Centre for Efficiency and Productivity Analysis

Working Paper Series
No. WP01/2024

A Homothetic and Additively Separable Production Frontier

Antonio Peyrache

Date: May 2024

School of Economics
University of Queensland
St. Lucia, Qld. 4072
Australia

ISSN No. 1932 - 4398

A Homothetic and Additively Separable Production Frontier

Antonio Peyrache[*]

a.peyrache@uq.edu.au

Abstract

I propose a computationally tractable and simple way of building a technology set that is homothetic and complete additevely separable. This results in a technology which is nonparametric in the graph and has an input isoquant of a constant elasticity of substitution (CES) functional form (not necessarily convex). The method introduced in this paper preserves good discrimination power when the number of inputs is large (thus addressing the curse of dimensionality), while preserving full flexibility in the graph of the technology and the form of scale economies. A numerical simulation is presented to show the drastic improvement in discrimination power compared to other methods. Two empirical illustrations are provided to show the usefulness of the approach.

Key Words: Data Envelopment Analysis, Free Disposal Hull, Homotheticity, Additive Separability, Efficiency.

1 Introduction

Data generated technologies, such as data envelopment analysis (DEA) and the free disposal hull (FDH), suffer from low discrimination power when the number of inputs is large. This means that even with datasets of moderate size, when the number of inputs is large most observations will be classified as fully efficient even if they are not. This is especially the case with the FDH since it dispenses of the assumption of convexity. The result of this state of affairs is that the applied

[*]Centre for Efficiency and Productivity Analysis (CEPA), School of Economics, The University of Queensland, Australia.

researcher has been discouraged to use DEA or FDH in favour of more parametric alternatives, since, otherwise, the analysis would simply reveal that most observations are efficient. This clearly extends to other aspects of the analysis, such as the assessment of economies of scale and their impact on the structure of industries.

An attempt in the direction of mitigating this lack of discrimination power is represented by Peyrache (2022) who proposed to use the assumption of homotheticity in DEA. Although there is indeed an improvement in discrimination power when using homotheticity, this is sometimes overstated by Montecarlo simulations that only look at small dimensional problems (i.e. 1 output and two or three inputs). Although homotheticity is a useful functional restriction on the technology that reduces dimensionality, it is really insufficient in large dimensional problems (let's say more than 3 inputs).

In this paper I propose to include together with homotheticity, complete additive separability (as proposed and discussed in the seminal, although neglected by the modern researcher, contribution of Blackorby et al. (1978)). The joint of homotheticity and complete additive separability implies that the input isoquant is of the constant elasticity of substitution (CES) type. Notice that although the CES is used for the input isoquant, in the discussion that will follow this is derived in the framework of functional restrictions on the technology set. Neither homotheticity nor complete additive separability are sufficient alone to impose a parametric form on the input isoquant. This two functional restrictions will transform the model into a semiparametric production model. It is somehow overlooked that a parametric functional form can, in principle, always be derived via functional restrictions on a nonparametric function. If one approaches the problem from this perspective, then the discussion is shifted towards what is the most appropriate functional restriction to impose, rather than what is the most appropriate parametric functional form.

As shown in the numerical simulation, while discrimination power collapses exponentially with the number of dimensions for the homothetic model, it remains intact for the homothetic additevely separable model. It seems therefore that the two assumptions of homotheticity and complete additive separability are at least sufficient to guarantee good discrimination power in almost all situations. Moreover, the graph of the technology set is preserved of a nonparametric form, therefore allowing the researcher to test alternative assumptions on the scale economies of the technology. I should end this small introduction by clarifying that the method preserves the computational speed proper of the DEA and FDH models, therefore making it an effective method to address discrimination power in all datasets sizes.

Rather than relying on *ad hoc* procedures or disputable rules of thumbs, what is proposed here makes explicit what assumption is needed to address the problem of the lack of discrimination power. Moreover, assuming a parametric form for the input isoquant can prove particularly useful in order to simplify the interpretation of the results. Since the input isoquant cannot be visualized in more than 3 dimensions, it is useful to be able to compute quantities such as the marginal rate of substitution by using the CES function directly. In fact, given the strong discrimination power of the model, the graph of the technology will be estimated more precisely as well, therefore providing more accurate information on scale economies.

2 Definitions and Assumptions

Consider a production process where inputs $\mathbf{x} \in \mathbb{R}_+^P$ are used to produce a single output $y \in \mathbb{R}_+$ and let T denote the production possibilities set or technology (see O'Donnell (2018)):

$$T = \{(\mathbf{x}, y) \in \mathbb{R}_+^P \times \mathbb{R}_+ \mid \mathbf{x} \text{ can produce } y\}$$

For any $y \in \mathbb{R}_+$ the input requirement set $L(y)$ can be defined as the set of all input vectors which yield at least y:

$$L(y) = \left\{\mathbf{x} \in \mathbb{R}_+^P \mid (\mathbf{x}, y) \in T\right\}.$$

The input isoquant is then defined as:

$$IL(y) = \left\{\mathbf{x} \in \mathbb{R}_+^P \mid (\mathbf{x}, y) \in T, (\mathbf{x}/\lambda, y) \notin T, \lambda > 1\right\}$$

The free disposal conical extension of the production set T is defined as:

$$C(T) = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \geq \delta\overline{\mathbf{x}}, y \leq \delta\overline{y}, (\overline{\mathbf{x}}, \overline{\mathbf{y}}) \in T, \delta \geq 0\} \tag{1}$$

The technology satisfies CRS if and only if $T \equiv C(T)$. Note that even if T does not satisfy CRS, the conical extension $C(T)$ can be defined as an enlargement of T according to the previous formula[1]. It is worth stressing that the technology T is not necessarily convex, and this means that when taking its conical extension $C(T)$ the cone so obtained will be non-convex as well. The reader accustomed to think in terms of convex sets, can do so in what follows, keeping in mind that none of the results are confined to the convex case and they apply as well to the non-convex case.

2.1 Distance Functions, Efficiency and Homotheticity

Input and output distance functions measure the technical efficiency in the input and output orientation and are defined as[2]:

$$D_I(\mathbf{x}, y) = \sup\{\lambda > 0 \mid (\mathbf{x}/\lambda, y) \in T\} \tag{2}$$

and

$$D_O(\mathbf{x}, y) = \inf\{\theta > 0 \mid (\mathbf{x}, y/\theta) \in T\}. \tag{3}$$

[1] This will imply some restrictions on the production set T in order for $C(T)$ to be a strict subset of the positive orthant. Since, in the next section, the technology will be generated by a set of given data points, what is required is that there are no observations that produce strictly positive output using a zero input vector. This is also known as the no free lunch assumption: $\nexists k : \mathbf{x}_k = \mathbf{0}_P \, and \, y_k > 0$.

[2] The assumptions of this paper are sufficient for all the functional properties discussed in this section to hold. See O'Donnell (2018) for an exhaustive account of the properties.

The input distance function is linearly homogeneous in the input vector and the output distance function is linearly homogeneous in the output. Linear homogeneity of the output distance function implies $D_O(\mathbf{x}, y) = yD_O(\mathbf{x}, 1)$. By focusing on the boundary of the set (i.e. $(\mathbf{x}, y) : D_O(\mathbf{x}, y) = 1$) and defining $F(\mathbf{x}) = 1/D_O(\mathbf{x}, 1)$, then the production function representation of the technology frontier is obtained:

$$y = F(\mathbf{x}) = \max \{y \mid (\mathbf{x}, y) \in T\} \tag{4}$$

The technology satisfies input homotheticity (see Blackorby et al. (1978)) if the input sets satisfy $L(y) = L(1)H(y)$, with $H(y)$ a non-decreasing, lower semi-continuous function of its argument.

This definition implies the following functional separation of the input distance function:

$$D_I(\mathbf{x}, y) = \frac{D_I(\mathbf{x}, 1)}{H(y)} = \frac{X(\mathbf{x})}{H(y)} \tag{5}$$

where $X(\mathbf{x})$ is a linearly homogeneous, non-decreasing function in the input vector. While the production function $F(.)$ gives the maximum output producible with a given input vector, under input homotheticity the function $H(.)$ gives the minimal aggregate input vector that is able to produce a given level of output; therefore the function $H(.)$ can be interpreted as an aggregate input requirement function. By choosing as a reference the unit output $y = 1$, the input aggregates can be obtained as:

$$X(\mathbf{x}) = D_I(\mathbf{x}, 1) \tag{6}$$

where due to the separability of the input distance function, the choice of the reference output is innocuous, since any other reference output vector would give rise to the same input aggregates (up to a re-scaling factor, i.e. a change in the unit of measurement of the input aggregates). Input homotheticity can be equivalently stated as the following functional restriction on the production function:

$$y = F(\mathbf{x}) = f[X(\mathbf{x})] \tag{7}$$

where $X(\mathbf{x})$ is the input aggregator function and $f(.)$ is non-decreasing and upper semi-continuous (a proof of this can be found in Blackorby et al. (1978)).

3 Separability and Dimensionality Reduction

Input homotheticity reduces the dimensionality of the problem by separating the technology set into the construction of a base isoquant and the building of the graph of the technology. The graph of the technology is a 2-dimensional problem involving the output and the input aggregate. Therefore the dimensionality of the overall problem is given by the number of inputs, since this will determine the dimensionality of the input aggregator function $X(\mathbf{x})$. Therefore, homotheticity reduces the dimensionality of the problem from $P + 1$ to P. This results to be very effective with a small number of inputs. In applications with a large number of inputs (say more than 3), unless additional assumptions are made about the input aggregator function, the benefits of homotheticity will be limited. The dimensionality of the problem is still equal to the dimensionality of the input aggregator function and the discrimination power of the homothetic model will likely deteriorate exponentially with the number of dimensions (in the Montecarlo simulation reported below this is very clear). Thus, although homotheticity is a step forward in reducing the dimensionality of the problem, some additional assumptions are needed in applications with a large number of inputs.

I start the discussion here by presenting what is known in economics as the two-level budgeting model (see Blackorby et al. (1978)). In the two level budgeting model, there exists a partition of the input variables into G groups $\mathbf{x} = [\mathbf{x}_1, \ldots, \mathbf{x}_g, \ldots, \mathbf{x}_G]$ of dimensionality $p_1 + \ldots + p_g + \ldots + p_G =$

P such that:

$$y = F(\mathbf{x}) = f\left[X_1(\mathbf{x}_1), \ldots, X_G(\mathbf{x}_G)\right] \tag{8}$$

It should be noted that assumptions on the aggregator functions $X_g(\mathbf{x}_g)$ and on the mother function $f(.)$ will determine the form of the functional restriction imposed on the technology. One obvious possibility of making the previous grouping correspond to a restriction on the technology is to assume that the aggregator functions are linearly homogeneous in their respective arguments. This means that the groups are homothetically separable, a property that I shall call group homotheticity. Notice that homotheticity of the groups does not imply homotheticity of the mother function and vice-versa. In fact, assuming homotheticity of the mother function $f(.)$, one obtains:

$$y = f\left\{I\left[X_1(\mathbf{x}_1), \ldots, X_G(\mathbf{x}_G)\right]\right\} \tag{9}$$

In this model the various groups of inputs and outputs are aggregated into sub-vector aggregators and these sub-vector aggregates are then aggregated into an overall input aggregate $I(.)$. The advantage of such a model is that the dimensionality of the computational problem is given by the function with the highest number of dimensions. This means that by choosing the groups carefully, once is able to reduce dimensionality substantially, as the next integer to the square root of P. For example, with 9 inputs one may form 3 groups of 3 inputs each, thus reducing the dimensionality from 9 to 3. Notice however that there are limits to this approach, either because of the large dimension of P or, even more so, because of computational issues associated with the estimation of the various aggregates (this is an aspect of the problem that is usually neglected, although numerical stability of the results is an important feature in applications). Clearly, such an approach could be extended by looking into more than two levels. For example, with three levels, one can have that a given sub-aggregate is disaggregated into 3 additional groups. Although theoretically interesting, it is not clear if such an approach could lead to numerically stable results, but for sure, at least theoretically, the dimensionality is reduced to the ceiling of the cubic root of P. For example in a model with 27 inputs, the three levels budgeting model reduces it to aggregator functions of dimension 3. The end result of this discussion is that increasing the number of levels, although conducive to a reduction in dimensionality, may be problematic in computational terms. To the best of my knowledge there are no significant attempts at computing this model directly. Notice that in principle, if a computational solution is found that is numerically stable, the three level model would accommodate most practical situations. In fact, as reported in Peyrache et al. (2020), with a dataset of 250 observations, the dimensionality of the problem should not exceed 3 (which is achieved in the three-level budgeting model with 27 inputs!).

Given the computational complexities that are likely to arise in the previous discussion, I now turn the attention to imposing alternative restrictions on the mother function $f(.)$ and in particular assume additive separability of the groups:

$$y = f\left[X_1(\mathbf{x}_1) + \ldots + X_G(\mathbf{x}_G)\right] \tag{10}$$

In the definition of additive separability, the group aggregator functions $X(\mathbf{x}_g)$ are not assumed to be linearly homogeneous in the group input vector, therefore additive separability is different from homothetic separability. Additivity reduces the dimensionality of the problem to the computation of the group functions, since the input aggregate is then obtained as the sum of those functions. Notice that in such an approach, irrespective of the number of inputs one can always define those groups such that they do not exceed a given dimension. In fact, at one extreme it is found that one can make each group correspond to a particular input, with the dimensionality of the group being equal to one. This case is known as the complete additive separable model and it is defined as:

$$y = f\left[\sum_p X_p(x_p),\right] \tag{11}$$

From a dimensionality perspective this model is extremely effective since it reduces the problem to look at one dimensional aggregator functions, with perfect substitution between each pair of aggregated inputs. This model is known in statistics as the generalized additive model. Again, to the best of my knowledge, no attempt has been done in computing the generalized model in a production context.

At this point one quite natural question to ask is what is the relationship between additive separability and homothetic separability. The most known result on this (see Blackorby et al. (1978) for a proof) is that the joint of complete additive separability and homotheticity implies that the aggregator function is a CES. If we restrict the attention to a given group of inputs g, then if inputs in this group are jointly homothetically separable and complete additively separable from the other inputs, then the aggregator function has the known form of being a CES function:

$$X_g(\mathbf{x}_g) = \left(\sum_p \lambda_{pg} x_p^{\rho_g}\right)^{1/\rho_g} \tag{12}$$

In other words, CES is the most general functional form that can satisfy additive separability and homotheticity jointly. This has a number of consequences for the choice of the aggregators. If one starts with the two-level group homothetic model and imposes complete additive separability within each group, then the resulting group aggregator functions will be CES. On the contrary, one can start with linearly homogeneous (homothetic) aggregator functions and assume input homotheticity, in which case the functional restriction results to be:

$$y = F\left[I\left(X_1(\mathbf{x}_1) + \ldots + X_G(\mathbf{x}_G)\right)\right] \tag{13}$$

We notice that since at the upper level one is assuming complete additive separability and homotheticity jointly, the aggregate of the sub-aggregates is a CES function, i.e.:

$$I\left(X_1 + \ldots + X_G\right) = \left(\sum_g \lambda_g X_g^{\rho}\right)^{1/\rho} \tag{14}$$

In such a specification the dimensionality is reduced to the highest dimension of the group aggregators and with appropriate choice this will be very effective, since the mother function $I(.)$ is parametric, i.e. increasing the number of groups will increase the number of parameters, not the dimensionality of the problem. Additionally, one may choose to specify complete additive separability for the group aggregates in which case:

$$X_G = \left(\sum_i \lambda_{ig} X_{ig}^{\rho_g}\right)^{1/\rho_g} \tag{15}$$

which gives rise to the nested-CES (NCES) function. The NCES function is here obtained as the result of a functional separability assumption on the production function. Notice however that attempting to estimate the NCES function directly is difficult since the functional form is non-linear and any non-linear optimization will likely be numerically unstable and not guaranteed to find a global optimum for the parameter values. Nevertheless, the NCES function corresponds to a two-level budgeting model that satisfies both homotheticity and complete additive separability at both levels of aggregation. In this sense the structure of the problem is reduced to the estimation of a parametric functional form for the input aggregator function and a two-dimensional nonparametric problem in the graph of the technology. This means that the dimensionality of the problem is in fact reduced to a dimension 2 problem.

The functional restrictions presented so far impose alternative ways of reducing the dimensionality of the problem, most of them are of academic interest only, since numerically stable and reliable procedures for their computation are not available. I therefore focus on an even more narrow case, where I should consider an input aggregator function that satisfies input homotheticity and complete additive separability with respect to all inputs, and for this case I am able to obtain some positive computational results. This will return a linearly homogeneous CES function for the input aggregator function:

$$X(\mathbf{x}) = \left(\sum_p \alpha_p x_p^\rho \right)^{\frac{1}{\rho}} \tag{16}$$

Notice how the dimensionality of the problem is here reduced drastically, since there are $P+1$ parameters that define the aggregator function, therefore increasing the dimensionality of the problem will increase complexity parametrically by adding additional parameters. This last option, although the most restrictive, is the one for which it is possible to find a simple and fast computational method (which I will describe in the next section). To summarize this section, the computationally tractable model that I choose to describe will be:

$$\begin{aligned} y &= F(\mathbf{x}) = f(X(\mathbf{x})) \\ X(\mathbf{x}) &= \left(\textstyle\sum_p \alpha_p x_p^\rho \right)^{\frac{1}{\rho}} \end{aligned} \tag{17}$$

Although all of the previous options are suitable ways of reducing dimensionality, imposing a CES functional form is the most effective and simple way of reducing dimensionality.

4 Data Generated Technologies

In order to give empirical meaning to the previous definitions consider the production possibilities set generated by a set of data points or decision making units (DMUs) $(\mathbf{x}_k, y_k)$ $(\forall k = 1, \ldots, K)$, where the input vectors are column vectors. The free disposal hull (FDH) is defined as:

$$T^V = \left\{ (\mathbf{x}, y) : \sum_{k=1}^{K} \lambda_k \mathbf{x}_k \leq \mathbf{x}, \sum_{k=1}^{K} \lambda_k y_k \geq y, \sum_{k=1}^{K} \lambda_k = 1, \lambda_k \in \{0, 1\} \right\} \tag{18}$$

Using the conical extension operator defined in equation (1) one can introduce the following

CRS non-convex technology (FDH-CRS):

$$T^C = \left\{ (\delta \mathbf{x}, \delta y) : \sum_{k=1}^{K} \lambda_k \mathbf{x}_k \leq \mathbf{x}, \sum_{k=1}^{K} \lambda_k y_k \geq y, \sum_{k=1}^{K} \lambda_k = 1, \lambda_k \in \{0,1\}, \delta \geq 0 \right\} \tag{19}$$

It should be noted that the conical extension of the FDH set is a non-convex cone. This non-convex cone technology has been studied in Kerstens and Eeckaut (1999), Podinovski (2004). Briec and Kerstens (2006) show that computation of distance functions using a non-convex cone technology can be accomplished by using an enumeration algorithm, as opposed to a linear program (LP) for the convex technology. This makes computation over large datasets for the non-convex case feasible and fast; in fact, orders of magnitude faster than in the convex case.

The convex VRS and CRS technologies are equivalent to the convex closure of the two sets introduced above. Therefore the convex VRS technology (known as the BCC technology) is equal to $T^V_{Conv} = Conv(T^V)$ and the CRS technology (also known as the CCR technology) is equal to $T^C_{Conv} = Conv(T^C)$[3]. In what follows, readers accustomed at thinking in terms of convex technologies can make this assumption without loss of generality. Readers not prone to think in terms of convex sets can dispense of this assumption with no harm.

4.1 Computing the CES Aggregates

Peyrache (2022) proposed a method to build a homothetic technology using the CRS isoquant as a base isoquant. If the input isoquant is of the CES form and the technology is CRS, then the

[3]The explicit definition of the VRS production technology under convexity is:

$$T^V_{conv} = \left\{ (\mathbf{x}, \mathbf{y}) : \sum_{k=1}^{K} \lambda_k \mathbf{x}_k \leq \mathbf{x}, \sum_{k=1}^{K} \lambda_k \mathbf{y}_k \geq \mathbf{y}, \sum_{k=1}^{K} \lambda_k = 1, \lambda_k \geq 0, \forall k \right\}$$

The CRS conical extension of this set is:

$$T^C_{conv} = \left\{ (\mathbf{x}, \mathbf{y}) : \sum_{k=1}^{K} \lambda_k \mathbf{x}_k \leq \mathbf{x}, \sum_{k=1}^{K} \gamma_k \mathbf{y}_k \geq \mathbf{y}, \lambda_k \geq 0, \forall k \right\}$$

Notice how the CRS technology under convexity can be obtained either as a conical extension $T^C_{conv} = C(T^V_{conv})$ or as the convex closure of the FDH-CRS technology $T^C_{conv} = Conv(T^C)$. Computation of distance functions with these technologies implies solving linear programs and it is computationally slower than the non-convex alternative.

production function is linearly homogeneous in inputs, returning the following form[4]:

$$y_k = \left(\sum_p \alpha_p x_{kp}^{\rho} \right)^{\frac{1}{\rho}} - u_k \tag{20}$$

To compute the parameters of this function, I adapt the parametric linear programming approach introduced by Aigner and Chu (1968), embedded in the following non-linear program:

$$\begin{aligned} \min_{\alpha_p,\rho} \quad & \sum_{k=1}^{K} u_k \\ st \quad & u_k \geq 0, \ \forall k = 1, \dots, K \end{aligned} \tag{21}$$

This program is trying to select the parameters of the input isoquant so to minimize the aggregate output loss. By substitution, the previous program can be written as:

$$\begin{aligned} \min_{\alpha_p,\rho} \quad & \sum_{k=1}^{K} \left(\sum_{p=1}^{P} \alpha_p x_{kp}^{\rho} \right)^{1/\rho} - \sum_k y_k \\ st \quad & \left(\sum_{p=1}^{P} \alpha_p x_{kp}^{\rho} \right)^{1/\rho} - y_k \geq 0, \ \forall k = 1, \dots, K \end{aligned} \tag{22}$$

In this program the objective function can be simplified (since $\sum_k y_k$ is a known constant, it

[4]An alternative objective function would look at the maximization of the sum of efficiency scores, by expressing efficiency in multiplicative terms as:

$$y_k = \theta_k \left(\sum_p \alpha_p x_{kp}^{\rho} \right)^{\frac{1}{\rho}}$$

This would return the following program:

$$\begin{aligned} \max_{\alpha_p,\rho} \quad & \sum_{k=1}^{K} \theta_k \\ st \quad & \theta_k \leq 1, \ \forall k = 1, \dots, K \end{aligned}$$

This program can be linearized along the same lines. The interpretation of the objective function is not clear in this case. Thus, I prefer to minimize the aggregate output loss.

can be eliminated). The constraint can be simplified as well, returning the following program:

$$\begin{aligned} \min_{\alpha_p,\rho} \quad & \sum_{k=1}^{K}\left(\sum_{p=1}^{P}\alpha_p x_{kp}^{\rho}\right)^{1/\rho} \\ st \quad & \left(\sum_{p=1}^{P}\alpha_p a_{kp}^{\rho}\right)^{1/\rho} \geq 1,\ \forall k=1,\ldots,K \end{aligned} \tag{23}$$

where $a_{kp} = x_{kp}/y_k$. This program is non-linear in the parameters, but it should be noted that the non-linearity is arising only because of the elasticity parameter ρ. If this parameter was known, then it would be possible to convert it to a linear program. One needs to distinguish two cases depending on the sign of ρ. In fact, for $\rho > 0$, notice that minimizing the objective is equivalent to minimizing $\sum_{k=1}^{K}\sum_{p=1}^{P}\alpha_p x_{kp}^{\rho}$, since the power function is a monotonic transform and the optimal solution in terms of the α_p would not change[5]. Similarly, for a given value of ρ, the K constraints can be linearized. Therefore for a known $\rho > 0$ the solution in terms of the decision variables will be obtained as the solution to the following linear program:

$$\begin{aligned} \min_{\alpha_p} \quad & \sum_{k=1}^{K}\sum_{p=1}^{P}\alpha_p x_{kp}^{\rho} \\ st \quad & \sum_{p=1}^{P}\alpha_p a_{kp}^{\rho} \geq 1,\ \forall k=1,\ldots,K \end{aligned} \tag{24}$$

On the contrary when $\rho < 0$, minimizing the objective function would be equivalent to maximizing $\sum_{k=1}^{K}\sum_{p=1}^{P}\alpha_p x_{kp}^{\rho}$ and the constraint can still be linearized but would have a different

[5]For $\rho > 0$ and any value of α_{p1}, α_{p2} such that $\sum_{k=1}^{K}\left(\sum_{p=1}^{P}\alpha_{p1} x_{kp}^{\rho}\right)^{1/\rho} \geq \sum_{k=1}^{K}\left(\sum_{p=1}^{P}\alpha_{p2} x_{kp}^{\rho}\right)^{1/\rho}$, then $\sum_{k=1}^{K}\sum_{p=1}^{P}\alpha_{p1} x_{kp}^{\rho} \geq \sum_{k=1}^{K}\sum_{p=1}^{P}\alpha_{p2} x_{kp}^{\rho}$.

inequality sign[6]. For $\rho < 0$ the program becomes:

$$\begin{array}{ll} \max\limits_{\alpha_p,\rho} & \sum\limits_{k=1}^{K}\sum\limits_{p=1}^{P} \alpha_p x_{kp}^{\rho} \\ st & \sum_{p=1}^{P} \alpha_p a_{kp}^{\rho} \leq 1, \ \forall k = 1,\ldots,K \end{array} \quad (25)$$

To summarize, the solution to the non-linear program can be obtained by a transformation of the data and the solution of a linear program. Since in general ρ is unknown, it can be selected in order to optimize the objective function. This can be accomplished via a grid search (or a golden line search or a random search) over the single parameter ρ by solving the linear program over several values of ρ and choosing it optimally.

Given the values of the CES parameters, the input aggregates can be easily computed by applying the CES function to the observed input vectors:

$$X_k = \left(\sum_{p=1}^{P} \alpha_p x_{kp}^{\rho}\right)^{1/\rho} \quad (26)$$

One can then proceed by computing efficiency scores in the graph of the technology directly by looking at the two-dimensional dataset (X_k, y_k).

5 Montecarlo Evidence

In tables 3, 4, 5, 6, 7, 8, 9, 10, I report the results of a Montecarlo simulation. The datasets were generated for sample sizes of $K = \{25, 50, 100, 250, 500, 1000, 2500, 5000\}$ and the number of inputs varied from $P = 2$ to $P = 20$. The inputs were generated in the following way. The P inputs were generated uniformly on the unit isoquant $\sum_p x_p = 1$ (which implicitly assumes perfect substitutes with $\rho = 1$ and $\alpha_p = 1$, $\forall p$). To do so, I generate random values from the multivariate

[6] For $\rho < 0$ and any value of α_{p1}, α_{p2} such that $\sum\limits_{k=1}^{K}\left(\sum\limits_{p=1}^{P} \alpha_{p1} x_{kp}^{\rho}\right)^{1/\rho} \geq \sum\limits_{k=1}^{K}\left(\sum\limits_{p=1}^{P} \alpha_{p2} x_{kp}^{\rho}\right)^{1/\rho}$, then $\sum\limits_{k=1}^{K}\sum\limits_{p=1}^{P} \alpha_{p1} x_{kp}^{\rho} \leq \sum\limits_{k=1}^{K}\sum\limits_{p=1}^{P} \alpha_{p2} x_{kp}^{\rho}$.

standard normal. For each random draw k, I standardize it by the value $r_k = \sum_p |z_{pk}|$. Then taking the absolute values of these draws $|\mathbf{z}_k/r_k|$ will return random draws uniformly distributed on the unit surface of a polyhedron (the unit isoquant $\sum_p x_p$). I then generate the aggregate output X_k from a beta distribution with both parameters equal to 2. Taking $x_k = |\mathbf{z}_k/r_k|/X_k$ will place the different input mixes at different inputs level. This will provide good coverage of the input mixes, but also insure that the aggregate output is distributed between zero and one. Notice that one could generate points uniformly directly inside the tetrahedron. The shortcoming of this alternative method is that the sum of the coordinates (the aggregate output) will not be distributed between zero and one (since the distribution would follow a Irwin-Hall distribution with probability in the left tail close to zero). I then generate the frontier output according to the following non-convex piecewise production function:

$$q = F(\mathbf{X}) = \begin{cases} \frac{1}{2}X & 0 \leq X < 0.25 \\ \frac{5}{2}X - \frac{1}{2} & 0.25 \leq X < 0.5 \\ \frac{1}{2}X + \frac{1}{2} & 0.5 \leq X \leq 1 \end{cases}$$

The output so generated is then perturbed by an efficiency score. The efficiency score is generated according to the following distribution:

$$\begin{cases} 1 & p = 0.1 \\ U(0.1, 1) & 1 - p = 0.9 \end{cases}$$

This means there is a 10% chance of obtaining a frontier point and a 90% chance of obtaining a random number distributed as a uniform between 0.1 and 1. This data generating process implies that the production technology is input homothetic and the input isoquant of the CES form with linear input isoquants. Notice also that the graph of the technology is non-convex.

In the simulation study I compare the performance of three different sets of efficiency scores. First, I compute the input aggregates using the CES function as described in this paper. Second, I compute the input aggregates using the Homothetic FDH model as described in Peyrache (2022). Last, I compute the efficiency scores directly with the FDH, ignoring the structure of the problem.

Two measures of performance are used: the proportion of incorrectly classified observations and the mean square error (MSE). The MSE is quite a standard measure. The proportion of incorrectly classified observations is taking the number of observations that have an efficiency score lower than one but are classified by the model as fully efficient:

$$\frac{\#(\theta_k < 1, \hat{\theta}_k = 1)}{\#(\theta_k < 1)}$$

This measure belong to the unit interval, with 0 meaning no mistakes are made and 1 meaning that the model fails classification completely and gives all efficiency scores equal to one. This second measure is in fact possibly the most relevant metric since the first test that an efficiency model should satisfy in terms of discrimination power is to demarcate between efficient and non efficient units.

A glance at the tables reported in the appendix will reveal that the CES model is in fact robust to increases in dimensionality, but also it will improve performance more sharply with an increase in the number of observations. The homothetic model, although useful in low dimensionality (say up to dimension 3 or 4), deteriorates very quickly in performance for larger dimensional problems. The tables also points to the problem with the FDH discrimination power, which possibly discouraged its use in the past, due to lack of discrimination power when already using 2 inputs.

In figure 1 I report, for a dataset of 250 observations, the deterioration in performance of the three different sets of efficiency scores. In the figure, on the x-axis the number of inputs are reported and on the y-axis the proportion of incorrectly classified observations, i.e. observation that are inefficient but are incorrectly classified as efficient. This is the most direct and crude measure of discrimination power that one can use, since it measure the basic task of demarcating between efficient and inefficient observations. This measure is also easy to interpret since it varies between zero and one. As it can be seen from this graph, in the case of a single input all three models classify correctly more than 90% of observations. With two inputs the FDH is already incorrectly classifying about 40%, while the homothetic FDH shows about the same performance as the CES method. It is with 3 dimensions and above that the homothetic FDH performance deteriorates quite fast and with 5 dimension the proportion of incorrectly classified observation is already close to 40%, while the CES model mostly retains its discrimination power. Although the homothetic FDH is a step forward compared to the FDH, it is still deteriorating quite fast in high dimension.

Another way of looking at this same phenomena is to look at improvement when the number of observations increases from 25 to 5,000. In figure 2 and 3 I report this improvement for the case of 5 and 10 inputs respectively. From these two figures it is very clear that the CES model falls below the 10% threshold very quickly and in fact close to zero (exponentially). On the contrary for both the FDH and the homothetic FDH, although there is an improvement at the beginning (mostly due to a poor performance in small samples), they very soon flatten out and any increase in performance relies on exponential growth in the sample size.

The bottom line of this analysis is that the homothetic FDH is certainly a significant improvement on the FDH and it performs very well in dimensions up to, say, 3 inputs. After that the homothetic FDH, while still maintaining a large advantage compared to the FDH, cannot match the performance of the CES. This points to the fact that homotheticity alone is insufficient to deal with the curse of dimensionality.

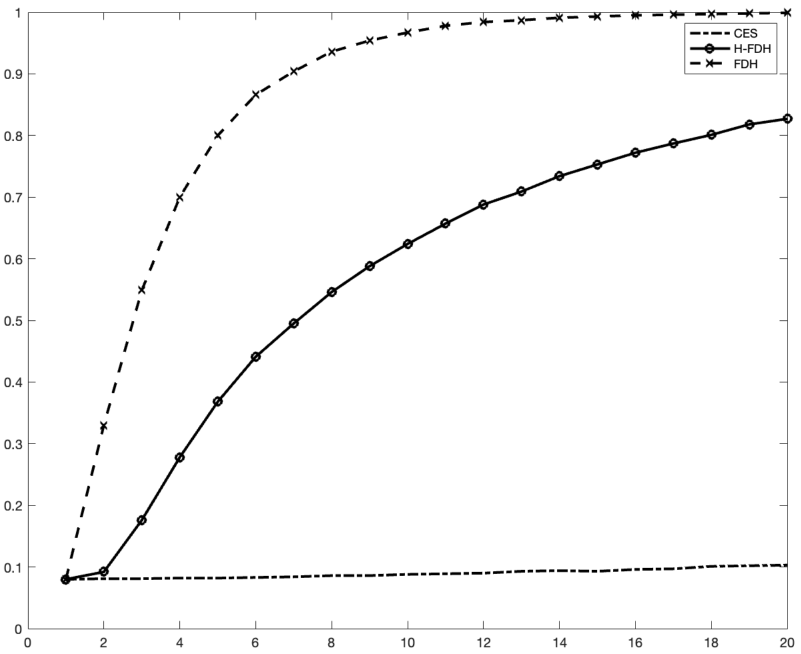

Figure 1: Proportion of incorrectly classified observations for a dataset with 250 observations and different number of inputs.

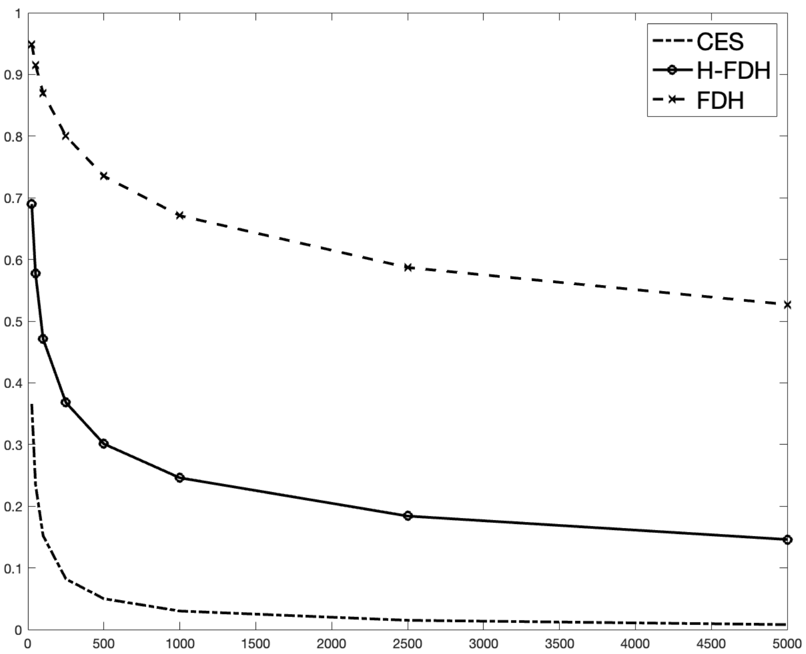

Figure 2: Proportion of incorrectly classified observations for a dataset with 5 inputs and different number of observations.

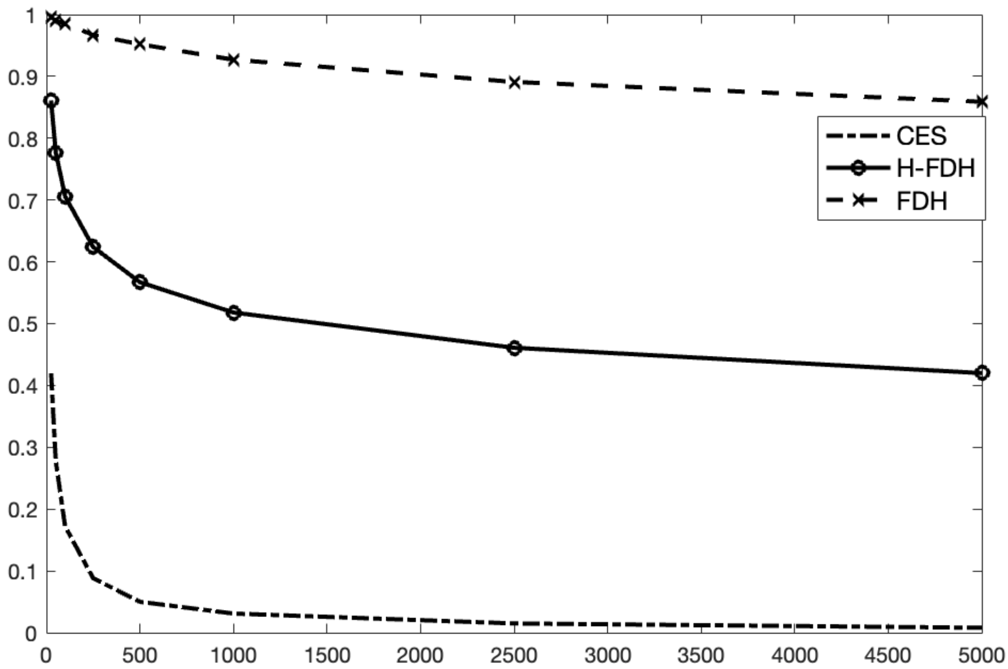

Figure 3: Proportion of incorrectly classified observations for a dataset with 10 inputs and different number of observations.

6 Empirical Illustration

In this section two real data examples are used to illustrate the practical use of the separability assumptions. The first one considers courts of justice in Italy and the data were collected by Professor Angelo Zago (see Peyrache and Zago (2016)). For simplicity, only the data in the year 2003 are considered. Input in the production process is the total number of employees of the court of justice. Two outputs are considered in order to differentiate between civil and criminal cases. Therefore a model with 1 input and 2 outputs is considered. Descriptive statistics are reported in Table 1.

In Figure 4 the unit output isoquant is plotted. Since the output isoquants are all parallel under output homotheticity, the shape of this isoquant is representative of the shape of the isoquant at all points of the technology set. In the figure the homothetic FDH isoquant is represented in blue and the CES isoquant in red. Interestingly enough the CES isoquant produces

	Civil Cases	Criminal Cases	Number of Employees
Mean	15,507	7,538	140
Std Dev	24,962	10,069	200
Min	1,392	363	22
Max	216,210	76,046	1651
Total	2,558,690	1,243,804	23,071

Table 1: Descriptive Statistics for 165 courts of justice in the year 2003.

non-convex output sets and by construction it contains the homothetic FDH isoquant. The substitution possibilities are large and a reduction of one hundred civil cases would correspond to an increase of about 150 criminal cases. The value of the substitution parameter is $\rho = 0.58$ and the output coefficients are respectively 0.0365 and 0.0222.

In Figure 5 a boxplot of seven sets of efficiency scores is reported. These efficiency scores are, respectively: the FDH; the homothetic FDH; the CES-FDH; the FDH under CRS; the DEA under VRS; the DEA under CRS; and the CES-FDH under CRS. The various distributions have been plotted from the one with the least (FDH) discrimination power to the one with the highest discrimination power (CES-CRS). It is interesting to note that the CES-FDH is returning almost the same distribution as the homothetic FDH. This should not surprise given the discussion about the Montecarlo results, since the homothetic FDH has high discrimination power when the number of dimensions is low (in this case one input and two outputs). It is important to point out that imposing CRS on the homothetic FDH or the CES-FDH will return quite a different set of efficiency scores. This difference is quite marked for the CES model, since the CES-CRS will contain by definition the FDH-CRS set. On the contrary if using a convex model these differences are lost: one can easily see from the boxplot of the DEA-VRS and DEA-CRS that there are not very significant differences in terms of the distribution of the efficiency scores.

Table 2 reports descriptive statistics for the Philippines rice farmers dataset (see Coelli et al. (2005) for a complete description of the dataset). Here rice (the single output) is produced using four inputs: land, labour, capital and fertilizers, and other inputs. Figure 6 reports the

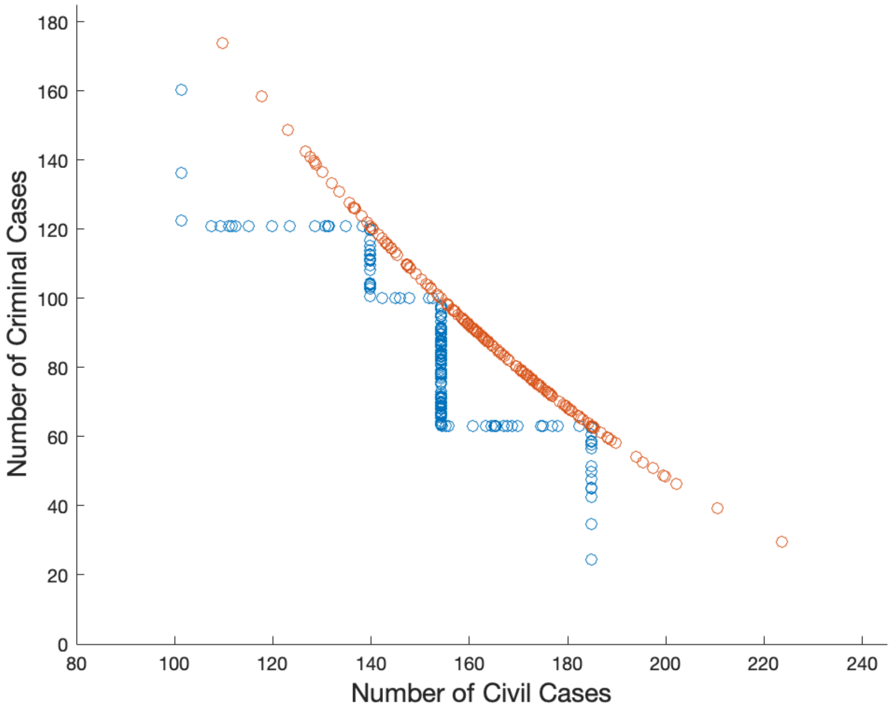

Figure 4: Output substitution possibilities for the Italian Courts of Justice (CES and Homothetic FDH isoquants).

	Rice Production	Land	Labour	Capital and Fertilizers	Other Inputs
Mean	6.54	2.14	108.34	189.23	125.34
Std Dev	5.11	1.46	77.19	169.80	158.24
Min	0.09	0.20	8.00	10.00	1.46
Max	31.1	7.00	437.00	1,030.90	1,083.40
Total	2,249.90	737.37	3727.00	6,509.70	4,311.90

Table 2: Descriptive Statistics for 344 Rice farmers in the Philippines.

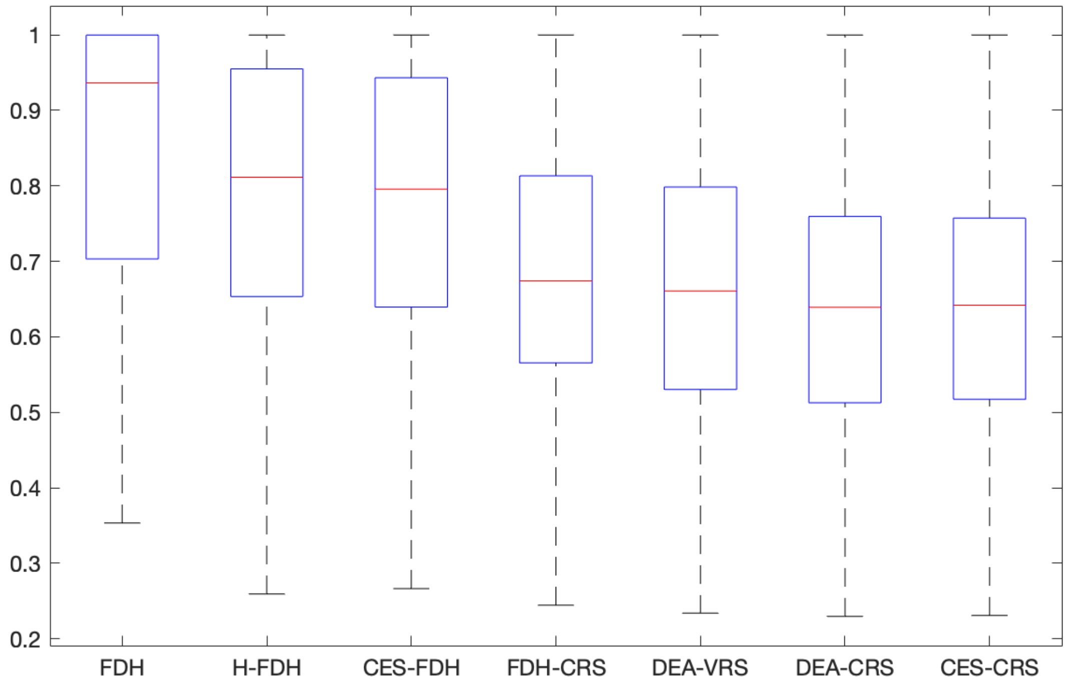

Figure 5: Output efficiency scores for the Italian Courts of Justice.

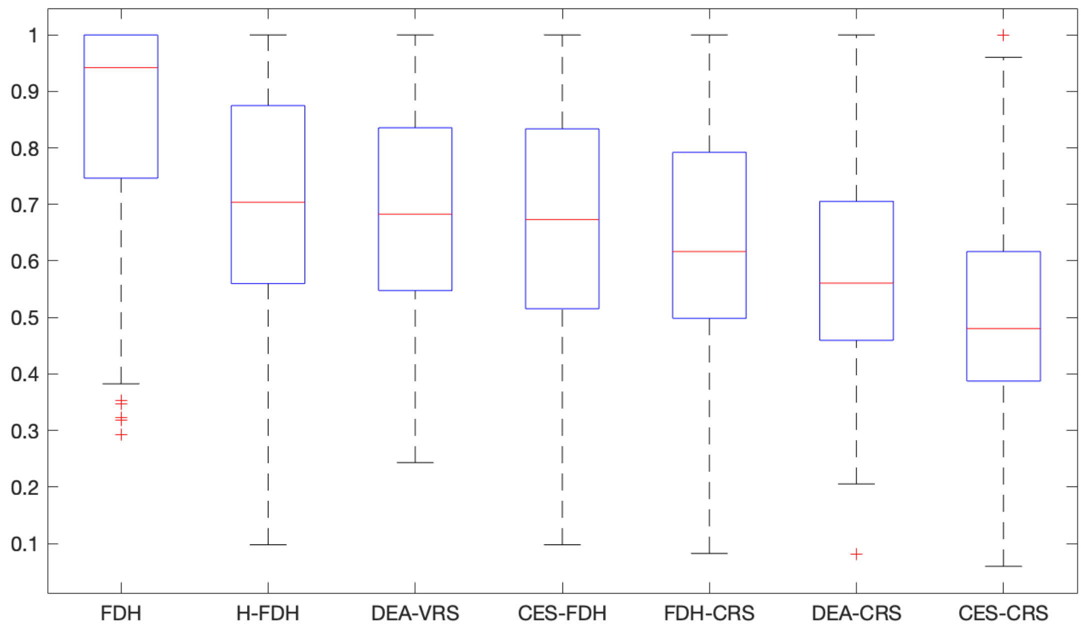

Figure 6: Output efficiency scores for the Philippines Rice Farmers.

boxplot of the same set of efficiency scores computed for the courts of Justice example and ordered in terms of discrimination power. Notice how the picture that emerges here is quite different. First of all, given that this dataset has four inputs the CES has more discrimination power than the homothetic FDH. Interestingly, the CES-FDH has more discrimination power than the DEA-VRS. Since the only difference between the FDH and the DEA-VRS is given by the convexity assumption, this means that the big shift in the efficiency scores is entirely due to convexity. Considering that the same outcome in terms of discrimination power can be obtained via separability conditions, one is induced to question if the convexity assumption is warranted in this case. Finally, notice how the CES model under CRS returns the highest discrimination power among all models: this is not necessarily the case, since the CES-FDH is not based on an assumption of convexity, in general. In this particular instance, though, the value of the elasticity parameter is $\rho = 0.073$, which means that the elasticity of substitution is close to unity and the shape of the input isoquant (in the four dimensional space) will resemble the Cobb-Douglas form. Since under CRS the only non-convexity will arise from the shape of the isoquant for the CES model, in this particular instance the CES under CRS will be an enlargement of the DEA-CRS, as it is clear from the boxplot of efficiency scores. Notice how convexity plays a much minor role in this empirical illustration and the major differences in the distribution of the efficiency scores come from the assumption of CRS (i.e. scalability of the production process).

All in all, the CES model provides a very nice tool to explore the efficiency of production and it is also providing some ease of interpretation, since its shape depends on a limited number of parameters.

7 Conclusion

In this paper I used the properties of homotheticity and complete additive separability to build a production technology that is nonparametric in the graph and has level sets of the CES form. In the paper several assumptions were discussed and the most interesting one is to tackle computation of the same model under the assumption that the input isoquants are of the nested CES form instead of the simple CES. This would allow more flexibility since the elasticity of substitution would vary from one group to the next. The cost of this additional flexibility is an increase in the number of parameters. And of course the additional burden of finding a suitable numerical implementation, which in this case would not be as simple. This may provide material for future research.

One question that stays unanswered is if it is possible to build a technology that satisfies complete additive separability without satisfying homotheticity, in which case the shape of the isoquant would not be a CES or of any parametric form.

Finally, all the previous computational methods have been delivered in the case of a single output technology. This can be easily extended to the case of a single input multiple output technology. Extension to multiple output multiple input technologies is definitely an important method that needs to be researched.

References

Aigner, D. J. and Chu, S.-f. (1968). On estimating the industry production function. *The American Economic Review*, 58(4):826–839.

Blackorby, C., Primont, D., and Russell, R. (1978). *Duality, Separability, and Functional Structure: Theory and Economic Applications.* North-Holland.

Briec, W. and Kerstens, K. (2006). Input, output and graph technical efficiency measures on non-convex fdh models with various scaling laws: An integrated approach based upon implicit enumeration algorithms. *Top*, 14(1):135–166.

Coelli, T. J., Rao, D. S. P., O'Donnell, C. J., and Battese, G. E. (2005). *An introduction to efficiency and productivity analysis*. springer science & business media.

Kerstens, K. and Eeckaut, P. V. (1999). Estimating returns to scale using non-parametric deterministic technologies: A new method based on goodness-of-fit. *European Journal of Operational Research*, 113(1):206–214.

O'Donnell, C. J. (2018). *Productivity and efficiency analysis*. Springer.

Peyrache, A. (2022). A homothetic data generated technology. *CEPA Working Paper*.

Peyrache, A., Rose, C., and Sicilia, G. (2020). Variable selection in data envelopment analysis. *European Journal of Operational Research*, 282(2):644–659.

Peyrache, A. and Zago, A. (2016). Large courts, small justice!: The inefficiency and the optimal structure of the italian justice sector. *Omega*, 64:42–56.

Podinovski, V. (2004). On the linearisation of reference technologies for testing returns to scale in fdh models. *European Journal of Operational Research*, 152(3):800–802.

8 Appendix: Tables from the Montecarlo

P \ K	25	50	100	250	500	1000	2500	5000
2	0.315	0.217	0.150	0.083	0.051	0.031	0.015	0.009
3	0.311	0.221	0.148	0.083	0.051	0.031	0.015	0.009
4	0.331	0.224	0.146	0.084	0.053	0.031	0.015	0.009
5	0.334	0.219	0.149	0.084	0.052	0.031	0.016	0.009
6	0.321	0.226	0.151	0.083	0.052	0.032	0.015	0.009
7	0.328	0.220	0.149	0.085	0.051	0.032	0.016	0.009
8	0.324	0.227	0.145	0.084	0.051	0.031	0.015	0.009
9	0.318	0.222	0.156	0.083	0.051	0.031	0.015	0.009
10	0.320	0.223	0.147	0.084	0.052	0.032	0.015	0.009
11	0.323	0.215	0.152	0.083	0.053	0.031	0.015	0.009
12	0.326	0.219	0.150	0.084	0.051	0.032	0.015	0.009
13	0.321	0.221	0.150	0.085	0.051	0.031	0.015	0.009
14	0.326	0.218	0.148	0.082	0.052	0.032	0.015	0.009
15	0.322	0.222	0.149	0.082	0.051	0.031	0.016	0.009
16	0.332	0.222	0.148	0.084	0.052	0.031	0.015	0.009
17	0.320	0.227	0.152	0.082	0.052	0.031	0.015	0.009
18	0.315	0.224	0.148	0.085	0.052	0.032	0.015	0.009
19	0.325	0.223	0.153	0.082	0.053	0.032	0.016	0.009
20	0.305	0.217	0.149	0.082	0.052	0.031	0.015	0.009

Table 3: Oracle - Proportion of Incorrectly Classified Observations

P \ K	25	50	100	250	500	1000	2500	5000
2	0.333	0.218	0.148	0.081	0.050	0.031	0.015	0.009
3	0.341	0.232	0.147	0.081	0.050	0.030	0.015	0.009
4	0.365	0.233	0.148	0.082	0.051	0.031	0.015	0.009
5	0.366	0.235	0.152	0.082	0.050	0.030	0.015	0.008
6	0.379	0.249	0.152	0.083	0.050	0.031	0.015	0.009
7	0.394	0.255	0.158	0.084	0.051	0.030	0.015	0.009
8	0.400	0.261	0.164	0.086	0.051	0.031	0.015	0.009
9	0.411	0.273	0.167	0.086	0.050	0.031	0.015	0.009
10	0.420	0.278	0.172	0.088	0.050	0.031	0.015	0.008
11	0.428	0.268	0.175	0.089	0.053	0.031	0.015	0.009
12	0.433	0.283	0.181	0.090	0.053	0.032	0.015	0.009
13	0.444	0.285	0.180	0.093	0.054	0.031	0.015	0.008
14	0.447	0.286	0.185	0.094	0.053	0.032	0.015	0.009
15	0.451	0.305	0.193	0.093	0.055	0.032	0.015	0.009
16	0.457	0.305	0.191	0.096	0.056	0.032	0.015	0.009
17	0.447	0.299	0.196	0.097	0.056	0.032	0.015	0.008
18	0.459	0.309	0.192	0.101	0.058	0.032	0.015	0.009
19	0.459	0.309	0.198	0.102	0.058	0.033	0.015	0.009
20	0.461	0.309	0.199	0.103	0.059	0.033	0.015	0.008

Table 4: CES - Proportion of Incorrectly Classified Observations

P \ K	25	50	100	250	500	1000	2500	5000
2	0.409	0.270	0.179	0.092	0.055	0.032	0.015	0.008
3	0.525	0.391	0.290	0.176	0.122	0.080	0.046	0.029
4	0.629	0.499	0.396	0.278	0.214	0.162	0.110	0.079
5	0.690	0.577	0.471	0.368	0.301	0.246	0.184	0.146
6	0.731	0.635	0.542	0.441	0.373	0.316	0.256	0.215
7	0.783	0.681	0.597	0.495	0.438	0.381	0.318	0.276
8	0.815	0.725	0.640	0.546	0.489	0.433	0.372	0.331
9	0.839	0.754	0.681	0.588	0.532	0.479	0.420	0.378
10	0.861	0.777	0.706	0.624	0.567	0.518	0.461	0.420
11	0.862	0.802	0.738	0.657	0.604	0.556	0.498	0.458
12	0.886	0.823	0.765	0.688	0.632	0.585	0.532	0.492
13	0.896	0.840	0.776	0.709	0.659	0.616	0.561	0.523
14	0.905	0.859	0.803	0.734	0.685	0.640	0.587	0.550
15	0.922	0.871	0.819	0.753	0.708	0.663	0.612	0.576
16	0.926	0.885	0.833	0.772	0.724	0.686	0.635	0.600
17	0.928	0.887	0.848	0.787	0.744	0.705	0.657	0.621
18	0.936	0.898	0.857	0.801	0.761	0.724	0.677	0.642
19	0.949	0.909	0.869	0.818	0.777	0.739	0.693	0.660
20	0.954	0.915	0.878	0.827	0.789	0.756	0.711	0.679

Table 5: Homothetic FDH - Proportion of Incorrectly Classified Observations

P \ K	25	50	100	250	500	1000	2500	5000
2	0.663	0.552	0.449	0.330	0.257	0.197	0.136	0.102
3	0.824	0.750	0.668	0.549	0.467	0.392	0.306	0.251
4	0.911	0.857	0.793	0.700	0.626	0.551	0.462	0.399
5	0.949	0.915	0.870	0.800	0.735	0.671	0.587	0.527
6	0.969	0.948	0.917	0.866	0.812	0.759	0.684	0.629
7	0.983	0.964	0.945	0.904	0.869	0.824	0.759	0.709
8	0.991	0.978	0.962	0.936	0.906	0.870	0.816	0.772
9	0.992	0.986	0.977	0.954	0.931	0.904	0.859	0.820
10	0.996	0.990	0.985	0.967	0.952	0.927	0.891	0.859
11	0.996	0.994	0.988	0.978	0.964	0.946	0.916	0.889
12	0.998	0.996	0.992	0.984	0.974	0.959	0.935	0.912
13	0.998	0.997	0.994	0.987	0.979	0.969	0.949	0.931
14	0.999	0.998	0.996	0.991	0.985	0.976	0.961	0.945
15	1.000	0.998	0.998	0.993	0.989	0.982	0.969	0.956
16	0.999	0.999	0.998	0.995	0.992	0.986	0.976	0.965
17	0.999	0.999	0.999	0.996	0.994	0.989	0.981	0.972
18	1.000	0.999	0.999	0.997	0.995	0.992	0.985	0.978
19	0.999	0.999	0.999	0.998	0.996	0.993	0.988	0.982
20	1.000	1.000	0.999	0.999	0.997	0.995	0.991	0.985

Table 6: FDH - Proportion of Incorrectly Classified Observations

P\K	25	50	100	250	500	1000	2500	5000
2	0.212	0.163	0.121	0.079	0.054	0.039	0.024	0.016
3	0.215	0.162	0.121	0.076	0.056	0.038	0.023	0.016
4	0.220	0.168	0.122	0.076	0.054	0.038	0.023	0.016
5	0.221	0.163	0.119	0.078	0.054	0.037	0.023	0.016
6	0.218	0.165	0.119	0.076	0.054	0.038	0.023	0.016
7	0.216	0.165	0.119	0.076	0.055	0.038	0.024	0.016
8	0.220	0.159	0.119	0.078	0.053	0.039	0.023	0.016
9	0.219	0.163	0.122	0.077	0.053	0.038	0.023	0.016
10	0.215	0.161	0.118	0.078	0.054	0.037	0.023	0.016
11	0.223	0.161	0.119	0.079	0.055	0.037	0.022	0.016
12	0.219	0.166	0.120	0.077	0.054	0.038	0.023	0.016
13	0.217	0.163	0.122	0.079	0.055	0.039	0.023	0.016
14	0.220	0.161	0.121	0.076	0.054	0.038	0.023	0.016
15	0.213	0.164	0.120	0.078	0.055	0.038	0.023	0.016
16	0.221	0.160	0.120	0.077	0.054	0.037	0.023	0.016
17	0.216	0.161	0.120	0.078	0.054	0.038	0.023	0.016
18	0.220	0.164	0.122	0.078	0.055	0.037	0.024	0.016
19	0.214	0.164	0.119	0.079	0.054	0.039	0.023	0.016
20	0.217	0.166	0.119	0.079	0.055	0.038	0.023	0.016

Table 7: Oracle - Mean Square Error

P \ K	25	50	100	250	500	1000	2500	5000
2	0.225	0.170	0.124	0.082	0.056	0.041	0.024	0.016
3	0.242	0.174	0.134	0.083	0.061	0.041	0.025	0.017
4	0.253	0.192	0.135	0.087	0.062	0.042	0.025	0.017
5	0.263	0.194	0.139	0.091	0.064	0.044	0.027	0.017
6	0.269	0.202	0.147	0.094	0.067	0.047	0.027	0.018
7	0.272	0.206	0.150	0.095	0.072	0.049	0.028	0.018
8	0.282	0.205	0.155	0.103	0.072	0.052	0.030	0.019
9	0.285	0.216	0.159	0.109	0.075	0.054	0.030	0.020
10	0.288	0.218	0.162	0.112	0.076	0.056	0.032	0.021
11	0.297	0.217	0.165	0.111	0.082	0.056	0.033	0.021
12	0.286	0.222	0.169	0.115	0.085	0.059	0.034	0.023
13	0.297	0.228	0.168	0.119	0.090	0.063	0.036	0.023
14	0.303	0.226	0.174	0.118	0.089	0.065	0.038	0.024
15	0.297	0.222	0.175	0.120	0.092	0.067	0.039	0.025
16	0.305	0.229	0.173	0.126	0.096	0.068	0.041	0.026
17	0.300	0.228	0.176	0.125	0.096	0.069	0.042	0.027
18	0.302	0.233	0.175	0.127	0.099	0.073	0.044	0.028
19	0.301	0.234	0.176	0.128	0.099	0.074	0.046	0.029
20	0.311	0.235	0.175	0.128	0.103	0.076	0.046	0.030

Table 8: CES - MSE

P \ K	25	50	100	250	500	1000	2500	5000
2	0.262	0.213	0.164	0.129	0.118	0.113	0.112	0.116
3	0.310	0.256	0.218	0.179	0.164	0.161	0.161	0.164
4	0.345	0.300	0.258	0.215	0.194	0.182	0.175	0.174
5	0.374	0.327	0.289	0.245	0.222	0.206	0.190	0.184
6	0.393	0.354	0.317	0.274	0.248	0.228	0.209	0.198
7	0.409	0.369	0.337	0.296	0.272	0.251	0.228	0.215
8	0.424	0.386	0.351	0.315	0.292	0.271	0.247	0.232
9	0.434	0.398	0.370	0.334	0.310	0.289	0.265	0.249
10	0.444	0.410	0.383	0.350	0.326	0.305	0.280	0.265
11	0.452	0.421	0.394	0.360	0.339	0.319	0.296	0.279
12	0.454	0.432	0.406	0.373	0.352	0.332	0.309	0.293
13	0.458	0.438	0.413	0.383	0.363	0.344	0.321	0.306
14	0.465	0.442	0.421	0.392	0.374	0.355	0.333	0.317
15	0.465	0.443	0.425	0.400	0.382	0.364	0.343	0.327
16	0.469	0.451	0.431	0.407	0.388	0.373	0.352	0.337
17	0.471	0.455	0.438	0.415	0.396	0.381	0.361	0.346
18	0.476	0.457	0.443	0.419	0.404	0.389	0.370	0.354
19	0.476	0.461	0.446	0.427	0.411	0.395	0.376	0.362
20	0.479	0.466	0.449	0.431	0.414	0.401	0.383	0.369

Table 9: Homothetic FDH - MSE

P \ K	25	50	100	250	500	1000	2500	5000
2	0.365	0.323	0.276	0.216	0.177	0.142	0.104	0.080
3	0.430	0.400	0.370	0.315	0.280	0.242	0.198	0.168
4	0.457	0.441	0.419	0.381	0.351	0.318	0.274	0.244
5	0.476	0.460	0.447	0.417	0.393	0.369	0.331	0.304
6	0.479	0.475	0.465	0.442	0.423	0.404	0.372	0.349
7	0.483	0.479	0.473	0.459	0.444	0.429	0.402	0.382
8	0.488	0.484	0.478	0.469	0.458	0.447	0.425	0.407
9	0.491	0.487	0.483	0.478	0.469	0.459	0.441	0.427
10	0.493	0.489	0.488	0.484	0.477	0.468	0.454	0.442
11	0.499	0.492	0.488	0.484	0.481	0.474	0.464	0.453
12	0.493	0.497	0.492	0.487	0.485	0.480	0.471	0.463
13	0.494	0.494	0.493	0.491	0.487	0.483	0.476	0.469
14	0.497	0.491	0.493	0.490	0.488	0.487	0.480	0.474
15	0.492	0.490	0.492	0.491	0.489	0.488	0.483	0.478
16	0.494	0.491	0.491	0.490	0.490	0.488	0.486	0.481
17	0.496	0.494	0.492	0.493	0.491	0.490	0.487	0.484
18	0.497	0.493	0.495	0.492	0.492	0.491	0.489	0.485
19	0.493	0.491	0.493	0.493	0.493	0.492	0.489	0.488
20	0.494	0.495	0.493	0.494	0.492	0.492	0.490	0.488

Table 10: FDH - MSE