. Using the definition of T_t^{NC}, $x/y \in L_t^{NC}(1)$ or $A_t x/y \in L^C(1)$. Then $\exists k, s : x_{ks}/y_{ks} \leq A_t x/y$, or $\exists k, s, \delta : \delta x_{ks} \leq A_t x, \delta y_{ks} \geq A_t y$. Feasibility of observations (A1) implies that $(x_{ks}, y_{ks}) \in T_s \subseteq T^C$; neutrality implies that $(x_{ks}/A_t, y_{ks}) \in T_t$. Since $T_t \subseteq C(T_t)$, $(x_{ks}/A_t, y_{ks}) \in C(T_t)$. Scalability of $C(T_t)$ implies that $(\delta x_{ks}/A_t, \delta y_{ks}) \in C(T_t)$ which means (using free disposability) $(x, y) \in C(T_t)$, proving that $T_t^{NC} \subseteq C(T_t)$. The proof for the convex case is similar. QED

It is interesting to note that it is impossible to satisfy both $R1$, neutrality and homotheticity at the same time. This can only happen if $T_t^{NC} = T_t^C$. In such a case the group homothetic technologies T_t^H would also be neutral and the search for a neutral homothetic group technology would be concluded. It is therefore assumed in the following analysis that $T_t^C \subset T_t^{NC}$ (again, the reason for this is that if $T_t^C T_t^{NC}$ then the problem becomes trivial). The following impossibility result is interesting at this point of the discussion.

Proposition 4.1. *There is no technology that satisfies $A1, A2, A3, R1$.*

Proof. This can be proved by contradiction. Suppose that there exists a technology T_t that is neutral homothetic, then by lemma 4.3 $T_t^{NC} \subseteq C(T_t)$. R1 requires that $T_t \subseteq T_t^C$ which implies $C(T_t) = T_t^C$ (a proof of this can be found in Peyrache (2024)). Since $T_t^C \subset T_t^{NC}$ this would imply $C(T_t) \subset T_t^{NC}$ in contradiction with $T_t^{NC} \subseteq C(T_t)$. QED

This proposition proves that it is not possible to build neutral homothetic group technologies that satisfy $R1$. Therefore a weakening of this condition is necessary for building such a technology. In order to find this weaker condition, the following lemma is useful.

Lemma 4.4. *$T_t^{NC} = C(T_t)$ if and only if $T_t \subseteq T_t^{NC}$.*

Proof. For the sufficiency part, note that $T_t \subseteq C(T_t)$, and since we are assuming $T_t^{NC} = C(T_t)$, this proves $T_t \subseteq T_t^{NC}$.

For the necessity part, note that Lemma 4.3 implies $T_t^{NC} \subseteq C(T_t)$. Then, $T_t \subseteq T_t^{NC}$ implies $C(T_t) \subseteq T_t^{NC}$ and this together with the previous condition implies $T_t^{NC} = C(T_t)$. QED

This lemma states that in order for the conical extension of T_t to be minimal, the technology must be contained in the neutral CRS group technology T_t^{NC}. It is therefore possible to state the following condition as a requirement that the neutral homothetic group technology needs to satisfy:

R3. $T_t \subseteq T_t^{NC}$.

This requirement is stronger than $R2$. $R3$ is also weaker than $R1$ since the neutral CRS technologies are enlargements of the CRS technologies. Call the class of technologies that satisfy this additional requirement R3, $\mathcal{N}_t \subseteq \mathcal{Z}_t$. It is now possible to find a minimal technology in this class. Requirement R3 also characterizes the class of technologies $\mathcal{N}_t$ as being those neutral homothetic technologies that generate the neutral CRS group technologies T_t^{NC} as their conical extensions. In this sense, this class of technologies contains all the homothetic generators of T_t^{NC}.

Lemma 4.5. *$\bigcup_t T_t \subseteq T^C$ if and only if $T^C = C\left(\cup_t T_t\right) = \cup_t C(T_t)$.*

Proof. $C(T) = C(\bigcup_t T_t) = \bigcup_t C(T_t) = \bigcup_t T_t^{NC} = T^C$. QED

There is a very useful property that all technologies in $\mathcal{N}_t$ share and it is useful in order to build the minimal technology. In order to derive this property, first define the set of efficient points with reference to T_t^{NC} is:

$$S_t^{NC} = \left\{(x_{kt}, y_{kt}) : D_O^{NC}(x_{kt}, y_{kt}, t) = 1\right\} \tag{28}$$

The sets S^C, S_t^{NC} do not contain points with zero output, unless the input vector is zero. Therefore the origin $(0, 0)$ can be part of these sets, but a point $(x, 0)$ with $x \neq 0$ will never be part of this set. Therefore, without loss of generality, the origin can be excluded from this set.

A generator of the input set $L_t^{NC}(y_{kt})$ can be derived from the generator of the neutral CRS technology D_t^{NC} (they generate the same technology):

$$d_t^{NC} = \left\{ (\overline{x}_{js}, y_{kt}) : \overline{x}_{js} = \frac{A_s}{A_t} \frac{y_{kt}}{y_{js}} x_{js},\ (x_{js}, y_{js}) \in S^C \right\} \tag{29}$$

The following lemma then follows.

Lemma 4.6 (Cone Isoquant Lemma). $\forall T_t \in \mathcal{N}_t,\ C(T_t) = T_t^{NC} \implies L_t(y_{kt}) = L_t^{NC}(y_{kt}) = L^C(y_{kt})/A_t,\ \forall y_{kt}$ *such that* $(x_{kt}, y_{kt}) \in S_t^{NC}$.

Proof. Graphical proof for the two inputs case.

$C(T_t) = T_t^{NC}$ implies $T_t \subseteq T_t^{NC}$ which implies $L_t(y_{kt}) \subseteq L_t^{NC}(y_{kt})$. For all $(x_{kt}, y_{kt}) \in S_t^{NC}$, it must be that $x_{kt} \in IL_t(y_{kt})$. If this were not the case, then it would be possible to find a point $(x_{kt}, y_{kt}) \in S_t^{NC}$ such that $x_{kt} \notin IL_t(y_{kt})$. Now, due to the feasibility of observations assumption (A1), this point must belong to T_t; thus this point must be in the interior of $L_t(y_{kt})$, meaning that $D_I^{NC}(x_{kt}, y_{kt}, t) \geq D_I(x_{kt}, y_{kt}, t) > 1$. Therefore this would mean $1/D_I^{NC}(x_{kt}, y_{kt}, t) = D_O^{NC}(x_{kt}, y_{kt}, t) < 1$, in contradiction with the fact that $(x_{kt}, y_{kt}) \in S_t^{NC}$. This proves the statement by contradiction. So it must be that $x_{kt} \in IL_t(y_{kt})$.

Consider now the points in set d_t^{NC}. For all $(\overline{x}_{js}, y_{js}) \in d_t^{NC}$, it must be that $\overline{x}_{js} \in IL_t(y_{kt})$. This can be proved by contradiction. Suppose that $\overline{x}_{js} \notin IL_t(y_{kt})$. Since it is assumed that $C(T_t) = T_t^{NC}$, then $(\overline{x}_{js}, y_{kt}) \in C(T_t)$. This means, using the definition of conical extension (and the fact that $(x_{kt}, y_{kt}) \neq (0, 0)$), that there exists $\delta > 0$ such that $(\delta\overline{x}_{js}, \delta y_{kt}) \in T_t$, which can be restated as $\delta\overline{x}_{js} \in IL(\delta y_{kt})$ (if $\delta\overline{x}_{js}$ were to be in the interior of $L(\delta y_{kt})$ this would contradict $C(T_t) = T_t^{NC}$, since $\overline{x}_{js} \in IL_t^{NC}(y_{kt})$). Homotheticity implies that there exists a constant α such that $\alpha IL_t(y_{kt}) = IL_t(\delta y_{kt})$, and $\overline{x}_{js} \notin IL_t(y_{kt})$ implies that $\alpha \neq \delta$. Now, if $\alpha > \delta$, then since $(\delta\overline{x}_{js}, \delta y_{kt}) \in T_t$, then $\left(\frac{\delta}{\alpha}\overline{x}_{js}, y_{kt}\right) \in T_t$. Since $\frac{\delta}{\alpha} < 1$, then $\left(\frac{\delta}{\alpha}\overline{x}_{js}, y_{kt}\right) \notin T_t^{NC}$ in contradiction with the assumption that $T_t \subseteq T_t^{NC}$ (or $C(T_t) = T_t^{NC}$). If, on the contrary, $\alpha < \delta$, then using homotheticity $\alpha x_{kt} \in IL_t(\delta y_{kt})$, and scalability of $C(T_t)$ implies $(\frac{\alpha}{\delta}x_{kt}, y_{kt}) \in C(T_t)$, but since $\alpha/\delta < 1$, then $(\frac{\alpha}{\delta}x_{kt}, y_{kt}) \notin T_t^{NC}$, in contradiction with the assumption that $T_t \subseteq T_t^{NC}$ (or $C(T_t) = T_t^{NC}$). Therefore it must be that $\overline{x}_{js} \in IL_t(y_{kt})$.

Take now $L_t^{NC}(y_{kt})$, for all y_{kt} such that $(x_{kt}, y_{kt}) \in S_t^{NC}$. Since the set of points $\overline{x}_{js}$, $\forall(\overline{x}_{js}, y_{kt}) \in d_t^{NC}$ is a generator of $L_t^{NC}(y_{kt})$, the input set can be described by free disposability as follows: $L^{NC}(y_{kt}) = \left\{x \mid \exists j, s : x \geq \overline{x}_{js},\ (\overline{x}_{js}, y_{kt}) \in d_t^{NC}\right\}$. Since all points in d_t^{NC} also belong to $L_t(y_{kt})$, this proves that $L_t^{NC}(y_{kt}) \subseteq L_t(y_{kt})$. Since it has been established already that $L_t(y_{kt}) \subseteq L^{NC}(y_{kt})$, this implies $L_t^{NC}(y_{kt}) = L_t(y_{kt})$ for all y_{kt} such that $(x_{kt}, y_{kt}) \in S_t^{NC}$. The proof is similar in the convex case. QED

The result of lemma 4.6 can be used to establish the following theorem that characterizes technologies $T_t \in \mathcal{N}_t$.

Theorem 4.2. *For any two technologies $T_t \neq T'_t$ satisfying axioms A1, A2, A3, R3, there exists a set of constants such that:*

$$IL'_t(y_{it}) = h_{ij} IL_t(y_{jt})\,,\ \ \forall i, j = 1, \ldots, K$$

In other words, the alternative homothetic technologies $T, T' \in \mathcal{N}_t$ are homothetic to each other (one is a homothetic transform of the other).

Proof. Too see this, pick two alternative technologies $T' \neq T$. From Lemma 4.6, it follows that $L'_t(y_{kt}) = L_t^{NC}(y_{kt}) = L_t(y_{kt})$, $\forall y_{kt}$ such that $(x_{kt}, y_{kt}) \in S_t^{NC}$. Consider now two points (x_{it}, y_{it}) and (x_{jt}, y_{jt}) $i, j, = 1, \ldots, K$ (not necessarily in S_t^{NC}). Since both T_t and T'_t are homothetic, using property A3 it must be that:

a) $H_{jt} IL_t(y_{it}) = H_{it} IL_t(y_{jt})$

b) $H'_{jt} IL'_t(y_{kt}) = H'_{kt} IL'_t(y_{jt})$;

c) $H_{jt} IL_t(y_{kt}) = H_{kt} IL_t(y_{jt})$

where $(x_{kt}, y_{kt}) \in S_t^{NC}$ and the choice of k is innocuous. Since it follows from lemma 4.6 that $IL'_t(y_{kt}) = IL_t(y_{kt})$. Using b) and c), one obtains:

$$\frac{H_{kt}}{H_{jt}} IL_t(y_{jt}) = IL_t(y_{kt}) = IL'_t(y_{kt}) = \frac{H'_{kt}}{H'_{jt}} IL'_t(y_{jt})$$

Using a) and substituting in the latter expression:

$$\frac{H_{kt}}{H_{it}} IL_t(y_{it}) = \frac{H'_{kt}}{H'_{jt}} IL'_t(y_{jt})$$

which returns the h_{ij} constants:

$$h_{ij} = \frac{H'_{jt}}{H'_{kt}} \frac{H_{kt}}{H_{it}}$$

QED

Since all technologies in set $\mathcal{N}_t$ are homothetic to each other (due to theorem 4.2) and since they satisfy neutral homotheticity, this also means that any two technologies $T_t \in \mathcal{N}_t$ and $T_s \in \mathcal{N}_s$ are also homothetic to each other. This does not extend to the set $\mathcal{Z}_t$ and technologies in this set are not necessarily homothetic to each other. This means that while it is possible to find a minimal technology in $\mathcal{N}_t$, it is not easy to define a minimal technology in the complement of $\mathcal{N}_t$ in $\mathcal{Z}_t$. Consider now the set of neutral homothetic technologies that are the complement of $\mathcal{N}_t$, $\overline{\mathcal{N}}_t = \mathcal{Z}_t \setminus \mathcal{N}_t$, so that $\mathcal{Z}_t = \overline{\mathcal{N}}_t \bigcup \mathcal{N}_t$.

Corollary 4.3. *$\forall T_t \in \overline{\mathcal{N}}_t$, $T_t^{NC} \subset C(T_t)$.*

Proof. This can be proved by contradiction. Lemma 3.2 states that $\forall T_t \in \mathcal{Z}_t$, $T_t^C \subseteq C(T_t)$. Suppose now that it were possible to find a technology $T_t \in \overline{\mathcal{N}}_t$ for which $C(T_t) = T^C$. This technology would satisfy requirement R3 and therefore also belong to $\mathcal{N}_t$ which leads to a contradiction, since $\mathcal{N}_t$ and $\overline{\mathcal{N}}_t$ are disjoint sets. QED

This corollary proves that if a neutral homothetic technology does not satisfy requirement R3, then its conical extension will be a strict superset of the neutral CRS technology. Thus, the conical extension of such a technology will not coincide with the minimal CRS technology.

The neutral homothetic technology is obtained by choosing as base isoquant the unit isoquant of T^C, $L^C(1)$. The input aggregates are computed against this isoquant and they will provide the group-specific graphs and the group specific technologies. This is equivalent to the procedure described in program (18). The second step in the procedure will now differ. Given the X_{kt} input aggregates so determined, the neutral homothetic technology can be represented in its graph (which I shall call G_t^{NH}). This graph is defined by applying definition (5) to the dataset (X_{kt}, y_{kt}):

$$G_t^{NH} = \left\{ (X, y) : \sum_{k=1}^{K} \lambda_k X_{kt} \leq X, \sum_{k=1}^{K} \lambda_k y_{kt} \geq y, \sum_{k=1}^{K} \lambda_k = 1, \lambda_k \in \{0, 1\} \right\} \tag{30}$$

This will provide the graphs of the neutral homothetic technologies. A generator for this neutral homothetic technology can be obtained by considering the following enlarged dataset:

$$\left(x_{kt} \frac{X_{jt}}{X_{kt}}, y_{jt} \right), \ \forall k, j = 1, \ldots, K \tag{31}$$

which is the same as the one described in equation (23). The frame of the neutral homothetic technology can be extracted using either the method presented in Peyrache (2024) or Dulá and Thrall (2001). This method of defining the neutral homothetic group technologies will also return a minimal set in $\mathcal{N}_t$ as shown in the next theorem.

Theorem 4.3. *The neutral homothetic technologies T_t^{NH} are minimal in $\mathcal{N}_t$.*

Proof. The proof is by contradiction. Minimality requires that $\forall T_t \in \mathcal{N}_t$, $T_t^{NH} \subseteq T_t$. Suppose, on the contrary, that there exists a technology $T_t \in \mathcal{N}_t$ that violates this condition: $\exists (x, y) \in T_t^{NH} : (x, y) \notin T_t$. Since the two technologies are homothetic to each other (they belong to $\mathcal{N}_t$, this last condition implies that $\exists (X, y) \in G_t^{NH} : (X, y) \notin G_t$, where $X = X(x) = D_I^C(x, 1)$ (where $L^C(1)$ has been selected as the reference input isoquant for the aggregates).

Using now the definition of G_t^{NH} in (30), we know that $(X, y) \in G_t^{NH}$ means $\exists k : X_{kt} \leq X, y_{kt} \geq y$, implying that $(X_{kt}, y_{kt}) \notin G_t$ (otherwise free disposability would be violated). This means a violation of assumption A1 (feasibility of observations) for technology T_t in contradiction with the stated assumptions. A similar argument can be easily used for the convex case. QED

Since the neutral homothetic group technologies are minimal and the union of their generators is equal to the generator of the pooled homothetic technology, this means that the metafrontier generated by the neutral homothetic group technologies is in fact the same as the minimal homothetic pooled technology.

Theorem 4.4. *The homothetic metafrontier T^H can be obtained as the union of the neutral homothetic group technologies:*

$$\bigcup_t T_t^{NH} = T^H$$

Proof. The proof of this is easy by noting that $\bigcup_t T_t^{NH} = T^H$ is equivalent to $\bigcup_t G_t^{NH} = G_t^H$. This is because the input aggregates are the same, computed against the isoquant $L^C(1)$. Another way of looking at this is to notice that the union of the generator sets of T_t^{NH} is indeed a generator set for T^H. QED

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The following corollary emphasizes that if the neutral CRS group technologies T_t^{NC} are the same as the CRS group technologies T_t^C, then the neutral homothetic group technologies T_t^{NH} will coincide with the homothetic group technologies T_t^H (as one would expect).

Corollary 4.4. $T_t^{NH} = T_t^H$ *if and only if* $T_t^{NC} = T_t^C$.

Proof. This is trivial, since condition R3 ($T_t \subseteq T_t^{NC}$) becomes equivalent to condition R1 ($T_t \subseteq T_t^C$). If on the contrary one assumes $T_t^{NH} = T_t^H$, then it must be due to requirements R2 and R3 that $T_t^{NC} = C(T_t^{NH}) = C(T_t^H) = T_t^C$. QED

	Capital	Labor	Materials	Output
0%	8.89	79.57	19.57	0.16
25%	43.4	151.37	72.98	1.58
50%	71.7	200.27	122	2.75
75%	117.16	283.79	209.37	5.39
100%	500.45	1682.2	1523.78	41.74
Mean	95.21	251.66	184.07	4.51
Std	79.85	172.31	187.95	5.34

Table 1: Descriptive Statistics for 405 Farms in the years 1984, 1985, 1986.

5 Empirical Illustration

To illustrate the proposed method, the agricultural dataset collected by Ivaldi et al. (1996) on French farmers has been used. There is a single output and three inputs: capital, labour and materials. The dataset is an unbalanced panel with 130 farms in 1984, 135 farms in 1985 and 140 farms in 1986. This is a good illustration of the fact that the method here proposed does not rely on the assumption of the panel being balanced.

Table 1 reports descriptive statistics for the four different variables used. The dataset shows a large dispersion with the largest farm (in terms of output produced) about 250 times largest than the smallest one.

All calculations have been performed in R using the solver GLPK for all linear programs. I report the results for the pooled frontiers only, without reporting the results for the union of convex group frontiers. Table 1 reports the boxplots of six sets of input efficiency scores: the pooled FDH T^V (FDH); the pooled DEA Ψ^V_{pooled} (DEA); the pooled homothetic FDH T^H (HFDH); the pooled homothetic convex DEA Ψ^H_{pooled} (HDEA); the pooled FDH under CRS T^C (FDH-CRS); the pooled DEA under CRS Ψ^C_{pooled} (DEA-CRS). The first thing that jumps to the eyes is the gain in discrimination power obtained from homotheticity as opposed to convexity. In fact the distribution of the pooled DEA scores is well above the distribution of the HFDH. Thus homotheticity buys more discrimination power than convexity (at least for this empirical application). The second point to notice is the effect of the assumption of scalability once we remove the effect of homotheticity. Since the CRS technologies are also homothetic, if one were to compare the efficiency scores of, say, the pooled FDH to the scores of the pooled FDH-CRS, this would point to the overall effect of scalability and homotheticity. Therefore not all the change can be attributed to the scalability assumption only. The homothetic technologies allow one to disentangle the effect of scalability from the effect of homotheticity. In fact, if one looks

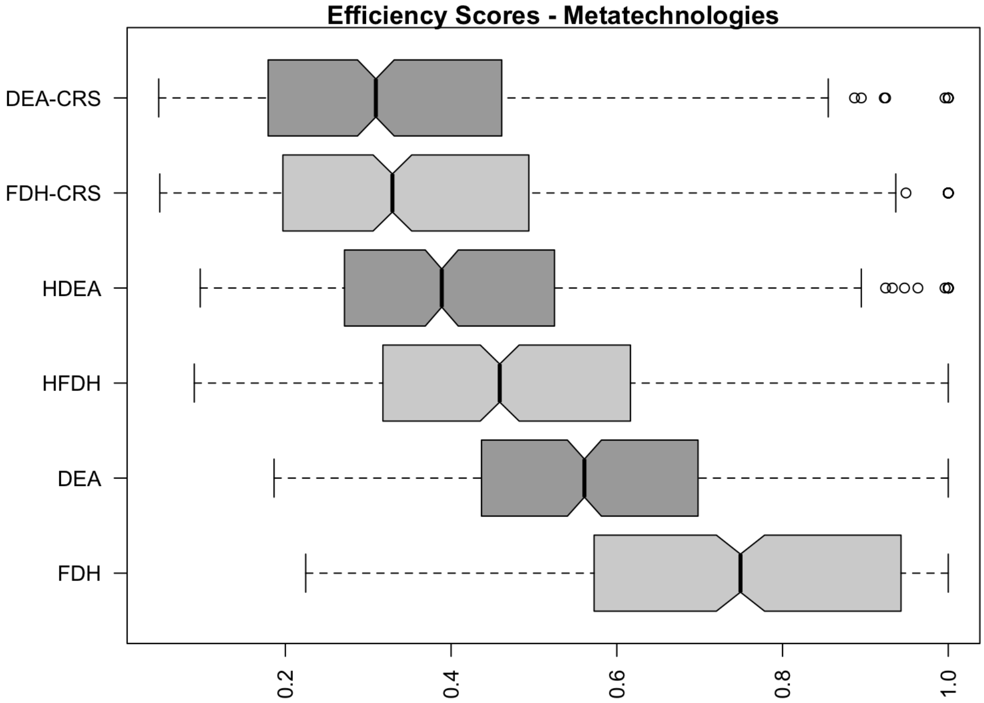

Figure 1: Efficiency Scores for the Metatechnologies.

at the distributions of the homothetic efficiency scores compared to the CRS scores and the VRS scores, it is quite evident that a large share of the shift in the distribution of the scores is given by the homotheticity assumption. These very same results have been found also in Peyrache (2024) using two alternative datasets.

Figure 2 reports the boxplots of a set of eight efficiency scores: the group FDH T_t^V (FDH); the group DEA Ψ_t^V (DEA); the group homothetic FDH T_t^H (HFDH); the group homothetic DEA Ψ_t^H (HDEA); the group neutral homothetic FDH T_t^{NH} (NH-FDH); the group neutral homothetic DEA Ψ_t^{NH} (NH-DEA); the group FDH under CRS (FDH-CRS); the group DEA under CRS (DEA-CRS). As for the previous boxplots it is clear that homotheticity increases discrimination power quite dramatically (especially for the non-convex technologies). Moreover the neutral homothetic technologies shifts the distribution to left even further, pointing out that the discrimination power is increased by the assumption of neutral homotheticity.

All in all, these results point to the fact that neutrality and homotheticity can play a big role in allowing to disentangle the pure effect of scale economies from the effects of homotheticity. Moreover, given the gains in discrimination power of the non-convex homothetic and neutral homothetic technologies, this also questions the widespread use of the convexity assumption. Notice for example that the homothetic FDH is superior to the standard DEA in terms of discrimination power. This fact, coupled with the faster computational time of the homothetic

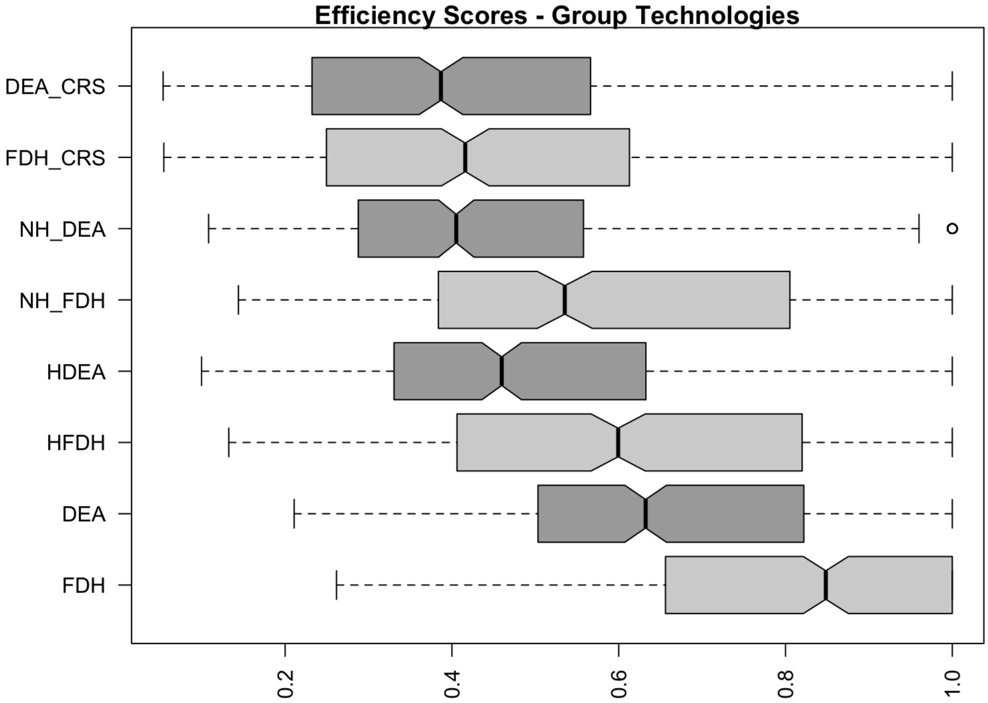

Figure 2: Efficiency Scores for the Group Technologies.

FDH points to this technology as a viable alternative to the use of the convexity assumption.

6 Conclusion

In this paper several new technology sets have been proposed under the assumptions of neutrality and homotheticity for a series of group technologies and their metafrontiers. The paper provides an axiomatic foundation for these technologies and discusses the relationships among them. An empirical application has been provided to show that the method can be easily implemented and has the same computational complexity as a standard efficiency model. The empirical application also points out to the usefulness of using homotheticity as a way of increasing the computational power of the model, without necessarily reverting back to using the convex model.

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