

Analysis of Price Manipulation and Dynamic Informed Trading in Securities Markets

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Abstract.

This paper studies the manipulation of prices by using a dynamic version of the Glosten and Milgrom (1985) model with a long-lived informed trader. We make a fundamental contribution by clarifying the conditions under which a unique equilibrium exists, and in what situations this equilibrium involves manipulation of prices by the informed trader. Furthermore, within the unique equilibrium, we characterize bid–ask spreads and show that bid and ask prices are monotonically increasing in the market maker’s prior belief. Finally, we propose a computational method to find equilibria in the model. Our simulation results confirm our theoretical findings and find multiple equilibria in some cases.

Key Words: Market microstructure; Glosten–Milgrom; Insider trading; Dynamic trading; Price formation; Sequential trade; Asymmetric information; Bid–ask spreads.

JEL Classification Numbers: D82, G12.

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1 Introduction

This paper studies a dynamic version of the Glosten and Milgrom (1985) model with a long-lived informed trader. We make two fundamental contributions: i.e., clarifying the conditions under which a unique equilibrium exists, and showing in what situations the dynamic informed trader trades against their information, which is defined as a *manipulative strategy* in the literature (see Chakraborty and Yilmaz (2004a)).¹

When the same individual with private information can buy an asset and sell the same asset in the future, it may not be optimal to always trade according to their information. In other words, it may be profitable for the informed trader who knows that in the terminal period, the value of the asset is high, to assign some positive probability to a sell order in some period. In this paper, we first show the conditions under which an equilibrium is unique, or there are multiple equilibria, and then we also show that a manipulative strategy is taken in equilibrium only when the positive change of the informed trader's future profit by a manipulative strategy is high.

The model in this paper works as follows. We adopt a sequential trade framework with trading of a risky asset over finitely many periods between the market maker, the strategic informed trader, and liquidity traders. At the beginning of the game, nature chooses the liquidation value of a risky asset to be high or low and informs the informed trader, who then trades dynamically. In each period, there is a random determination of whether the informed trader or a liquidity trader trades. Liquidity trade is determined by an exogenous probability. The “market maker” represents a continuum of competitive dealers, and sets bid and ask prices that satisfy the zero-profit condition.

The paper consists of two parts: i.e., theoretical and computational analyses. In the theoretical part, we study the Markov equilibrium such that the market maker's belief and the number of remain-

¹ Huberman and Stanzl (2004) define price manipulation as a round-trip trade. In this paper, it happens as a round-trip trade in equilibrium. Kyle and Viswanathan (2008) define “illegal price manipulation” as a strategy such that the violator intends to pursue a scheme that undermines economic efficiency both by making prices less accurate as signals for efficient resource allocation and by making markets less liquid for risk transfer. In this sense, the manipulative strategy that is studied in this paper is not classified as an illegal price manipulation (although insider trading based on nonpublic information is illegal because the Securities Exchange Commission defines insider trading as “any securities transaction made when the person behind the trade is aware of nonpublic material information” under Rule 10b5-1). Simultaneously, Kyle and Viswanathan (2008) explain that the finance and economics literature describes a trader's optimal strategy as manipulating the beliefs of other traders. In this sense, this paper uses the terminology of *manipulation* following the literature after Jarrow (1992); Chakraborty and Yilmaz (2004a,b); Huberman and Stanzl (2004). We say that a strategy is manipulative if with a strictly positive probability, it involves the informed trader undertaking a trade in any period that yields a strictly negative short-term profit.

ing trading periods determine the equilibrium strategy. By using the Markov equilibrium property, we truncate a multi-period serial problem into that of two-period decision-making and study the nature of equilibrium recursively. We provide a sufficient condition for the unique existence of the Markov equilibrium and show that a manipulative strategy exists in equilibrium only when the value functions are steep. A sufficient condition is provided as a set of properties that each period's value function satisfies. A Markov equilibrium that satisfies the sufficient condition for the uniqueness of an equilibrium is called a *tame equilibrium*.

In our attempt to obtain the conditions for the existence of the unique equilibrium, the key element is the probability of informed trading. Our approach is to split a unit time interval into subintervals, where the length of each subinterval is a function of the informed trading probability, and to consider the situation where the probability of informed trading is small and the number of trading periods is large. In this way, we study the informed trader's relative opportunity to re-trade and the possibility of manipulation, and provide a particular cut-off of the number of trading periods such that a unique or multiple equilibria arise. A tame equilibrium is a characterization of the unique Markov equilibrium in this setting. In other words, we also characterize the unique equilibrium when it exists uniquely.

Our results show that when there are relatively few trading periods, the equilibrium is unique, whereas when there are many trading periods, there are multiple equilibria. As there are two types of informed trader, there are four possible regimes, depending on whether each type manipulates. By a single crossing property of the payoff difference between buy and sell orders, we can prove that there is one equilibrium strategy within each regime when only one type manipulates. However, when there are too many chances to re-trade, both types simultaneously manipulate, and this gives more "freedom" for multiple regimes to coexist. This analysis explicitly derives bounds for the number of trading periods for which these different situations arise.

In our computational analysis, we show that in the interval of prior beliefs, there are two regions depending on how steep the value functions are. As the number of trading periods grows, the value functions start to become steep at the edges of the interval. In other words, there are two separate intervals in which the value functions can be steep, which we call cases (a) and (b). In case (a) when the market maker assigns a very low probability to the true state, the value function is very steep, and the informed trader can exploit this opportunity to fool the market. In case (b) when the market maker assigns a very high probability to the true state, the cost of manipulation is very low and it is easier for the informed trader to recoup the loss caused by manipulation.

In our theoretical analysis, we show that when the number of trading periods is small, then neither case arises. When the number of trading periods becomes large enough, case (a) starts to arise, and finally as the number of trading periods becomes even larger, cases (a) and (b) arise simultaneously

for some prior beliefs, because there can be the interval of market maker's belief at which both types' value functions are steep, and multiple different regimes arise. Then multiple equilibria coexist. Otherwise, the equilibrium is unique. We explicitly derive a cut-off of the number of trading periods for which these different situations arise.

The theoretical literature begins with manipulation by uninformed traders.² Allen and Gale (1992) provide a model of strategic trading in which some equilibria involve manipulation. Allen and Gorton (1992) consider a model of pure trade-based uninformed manipulation in which asymmetry in buys and sells by liquidity traders creates the possibility of manipulation. The first paper to consider manipulation by an informed agent within the discrete-time Glosten–Milgrom framework is Chakraborty and Yilmaz (2004a), who show that when the market faces uncertainty about the existence of informed traders, and when there are many trading periods, long-lived informed traders will manipulate in every equilibrium. Takayama (2010) furthers this analysis by providing a lower bound for the number of trading periods necessary for the existence of manipulation in equilibrium and shows that if the number of trading periods exceeds this lower bound, every equilibrium involves manipulation. The model in Chakraborty and Yilmaz (2004a) and Takayama (2010) is different from our model. In their model, at the beginning of the entire game, an informed or uninformed trader is chosen. Once the informed trader is chosen, he trades in every period. In our model, in every period, there is a random determination on the trader's type. Still, we obtain a similar result such that when the number of trading periods is large, an equilibrium involves manipulation.

The literature has investigated conditions based on the relations between prices and trades that rule out manipulation (Jarrow, 1992; Huberman and Stanzl, 2004). This paper also relates to that issue: i.e., our computational method provides a way to study these relations. We then respond to the questions of when manipulation arises and when the equilibrium with manipulation is unique.³

Another line of literature that this paper contributes to is the one that studies a problem about the uniqueness of equilibrium in canonical market microstructure models. In addition to the Glosten–Milgrom framework, another reference framework is proposed by Kyle (1985). Back (1992) extends the analysis in Kyle (1985) to a continuous-time version. While the uniqueness of the optimal informed trader's strategy either in the original Kyle model or Back (1992) remains unknown, McLennan *et al.* (2017), Boulatov and Taub (2013), and Boulatov *et al.* (2005) prove the uniqueness under

²Although our main concern in this paper is the dynamic strategic informed trader, there are works that study the dynamic informed market maker. For example, see Calcagno and Lovo (2006).

³Our focus in this paper is trade-based manipulation and its effect on prices. Goldstein and Guembel (2008) study manipulation and its effect on a firm's investment problem. Then, they show that the informed trader can profit from manipulation, and then identify a fundamental limitation inherent in the allocational role of stock prices.

some technical assumptions. Back and Baruch (2004) study the equivalence of the Glosten–Milgrom model and the Kyle model in a continuous-time setting and show that the equilibrium in the Glosten–Milgrom model is approximately the same as that in the Kyle model when the trade size is small and uninformed trades occur frequently. Given these two closely related frameworks as a proxy for a continuous-time model, our analysis shows the possibility of multiple equilibria and provides insights concerning uniqueness within these frameworks. Finally, Back and Baruch (2004) conclude that the continuous-time Kyle model is more tractable than the Glosten–Milgrom model, although most markets follow a sequential trade model. The model in this paper is a discrete-time version of Back and Baruch (2004)’s Glosten–Milgrom model.

Furthermore, the Glosten–Milgrom framework in Back and Baruch (2004) is a continuous-time stationary case and their program attempts to find the value functions as a fixed point. To do this, they use an extrapolation method that requires calculation of the slopes of the value functions. Because of this problem, Back and Baruch (2004) wrote that even though all the equilibrium conditions hold with a high degree of accuracy, the strategies were not estimated very accurately when manipulation arises. Our computational method proposes an alternative approach to study manipulation.

The remainder of the paper is organized as follows. Section 2 details the model. Section 3 presents the theoretical results. Section 4 describes the computational procedure and illustrates the results from the numerical simulations. Section 5 concludes by discussing further research directions.

2 The Model

In this section, we present our model, which is a discrete-time version of the Back and Baruch (2004)’s Glosten–Milgrom model. There is a single risky asset and a numeraire. The terminal value of the risky asset, denoted $\tilde{\theta}$, is a random variable that can take a low or high value, i.e., L or H , where $L = 0$ and $H = 1$. We assume that $\Pr(\tilde{\theta} = H) = \delta_0$ for some $\delta_0 \in (0, 1)$. There is a single long-lived informed trader who learns $\tilde{\theta}$ prior to the beginning of trading.

Trade occurs in finitely many periods t . We use the number of remaining trading periods to identify a period; thus, in period t , there are t trading periods remaining. Thus, period 1 is the last trading round. In each period, a single trader comes to the market, the market maker quotes bid and ask prices for the risky asset, and the trader either buys one unit or sells one unit. The agent who goes to the market in period t is a random variable unobserved by the market maker, such that with probability μ the informed trader is selected. If the informed trader is not selected, the agent selected is a “noise” or “liquidity” trader who (regardless of the quoted prices) buys with probability γ and sells with probability $1 - \gamma$. The identities of the selected traders, and the values of the liquidity

trader's trades, in the various periods are independent random variables.

We focus on equilibria where the market maker's belief and the number of remaining time periods determine an equilibrium strategy. The set of possible actions for the informed trader is denoted by $\{+1, -1\}$, in which $+1$ is a buy order and -1 is a sell order. Here, the market maker's belief b is the probability, from the market maker's point of view, that the state is high, going into period t . The market maker's ask and bid prices are functions of the market maker's belief and are given by the functions $\alpha_t : [0, 1] \rightarrow [0, 1]$ and $\beta_t : [0, 1] \rightarrow [0, 1]$, respectively. For each type θ of informed trader, a trading strategy $\sigma_{\theta t} : [0, 1] \rightarrow [0, 1]$ for $\theta \in \{H, L\}$ specifies a probability distribution over trades in period t with respect to the bid and ask prices posted in period t . In period t , the type- H informed trader buys the security with probability $\sigma_{Ht}(b)$ and sells with probability $1 - \sigma_{Ht}(b)$, and the type- L trader buys and sells with probabilities $1 - \sigma_{Lt}(b)$ and $\sigma_{Lt}(b)$, respectively.

The market maker's posterior belief after observing an order is updated using Bayes' rule on the posterior probability that $\tilde{\theta} = H$. As the value of the asset is either 0 or 1, the market maker's prior belief b is equal to the expected value of the risky asset conditional on his information going into period t . Define the bid and ask functions $A, B : [0, 1]^3 \rightarrow [0, 1]$ by the formulas:

$$A(b, x, y) = \frac{[(1-\mu)\gamma + \mu x]b}{(1-\mu)\gamma + \mu bx + \mu(1-b)(1-y)} \quad \text{and} \quad B(b, x, y) = \frac{[(1-\mu)(1-\gamma) + \mu(1-x)]b}{(1-\mu)(1-\gamma) + \mu b(1-x) + \mu(1-b)y},$$

where x is the probability that the type- H informed trader buys, while y is the probability that the type- L trader sells.

Now, the market maker is really to be thought of as a competitive market of risk-neutral market makers, e.g., a pair of market makers in Bertrand competition or a continuum of identical market makers. The equilibrium condition for the market maker is zero expected profits, which amounts to setting ask and bid prices equal to the posterior expected values of the asset.⁴

Now, we define the Markov equilibrium as follows.⁵

Definition 1. A Markov equilibrium is a collection of functions $\{\alpha_t, \beta_t, \sigma_{Ht}, \sigma_{Lt}\}_t$ with $\alpha_t, \beta_t : [0, 1] \rightarrow [0, 1]$, $\sigma_{Ht}, \sigma_{Lt} : [0, 1] \rightarrow \Delta(\{+1, -1\})$ and $J_t, V_t : [0, 1] \rightarrow \mathbb{R}$ such that for each t and $b \in [0, 1]$,

(E1) $\alpha_t(b) = A(b, \sigma_{Ht}(b), \sigma_{Lt}(b))$ and $\beta_t(b) = B(b, \sigma_{Ht}(b), \sigma_{Lt}(b))$.

⁴Biais *et al.* (2000) justify this assumption by showing that when there are infinitely many market makers, their expected profit converges to zero. More recently, Calcagno and Lovo (2006) show that a market maker's equilibrium expected payoff is zero if he is uninformed.

⁵Imagine that the market maker observes the sequence of realized trades for up until period t and updates his belief. Observing the history of realized trades, the informed trader can also find out what the market maker's belief b is. In this sense, this is a reduced form of equilibrium.

(E2) $\alpha_t(b) \geq \beta_t(b)$.

(E3)

$$\sigma_{Ht}(b) = \begin{cases} 0, & 1 - \alpha_t(b) + J_{t-1}(\alpha_t(b)) < \beta_t(b) - 1 + J_{t-1}(\beta_t(b)), \\ 1, & 1 - \alpha_t(b) + J_{t-1}(\alpha_t(b)) > \beta_t(b) - 1 + J_{t-1}(\beta_t(b)), \end{cases}$$

and

$$\sigma_{Lt}(b) = \begin{cases} 0, & -\alpha_t(b) + V_{t-1}(\alpha_t(b)) > \beta_t(b) + V_{t-1}(\beta_t(b)), \\ 1, & -\alpha_t(b) + V_{t-1}(\alpha_t(b)) < \beta_t(b) + V_{t-1}(\beta_t(b)). \end{cases}$$

(E4)

$$J_t(b) = \mu [\sigma_{Ht}(b)(1 - \alpha_t(b) + J_{t-1}(\alpha_t(b))) + (1 - \sigma_{Ht}(b))(\beta_t(b) - 1 + J_{t-1}(\beta_t(b)))] \\ + (1 - \mu) [\gamma J_{t-1}(\alpha_t(b)) + (1 - \gamma)J_{t-1}(\beta_t(b))],$$

and

$$V_t(b) = \mu [(1 - \sigma_{Lt}(b))(-\alpha_t(b) + V_{t-1}(\alpha_t(b))) + \sigma_{Lt}(b)(\beta_t(b) + V_{t-1}(\beta_t(b)))] \\ + (1 - \mu) [\gamma V_{t-1}(\alpha_t(b)) + (1 - \gamma)V_{t-1}(\beta_t(b))].$$

We see that (E1) states that the ask and bid prices are Bayesian updates of b conditional on the type of order received; (E2) states that there is no arbitrage opportunity; (E3) states that both types of informed seller optimize their order, considering the effect on the expected profits from future trades; and (E4) specifies the recursive computation of value functions. Implicitly we are assuming that the functions J_0 and V_0 are identically zero.

3 Theoretical Analyses

This section consists of three parts. In the first subsection, we provide the set of conditions for the unique existence of an equilibrium. In the second subsection, we state our main theorems. The first theorem proves the existence of equilibrium under the monotonicity and continuity of the value functions with respect to the market maker's belief. The second theorem shows that there is a $\bar{\mu}$ such that for every $\mu \leq \bar{\mu}$, a unique *tame* equilibrium, which satisfies the set of desired conditions for the unique equilibrium, exists. In the last subsection, Proposition 2 provides a condition for the existence of multiple equilibria.

As we see from the following analysis, solving analytically for an equilibrium strategy in the general model is not tractable. Our idea of finding equilibria in this model is backward induction. We will use some properties of value functions to identify the conditions under which there is a unique equilibrium. In period 1, the informed trader does not have a chance to re-trade and therefore does not manipulate. Thus, we know the period 1 value functions of prior beliefs. As we explain further in Section 4, given the next period value functions, we solve for the current period value functions. In this way, we focus on the decision-making problem of the informed trader in period t , given the period- $(t-1)$ value functions. As Theorem 1 states, we will obtain the set of conditions for the period- $(t-1)$ value functions, and given these conditions, we will have a desired equilibrium in period t . We first provide the set of conditions for the existence of a unique equilibrium.

As we show in Proposition 1, manipulation arises only when the value functions are steep. In general, when the number of trading periods becomes large, there are two separate intervals in which the value functions are steep, which we call cases (a) and (b). In case (a) when the market maker assigns a very low probability to the true state, the value function is very steep and the informed trader can exploit this opportunity to fool the market. In case (b) when the market maker assigns a very high probability to the true state, the cost of manipulation is very low and it is easier for the informed trader to recoup the loss caused by manipulation. When cases (a) and (b) arise simultaneously, there can be an interval of market maker's beliefs over which both types' value functions are steep and both types can manipulate. Then multiple equilibria exist. Otherwise, the equilibrium is unique.

In Theorem 2, a key element is the number of remaining trading periods. Because t is the number of trades that the informed trader could possibly make, we segment the time interval $[0, 1]$ into time periods of length Δt , i.e., the time between two trades. When μ goes to 0 in this setting, it is natural to ask whether the equilibrium prices in our discrete time setting converge to Brownian motion in a continuous time setting. Bearing in mind this question, in Proposition 2, we consider the case such that μ is small. Let $r > 0$ and $\Delta t = \mu^r$. As we show at the end of this subsection, the setting when $r = 2$ fits the situation where the belief process has increments, which has mean zero and variance Δt . This setting at the limit corresponds to Brownian motion. In Proposition 2, we show that indeed $r = 2$ is the boundary case between the unique and multiple equilibria in the symmetric case where the likelihood of the liquidity buy order γ is equal to $\frac{1}{2}$.

3.1 Manipulative Strategy and Tame Equilibrium

We start by defining two important concepts: i.e., a *manipulative strategy* and a *tame equilibrium*. We say that a strategy is manipulative if with a strictly positive probability, it involves the informed trader

undertaking a trade in any period that yields a strictly negative short-term profit.

Definition 2. For $\theta \in \{H, L\}$ we say that the type- θ trader takes a manipulative strategy at b in period t if $\sigma_{\theta t}(b) < 1$.⁶

Define the symbol ∂_+ such that for each $b \in [0, 1]$ and a function $f : [0, 1] \rightarrow \mathbb{R}$,

$$\partial_+ f(b) = \lim_{\epsilon \rightarrow 0^+} \frac{f(b + \epsilon) - f(b)}{\epsilon}. \quad (1)$$

Then we define the following four conditions:

C_s J_s and V_s are continuous and piecewise differentiable;

M_s J_s is strictly decreasing and V_s is strictly increasing;

SH_s if b_H satisfies $\partial_+ J_s(b_H) \leq -1 - \delta_H(\mu)$, then $\partial_+ J_s(b) \leq -1 - \delta_H(\mu)$ for every $b \in [0, b_H)$;

SL_s if b_L satisfies $\partial_+ V_s(b_L) \geq 1 + \delta_L(\mu)$, then $\partial_+ V_s(b) \geq 1 + \delta_L(\mu)$ for every $b \in (b_L, 1]$.

Definition 3. We say that a period- t Markov equilibrium is tame if the value functions in period t satisfy conditions **C_t**, **M_t**, **SH_t** and **SL_t**. We say that a Markov equilibrium is tame if for every s , a period- s Markov equilibrium is tame.

C_s states that the value functions are continuous and **M_s** states that the value function is monotone with respect to the market maker's belief. Theorem 1 of the next subsection will show that an equilibrium exists under these two conditions. **SH_s** and **SL_s** state that the value functions can be split into two regions where in one region, the slope is steep, while in the other regions, it is not.

Fix $b \in (0, 1)$ and $\sigma = (\sigma_H, \sigma_L) \in \mathcal{E}_t(b)$, and let $\alpha = A(b, \sigma_H, \sigma_L)$ and $\beta = B(b, \sigma_H, \sigma_L)$ be the pair of equilibrium ask and bid prices associated with b and σ in period t .

Lemma 1. In equilibrium, $\alpha > b > \beta$ and $\sigma_H + \sigma_L > 1$ in period t .

Proof. When $t = 1$, nobody manipulates in equilibrium and we obtain the first statement by Bayes' rule. Now, suppose that $t > 1$. By (E2), we must have $\alpha \geq \beta$. To show the inequality is strict, suppose that $\alpha = \beta$. Then by Bayes' rule, $\sigma_H = 1 - \sigma_L$, which leads to $\alpha = \beta = b$. Then the payoffs (3) are:

$$\begin{aligned} 1 - b + J_{t-1}(b) &> \beta - 1 + J_{t-1}(b); \\ -b + V_{t-1}(b) &< \beta + V_{t-1}(b). \end{aligned}$$

Then (E3) implies that $\sigma_H = 1$ and $\sigma_L = 1$, which contradicts $\sigma_H = 1 - \sigma_L$. \square

⁶This is the same definition used by Chakraborty and Yilmaz (2004a). Back and Baruch (2004) use the term "bluffing" instead, while Huberman and Stanzl (2004) define "price manipulation" as a round-trip trade. Also see Footnote 1.

In equilibrium, the type- H trader does not sell with probability one and the type- L trader does not buy with probability one. This means that the informed trader either trades on his information or assigns a positive probability to both buy and sell orders. In the latter case, the informed trader is indifferent between buy and sell orders.

Lemma 1 indicates that even when the informed trader manipulates, a strictly positive amount of information comes into the market in each period, which leads to a strictly positive bid and ask spread. This motivates consideration of the slopes of the value functions in the interval between these prices. By Lemma 1, the bid–ask spread $\alpha - \beta$ is strictly positive.

The next proposition provides a condition for manipulation by \mathbf{SH}_{t-1} and \mathbf{SL}_{t-1} , and Lemma 1. Because $\beta < \alpha$ by Lemma 1, when the type- H manipulates, β must be in the region where the slope of the type- H 's value function is smaller than $-1 - \delta_H(\mu)$. Therefore, if there are two bid prices for which the type- H manipulates, the slope between the two points must be steeper than -1 . This is formally summarized into the next proposition.

Proposition 1. *Suppose that \mathbf{C}_{t-1} , \mathbf{M}_{t-1} , \mathbf{SH}_{t-1} and \mathbf{SL}_{t-1} hold in period $t - 1$. In equilibrium, the following hold.*

H. *If the type- H trader manipulates at b , then $\frac{J_{t-1}(\beta_0) - J_{t-1}(\beta_1)}{\beta_0 - \beta_1} < -1$ for any $\beta_0, \beta_1 \leq \beta$.*

L. *If the type- L trader manipulates at b , then $\frac{V_{t-1}(\alpha_0) - V_{t-1}(\alpha_1)}{\alpha_0 - \alpha_1} > 1$ for any $\alpha_0, \alpha_1 \geq \alpha$.*

Proposition 1 indicates that manipulation could arise only in a region where the value function is steep. This is intuitive: i.e., the informed trader manipulates when the change in the future payoff from manipulating is large. In other words, the informed trader knowingly makes a loss when the value function is steep because the manipulation makes a difference to the future payoff, which is represented by the slope of the value function.

As discussed in Back and Baruch (2004), the informed trader trades gradually in the Kyle (1985) model as well as their model, because the informed trader may otherwise lose his informational advantage. In our model, manipulation arises in a similar way. It is not optimal for the informed trader to trade with probability one according to their information. Furthermore, manipulation or bluffing may be possible in the Kyle (1985) model. In the Kyle (1985) model, the market maker cannot distinguish between buy and sell orders, but can only see aggregate orders. Back and Baruch (2004) show that as the trade size decreases, the trade rate of each informed trader in the Kyle (1985) model converges to the *net* trade rate of each informed trader in a dynamic version of the Glosten and Milgrom (1985) model, which is given by the difference between buy and sell strategies, $\sigma_H - (1 - \sigma_H)$ in our model.

3.2 Main Results

Here, we state our two main theorems. For $b \in [0, 1]$ and $t < +\infty$, let

$$\mathcal{E}_t(b) = \{(\sigma_{Ht}(b), \sigma_{Lt}(b)) : \text{for the } \alpha_t(b) \text{ and } \beta_t(b) \text{ given by (E1), (E2) and (E3) hold}\}.$$

Theorem 1. *Suppose that a period- $(t-1)$ Markov equilibrium exists, and \mathbf{C}_{t-1} and \mathbf{M}_{t-1} are satisfied in period $t-1$. Then $\mathcal{E}_t(b)$ is nonempty.*

The proof of Theorem 1 uses Kakutani's fixed point theorem. The proofs appear in Appendix I, unless stated immediately after each result.

We now classify equilibria according to the types of trader that sometimes trade against their information. As there are two types of informed trader, there are four possible scenarios in terms of who manipulates. In every t , an equilibrium σ_t is in *Regime \emptyset* if $\sigma_t = (1, 1)$. It is in *Regime L* if $\sigma_{Lt} < 1$ and $\sigma_{Ht} = 1$; it is in *Regime H* if $\sigma_{Lt} = 1$ and $\sigma_{Ht} < 1$; and it is in *Regime HL* if $\sigma_{Lt} < 1$ and $\sigma_{Ht} < 1$. We say that a regime *arises* at a belief b in period t if $\mathcal{E}_t(b)$ contains an equilibrium in that regime.

Theorem 2. *Fix $r > 0$. Let $t \in \{1, \dots, \lfloor \frac{1}{\mu^r} \rfloor\}$. There is $\bar{\mu}$ such that for every $\mu \leq \bar{\mu}$, a tame Markov equilibrium exists, and satisfies the following description.*

- *If $r < 2$, the Markov equilibrium exists uniquely and is tame. Furthermore, the equilibrium bid and ask prices are monotonically increasing with respect to prior beliefs in every period.*
- *If $r \in (0, 1]$, manipulation does not arise;*
- *If $r \in (1, 2)$, manipulation arises at some period $t \in \{\lfloor \frac{1}{\mu} \rfloor + 1, \dots, \lfloor \frac{1}{\mu^r} \rfloor\}$, while Regime HL never arises.*

A key insight of our main result in this section is as follows. When μ is small, the slope of the value function becomes similar in magnitude across beliefs. When r is smaller than 1, the number of trading periods is not large enough for manipulation to arise. When r is between 1 and 2, the number of trading periods is large enough that one type of informed trader manipulates, and manipulation could arise in some period when the cost of manipulation is not large. This is case (b) manipulation that we have described in the previous section. Now, when r is greater than 2, the number of trading periods is large enough for both types to manipulate in some period t . This is the situation where case (a) and case (b) manipulations arise simultaneously.

Another interesting part of Theorem 2 is the condition for the unique equilibrium. When the number of trading periods is too small, manipulation itself does not arise. The second theorem specifies the intervals of μ^r for which the unique equilibrium with market manipulation arises.

In general, \mathbf{SH}_t and \mathbf{SL}_t do not hold when the number of remaining trading periods t is large compared with μ , i.e., μt is large. When b is equal to zero there will be no manipulation. By using Bayes' rule, it is not difficult to calculate⁷ that for all t ,

$$\partial_+ V_t(0) = \frac{t\mu(1-\mu)(1-\gamma)}{\mu + (1-\mu)(1-\gamma)} \quad \text{and} \quad \partial_+ J_t(1) = \frac{-t\mu(1-\mu)\gamma}{\mu + (1-\mu)\gamma}. \quad (2)$$

By (2), even when the market maker knows the true state, the slopes of the value functions increase geometrically. This is useful in understanding how \mathbf{SH}_s and \mathbf{SL}_s fail. By Proposition 1, the type- H or the type- L trader manipulates when the slopes of the value functions are steep. When \mathbf{SH}_s and \mathbf{SL}_s hold, manipulation arises only when the market maker is very wrong. However, when t is large compared with μ , then as in (2), the value functions can be steep near the points where the market maker is almost correct. Then there are two separate regions in which manipulation arises. One is when the market maker is almost correct, while the other is when the market maker is very wrong. Thus, these two types of manipulation can co-exist when this happens and there are multiple equilibria as a result.

In the proof of Theorem 2, we use the following two conditions instead of \mathbf{SH}_s and \mathbf{SL}_s . The idea is that when μ is small, the value functions become close to linear. Then we can analyze when the slopes become steep enough for manipulation to arise. By showing that when \mathbf{DH}_s and \mathbf{DL}_s hold (see Lemma I-7 in Appendix I), \mathbf{SH}_s and \mathbf{SL}_s also hold, we can gain the tractability of the situation where manipulation may arise.

Formally, we say that conditions \mathbf{DH}_s and \mathbf{DL}_s are satisfied if for every ϵ_H and ϵ_L ,

\mathbf{DH}_s for every $b \in (0, 1)$, $|\partial_+ J_s(b) + \mu s| < \epsilon_H$;

\mathbf{DL}_s for every $b \in (0, 1)$, $|\partial_+ V_s(b) - \mu s| < \epsilon_L$.

The proof of Theorem 2 is in Appendix I, and works in the following way. As discussed earlier, in period 1, the informed trader does not manipulate and we know what the period 1 value functions are. By using these period 1 value functions, we can show that the period 1 value functions satisfy the conditions for a tame equilibrium. Thus, mathematical induction is applicable. Fix $r > 0$. Let the total number of trading periods be $\lfloor \frac{1}{\mu^r} \rfloor$ and $t > 1$. By mathematical induction, assume that there is $\bar{\mu}$ such that for every $\mu < \bar{\mu}$, \mathbf{C}_{t-1} , \mathbf{M}_{t-1} , \mathbf{DH}_{t-1} and \mathbf{DL}_{t-1} hold in period $t-1$. Fix $\mu < \bar{\mu}$ arbitrarily.

⁷The detailed calculations are available from the author.

We will show that C_t , M_t , DH_t and DL_t hold in period t . In this way, we can truncate the decision problem of the informed trader into two periods: i.e., the next period (period $t - 1$) and the current period (period t). Let $r < 2$. Then we can show that the equilibrium strategy is unique in period t . Then we will show that the value functions $\{J_t, V_t\}$ also satisfy C_t , M_t , DH_t and DL_t .

3.3 Multiple Equilibria

Fix $r > 0$. As our approach is to split a unit interval into $\frac{1}{\Delta t}$ segments, it is natural to question what the equilibrium prices converge to, when μ converges to 0, and whether the equilibrium prices in our discrete time setting converge to Brownian motion in a continuous time setting. To see the connection, let $\mu = \sqrt{\Delta t}$ (which is $r = 2$) and from (11) and (12), when μ is sufficiently small,

$$\alpha_t(b) - b \approx \frac{\Delta}{\gamma} \sqrt{\Delta t}$$

and

$$b - \beta_t(b) \approx \frac{\Delta}{1 - \gamma} \sqrt{\Delta t}$$

where

$$\Delta = \Delta(b, \sigma^H, \sigma^L) = \mu b(1 - b)(\sigma^H - 1 + \sigma^L).$$

Notice that when the belief in period t is b , the belief in period $t - 1$ is either $\alpha_t(b)$ or $\beta_t(b)$. When the belief process is Brownian motion, the belief process has increments, which has mean zero and variance Δt . Because $r = 2$ fits this setting, it is worth exploring this case. When $r = 2$ and μ is small, by some calculation (see the proof of Proposition I-6 in Appendix II for the details), we obtain:

$$\frac{(\alpha_t(b) - \beta_t(b))}{\mu^{r-1}} \approx b(1 - b)(\sigma_H - 1 + \sigma_L) \left(\frac{1}{\gamma} + \frac{1}{1 - \gamma} \right).$$

By substituting the above into (30), we can see that whether a pair of bid and ask prices to satisfy (29) exists depends on γ . Still, the following result holds.

When $r > 2$, the number of trading periods is large enough relative to the informed trading probability and then Regime HL can arise. We prove this result when $\gamma = \frac{1}{2}$. Indeed, Proposition 2 shows that $r = 2$ is the boundary case between the unique and multiple equilibria in the symmetric case of $\gamma = \frac{1}{2}$.

Proposition 2. Fix $r > 0$ and $\mu \leq \bar{\mu}$. Let $\gamma = \frac{1}{2}$. Consider period $t \leq T$.

- When $r = 2$ and $t < \lfloor \frac{1}{\mu^r} \rfloor$, the unique equilibrium is tame. Moreover, in equilibrium, one type of trader manipulates in period t while Regime HL never arises.

- When $r > 2$ and $t > \lfloor \frac{1}{\mu^r} \rfloor + 1$ is sufficiently large, Regime H , Regime L , and Regime HL arise at $b = \frac{1}{2}$ in period t .

When $\gamma = \frac{1}{2}$, it is not difficult to prove that there is a symmetric equilibrium in the sense that when $\sigma \in \mathcal{E}_t(b)$, $\tilde{\sigma}_t(1 - b)$ where $\tilde{\sigma}_L = 1 - \sigma_H$. Then, when $b = \frac{1}{2}$, if the type- H manipulates, the type- L also manipulates. In our proof given in Appendix I, we show that within each regime, there is at most one equilibrium, and in equilibrium either only one regime exists, or three regimes coexist. Thus, when regimes H and L coexist, there are indeed three equilibria. Intuitively, when the number of trading periods becomes large, the value functions become steep so that Regime HL arises and three regimes coexist when $b = \frac{1}{2}$. However, until this happens, there is only one equilibrium.

4 The Computation Procedure and Simulation Results

In this section, we lay out the procedure to compute the equilibrium in our model. The following summarizes our procedure. First, we segment $[0, 1]$ into a grid with N cells of equal sizes.

1. We start with the terminal period 1. In period 1, nobody manipulates; thus, we can compute the period 1 value functions using Bayes' rule. We linearly interpolate the period 1 value functions into N grid points.
2. Given linearly interpolated J_1 and V_1 for each grid point, we solve a system of equations such that each type of trader is indifferent between buying and selling by using Bayes' rule. We further check if the strategy we compute for each trader satisfies optimality. In this way, we check which regime arises in each grid point belief.
3. We find a pair of equilibrium prices for each grid point belief.
4. We compute the period 2 value functions. Then, we repeat the same procedure from Step 2.

In Appendix III, we describe the computation procedure in more detail including equations that in each regime, the equilibrium strategy must satisfy in the simulation.

Characteristics of Equilibrium

We first consider $\gamma = \frac{1}{2}$, so that the equilibrium is symmetric. Figure 1 exhibits the equilibrium bid and ask prices with respect to the market maker's prior belief for periods 1 and 400. The solid curves that present the highest and lowest points represent the ask and bid prices for the case where

there is no manipulation. In the bid and ask price figures, there is a region of beliefs in which the bid or ask prices differ between the periods. It is in this region of beliefs that manipulation arises in equilibrium. As the informed trader’s strategy differs between periods because the manipulation rate is time-dependent, the bid and ask prices also differ between periods. Although it is obvious from Bayes’ rule, from this figure we can also see that manipulation indeed decreases the bid–ask spread for every belief in $(0, 1)$.

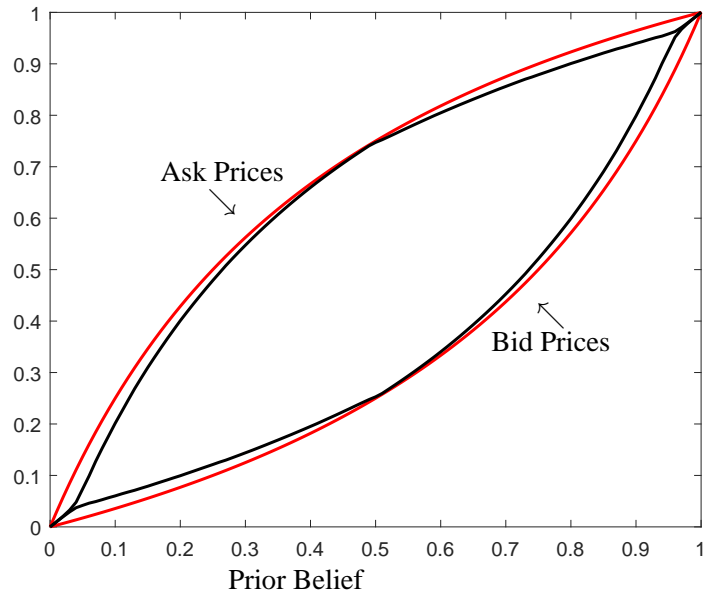


Figure 1: Bid and ask prices when $\mu = 0.5, \gamma = 0.5, t \in \{1, 400\}$

Note: The figures show bid and ask prices in two periods $t = 1$ and 400 . The outer lines are for $t = 1$ and the inner lines are for $t = 400$.

There is rich empirical evidence that shows that the adverse selection component in the bid–ask spread increases as the announcement date gets close. Krinsky and Lee (1996) use the sample of quarterly earnings announcements taken from the PR Newswire database and find that the adverse selection cost increases as earnings announcements approach. By using the data on New York Stock Exchange-listed companies, Koski and Michaely (2000) find that information asymmetry as manifested in trade size and the information environment of the trade has an impact on both prices and liquidity, and the price impact of trades is largest in magnitude during announcement periods. Venkatesh and Chiang (1986) studies three groups of announcements: i.e., (a) joint announcements, (b) initial announcements, and (c) second announcements by using a random sample of 75 stocks in the New

York Stock Exchange, and find a significant increase in information asymmetry before second announcements. Their interpretation is that market participants suspect an unusual announcement and increase in information asymmetry when the second announcement is separated from the first one.

Our simulation results have a similar feature in that bid–ask spreads for a given belief are the largest in period 1 just before a public announcement. This is also similar to the theoretical finding in Calcagno and Lovo (2006). In Calcagno and Lovo (2006), we observe this because the winner’s curse increases when the terminal period gets closer. In our analysis, this is a direct consequence of the fact that there is no manipulation in period 1. Although our mechanisms differ, we observe a similar result here as well. Furthermore, because of our setting, $b - \beta_t(b)$ or $\alpha_t(b) - b$ can be thought of as a price impact, and the price impacts for a given belief are also the largest in period 1. This is a similar feature to the empirical finding in Koski and Michaely (2000).

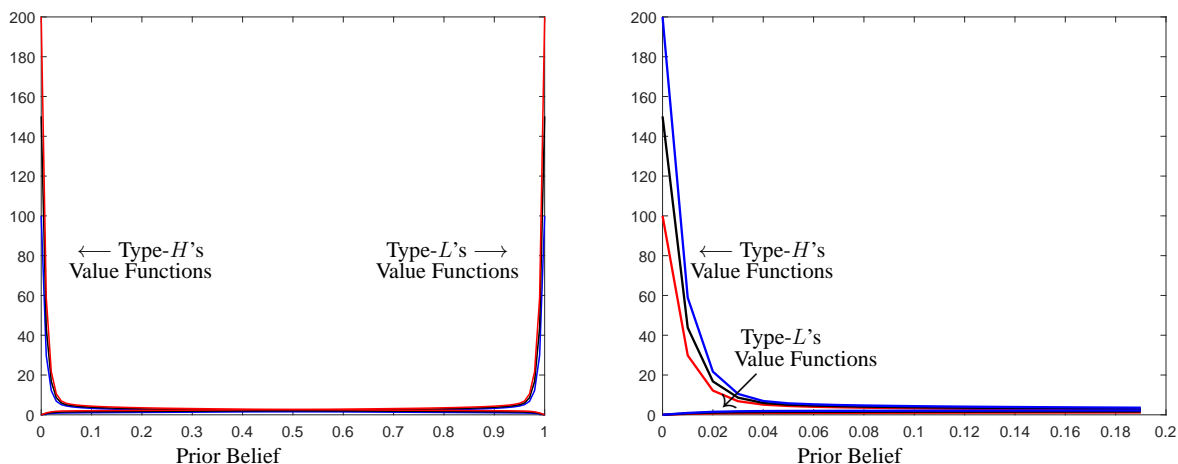
In equilibrium, the type- H trader does not sell with probability one and the type- L trader does not buy with probability one. This means that the informed trader either trades on his information or assigns a positive probability to both buy and sell orders. In the latter case, the informed trader is indifferent between buy and sell orders. This motivates consideration of the slopes of the value functions.

Manipulation

The results of the simulation also show that both cases (a) and (b), which we have discussed in Subsection 3.1, arise. By symmetry, we focus on the type- H trader. In case (a), the type- H trader manipulates in a region of beliefs close to 0. This result may sound somewhat counterintuitive because, e.g., if the type- H trader manipulates in a region of beliefs close to 0, the bid price will be very low and the trader can only obtain a little money. However, to affect the future payoffs through the updating of the market maker’s beliefs, they manipulate when the bid–ask spread is small and the slope of the next-period value function is steep. From the lower panels of Figure 2, we can see that when the market maker assigns a very low probability to the true state, the value functions are very steep.

To take a more careful look at a manipulative strategy, the second figure of the lower panel in Figure 2 shows the equilibrium strategy for the type- H in the $[0, 1]$ interval of prior beliefs. Interestingly, we can see that when the prior belief is close to 1, the type- H trader also manipulates. This is case (b) manipulation and consistent with our following calculation in 2. When the belief is very close to 0, the market maker almost knows the value of the asset. As shown in Proposition 2, the slopes of the value functions increase geometrically. As a result, the type- H starts to manipulate as the number

(a) Value functions when $t \in \{200, 300, 400\}$



(b) Value functions when $t \in \{1, 100, 200, 300, 400\}$ and manipulation rates when $t \in \{200, 300, 400\}$

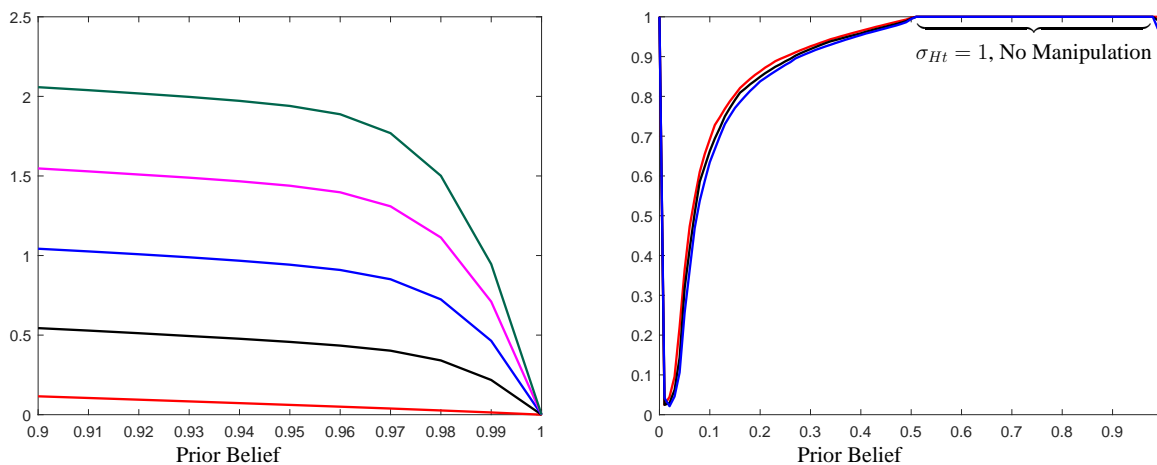


Figure 2: Value functions and manipulation rates when $\mu = 0.5$ and $\gamma = 0.5$

Note: The left-hand side figure of the upper panel shows value functions in three periods $t = 200, 300, 400$. In the second figure, the lowest line is for $t = 200$ and the highest line is for $t = 400$. In the lower panel, the left-hand side figure shows value functions at five periods $t = 1, 100, 200, 300, 400$. The lowest line is for $t = 1$ and the highest line is for $t = 400$. The second figure of the lower panel shows informed trading strategies in three periods $t = 200, 300, 400$. The lowest line is for $t = 400$ and the highest line is for $t = 200$.

of remaining trading periods increases. From the left-hand side of the lower panel in Figure 2, we can see that the value functions become steeper near the edges when the number of remaining periods increases.

Table 1 describes how manipulation starts to arise when the market maker is almost correct or very wrong. In a sense, there are two types of manipulation. Manipulation for $r \in (1, 2)$ corresponds to that which arises when the market maker is almost correct. Our simulation shows that as the slope becomes steeper, manipulation arises only when the market maker is very wrong. In other words, the other type of manipulation disappears and the remaining type of manipulation expands. This conveys the intuition that the slopes of the value functions are indeed an incentive for the informed trader, and as they increase, manipulation begins to take place over a wider range. In this way, Regime HL

	$b = 0.01$	$b = 0.02$	\dots	$b = 0.98$	$b = 0.99$
$t = 11$	L	\emptyset	\dots	\emptyset	H
$t = 12$	L	H	\dots	L	H
$t = 13$	H	H	\dots	L	L

Table 1: Regimes and beliefs for $t = 11, 12, 13$

starts to arise in our simulation as each region that each type of trader manipulates overlaps with the other. In our simulation, in period $t = 152$, Regime HL starts to arise at belief 0.5, and it starts to arise similarly at beliefs $b = 0.01$ or 0.99 in period $t = 271$.⁸ As the market maker is completely wrong at $b = 0$ or 1, the informed trader does not manipulate because the prices are favorable for the type- H ($b = 0$) trader. However, in the region close to $b = 0$, as the value function is steep, the type- H trader manipulates around $b = 0$. Therefore, there is a spike in the rate of informed trading near the edges. As we can see from the right-hand side figure of the lower panel in Figure 2, there appears to be a discontinuity in the informed strategy when the market maker is very wrong. The important idea in our theory of a tame equilibrium is to make manipulation arise near the other edge (i.e., when the market maker is almost correct) and not near this edge (i.e., when the market maker is very wrong).

As we can see in Figure 2, the value functions are not globally convex. Near $b = 0$, some parts appear to be concave. Thus, manipulation arises as the slope near $b = 0$ becomes sufficiently steep. This manipulation remains in the unique equilibrium when μ becomes sufficiently small and t is

⁸From Figure 1, it may be difficult to see that at $b = 0.5$, the bid and ask prices in Regime \emptyset differ from the simulated prices for $t = 201, \dots, 400$, because the manipulation rates at $b = 0.5$ are quite small. Indeed, the bid price without manipulation is 0.25 and the ask price without manipulation is 0.75, while the simulated equilibrium bid prices for these periods range from 0.2511 to 0.2528 and the ask prices for these periods range from 0.7489 to 0.7472.

sufficiently large. In other words, when the value functions become sufficiently “linear,” the moment when the market maker begins to make a mistake is the only opportunity to manipulate.

Multiple Equilibria

In our simulation results, we also observe multiple equilibria. A unique equilibrium without manipulation arises in the earlier periods, then a unique equilibrium with manipulation starts to arise. Finally, multiple equilibria arise in the later periods. For example, when $\mu = 0.2$, a unique equilibrium with manipulation starts to arise in period $t = 22$, and then multiple equilibria arise in period $t = 52$. When the number of trading periods grows more rapidly than the informed trading probability, Regime HL arises. However, if there are not enough trading periods, manipulation itself does not arise.

Effects of an Asymmetric Liquidity Distribution

To this point, we have considered the symmetric case in the sense that the liquidity for a buy is equally likely as the liquidity for a sell. Here we consider how an asymmetric liquidity distribution affects bid and ask prices and who manipulates in equilibrium. The following two figures in Figure 3 show the bid and ask prices, as well as value functions, when $\gamma = 0.2$. Note that the type- H trader at belief b when $\gamma = 0.2$ is a mirror image of the type- L trader at belief $1 - b$ when $\gamma = 0.8$. We also observe that the informed trader manipulates when the slope is steep. As we can predict from Bayes’ rule, as γ decreases, the type- H trader’s value functions are positioned closer to 0. As such, the region where its value functions are steep becomes smaller and the region in which the type- H trader manipulates become more restricted.

5 Concluding Remarks

In this paper, we developed a model of dynamic informed trading from a canonical framework in the market microstructure literature. We make a fundamental contribution to the literature by providing a theorem describing conditions under which a unique equilibrium exists. We also provided a computational method to find equilibria.

The model in this paper is a theoretical depiction of quote-driven markets with competitive market-making, as in Glosten and Milgrom (1985) and Ozsoylev and Takayama (2010). Quote-driven markets are common in over-the-counter (OTC) markets. The Nasdaq and London Stock Exchange Automated Quotation system are two examples of quote-driven markets with competitive market-making,

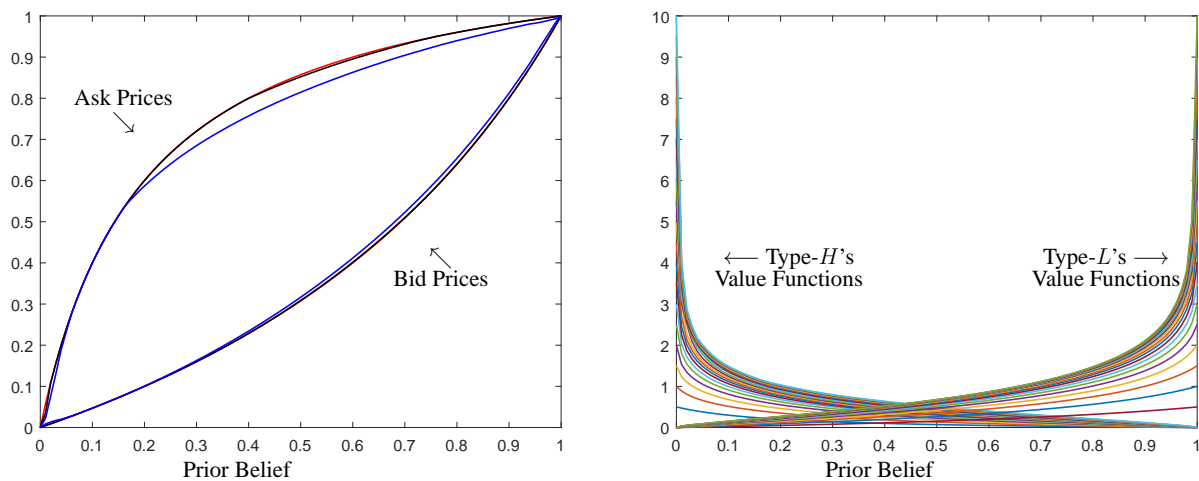


Figure 3: Bid and ask prices when $\mu = 0.5, \gamma = 0.2, t \in \{1, 10, 20\}$

Note: The figures show bid and ask prices in three periods $t = 1, 20, 400$. The innermost lines are for $t = 20$ and the outermost lines are for $t = 1$. The lowest lines correspond to $t = 1$ and as the number of periods increase, the lines move upward.

although the Nasdaq has become more of an order-driven market such as the New York Stock Exchange. OTC markets have created renewed interest in the literature, particularly because after the 2007–2008 subprime lending crisis, OTC markets were blamed as opaque financial markets (for further discussion, see Monnet and Quintin, 2017a,b; Glode and Opp, 2018). Pagano and Volpin (2012) develops a model to study the impact of transparency on the market for structured debt producers within the primary and secondary markets. Bolton *et al.* (2016) describes how OTC markets emerge even in the presence of well-functioning centralized exchanges. Babus and Hu (2017) proposes a theory of endogenous intermediation in OTC markets, while Babus and Kondor (2018) proposes a model of trading in OTC markets and studies the effect of trade decentralization and adverse selection on information diffusion.

Although this paper provides a framework for studying the dynamic strategy of the informed trader and price formation within quote-driven markets with competitive market-making, one limitation of the model is that the informed trader cannot choose to not trade. Allowing this strategy significantly complicates the analysis. Alternatively, in our model, we can introduce the situation in which one type of informed trader trades more frequently than the other type, i.e., in the high state, the informed trading probability is μ_H and in the low state, it is μ_L . Our analysis still holds with this setting of

different informed probabilities because our analysis relies mainly on the shapes of the value functions and, as we see in the two-period case, bid and ask prices can be expressed by using the probability of buy orders in each state. However, it simply complicates the formulation of Bayes' rule and this generalization does not change the results substantially. Therefore, we have kept the current setting.

From our analysis, several important research questions arise. While our model uses a discrete setting, as the limit of our setting with sufficiently small μ could converge to a continuous-time setting, our paper can add some insights into the literature on continuous-time models. One fundamental problem in financial econometrics is to identify the stock price dynamics accurately and analyze how a different market structure affects these dynamics. Many more studies address the relationship between prices and dynamic trading in a continuous-time setting.⁹ De Meyer (2010) studies an n -times repeated zero-sum game of incomplete information and shows that the asymptotics of the equilibrium price process converge to a continuous martingale of maximal variation (hereafter CMMV). As De Meyer (2010) points out, this CMMV class could provide natural dynamics that may be useful in financial econometrics, although it remains an open question as to whether the equilibrium dynamics in a nonzero-sum game still belong to the CMMV class.

First, given the association with De Meyer (2010), our paper provides a fundamental framework for a nonzero-sum trading game. Adding a time discount factor to the informed trader's profit to bring our analysis into a continuous-time setting is an obvious extension. Extending our analysis for the case of sufficiently small μ to see how our results are related to those works is an interesting research agenda.

More specifically, it is not difficult to prove that when there are multiple tame Markov equilibria in period t , and the equilibria strategies are given by $(\bar{x}_{HL}, \bar{y}_{HL})$ (Regime HL), $(1, \bar{y}_L)$ (Regime L), and $(\bar{x}_H, 1)$ (Regime H).¹⁰ Then the following holds:

$$V_t(b, \bar{x}_H, 1) > V_t(b, \bar{x}_{HL}, \bar{y}_{HL}) > V_t(b, 1, \bar{y}_L); \text{ and}$$

$$J_t(b, \bar{x}_H, 1) < J_t(b, \bar{x}_{HL}, \bar{y}_{HL}) < J_t(b, 1, \bar{y}_L),$$

⁹Brunnermeier and Pedersen (2005) consider the dynamic strategic behavior of large traders and show that “overshooting” occurs in equilibrium, while Back and Baruch (2007) analyze different market systems by allowing the informed traders to trade continuously within the Glosten–Milgrom framework. Lastly, within an extended Kyle framework, Collin-Dufresne and Fos (2012) study insider trading where the liquidity provided by noise traders follows a general stochastic process and show that even though the level of noise trading volatility is observable, the measured price impact is stochastic in equilibrium.

¹⁰The proof is available upon request from the author.

where

$$V_t(b, x, y) = \mu B(b, x, y) + (1 - \mu)\gamma V_{t+1}(A(b, x, y)) + ((1 - \gamma)(1 - \mu) + \mu)V_{t+1}(B(b, x, y));$$

$$J_t(b, x, y) = \mu(1 - A(b, x, y)) + ((1 - \mu)\gamma + \mu)J_{t+1}(A(b, x, y)) + (1 - \gamma)(1 - \mu)J_{t+1}(B(b, x, y)).$$

This implies that when the other type manipulates, then the first type obtains the highest payoff in period t . We have modified the computer program to choose one particular regime whenever there are multiple equilibria and compare the payoffs from always choosing Regime H , Regime L , and Regime HL whenever there are multiple equilibria. Our simulation also confirms the theoretical observation, although it remains unknown whether in the continuous time limit, the value functions from the different sequence of choosing different regimes converge to one value function or remain different.

Second, the existence of a unique equilibrium in the Kyle model remains an open question in the literature. As shown in Back and Baruch (2004), the equilibrium in the Glosten–Milgrom model converges to that in the Kyle model. Our analysis shows that there is a possibility of multiple equilibria. We could use our analysis to understand how a unique equilibrium in the dynamic Glosten–Milgrom model converges to the equilibrium in the Kyle model.

Third, one may question whether the market maker’s belief concerning the risky asset’s payoff converges to the truth as the number of trading periods tends to infinity. Recently, there has been renewed interest in private information and learning. Examples include Golosov *et al.* (2009) and Loertscher and McLennan (2018). Although their settings are quite different from ours, both question whether uninformed agents discover private information. In the market microstructure literature, Glosten and Milgrom (1985) originally show that such convergence is obtained almost certainly if the only available trade size is the unit trade size and an informed trader can trade only once. Ozsoylev and Takayama (2010) show a similar result where the informed trader can trade only once but in multiple sizes. We expect that this result will also hold in our framework, as an intuition similar to the Martingale convergence theorem holds when the equilibrium is unique. However, as there is a possibility that multiple equilibria will arise, and especially that both types of trader will manipulate simultaneously, it would be interesting to see how this type of manipulation affects the market’s learning.

Appendix I. Lemmas and Proofs

In Appendix I, we fix $b \in (0, 1)$ and $\sigma = (\sigma_H, \sigma_L) \in \mathcal{E}_t(b)$, and let $\alpha = A(b, \sigma_H, \sigma_L)$ and $\beta = B(b, \sigma_H, \sigma_L)$.

Proof of Theorem 1. For $(\sigma_H, \sigma_L) \in [0, 1]^2$ let $B(\sigma_H, \sigma_L)$ be the pair of posterior beliefs given by (E1). Evidently $B : [0, 1]^2 \rightarrow [0, 1]^2$ is a continuous function. For $(\alpha, \beta) \in [0, 1]^2$ let $BR^b(\alpha, \beta)$ be the set of pairs (σ_H, σ_L) satisfying (E3). Given that J_{t-1} and V_{t-1} are continuous, BR^b is an upper semicontinuous correspondence. Its value is always a Cartesian product of two elements of the set $\{\{0\}, [0, 1], \{1\}\}$, so it is convex valued. The composition $BR^b \circ B$ is thus an upper semicontinuous convex-valued correspondence, so Kakutani's fixed point theorem implies that it has a fixed point.

Suppose that a fixed point (σ_H, σ_L) satisfying (E3) does not satisfy (E2) and then $\alpha < \beta$. Then \mathbf{M}_{t-1} gives:

$$\begin{aligned} 1 - \alpha + J_{t-1}(\alpha) &> \beta - 1 + J_{t-1}(\beta); \\ -\alpha + V_{t-1}(\alpha) &< \beta + V_{t-1}(\beta). \end{aligned} \quad (3)$$

Now (E3) implies that $\sigma_H = 1$ and $\sigma_L = 1$, and Bayes' rule gives $\alpha > b > \beta$, a contradiction. Thus (σ_H, σ_L) must satisfy (E2). \square

Lemma I-1. $A(b, x, y)$ is a strictly increasing function of x and y , $B(b, x, y)$ is a strictly decreasing function of x and y , and $A(b, x, y) > (=, <) B(x, y)$ if and only if $x + y > (=, <) 1$. If $x + y > 1$, then

$$0 < \frac{\partial A}{\partial x}(b, x, y) < \frac{\partial A}{\partial y}(b, x, y) \quad \text{and} \quad 0 > \frac{\partial B}{\partial y}(b, x, y) > \frac{\partial B}{\partial x}(b, x, y).$$

Proof. The monotonicity claims are obvious and it is easy to see that $A(b, z, 1 - z) = b = B(b, z, 1 - z)$. If $x + y > 1$, then

$$A(b, x, y) > A(b, x, 1 - x) = b = B(b, x, 1 - x) > B(b, x, y)$$

by monotonicity, and similarly if $x + y < 1$. The claims concerning the partial derivatives can be verified easily by computing them. \square

Proof of Proposition 1. Suppose that $t > 1$. If the type- H trader manipulates, it must be the case that $1 - \alpha + J_{t-1}(\alpha) = \beta - 1 + J_{t-1}(\beta)$, so we have:

$$\frac{J_{t-1}(\alpha) - J_{t-1}(\beta)}{\alpha - \beta} = \frac{\alpha + \beta - 2}{\alpha - \beta} = -1 - \frac{2 - 2\alpha}{\alpha - \beta}. \quad (4)$$

Similarly, if the type- L trader manipulates, we have:

$$\frac{V_{t-1}(\alpha) - V_{t-1}(\beta)}{\alpha - \beta} = \frac{\alpha + \beta}{\alpha - \beta} = 1 + \frac{2\beta}{\alpha - \beta}. \quad (5)$$

Then (4) and (5) indicate the required results. \square

Proposition I-1. *The period 1 value functions $\{J_1, V_1\}$ satisfy C_1 , M_1 , SH_1 and SL_1 .*

Proof. Note that in period 1, Regime \emptyset arises in the whole interval $[0, 1]$ as there is no chance to re-trade. That σ_{H1} and σ_{L1} are identically one is an immediate consequence of optimization, and the equations are derived by substituting and simplifying.

In any Markov equilibrium, σ_{H1} and σ_{L1} are identically one, and:

$$\alpha_1(b) = \frac{b(\mu + (1 - \mu)\gamma)}{\mu b + (1 - \mu)\gamma} \text{ and } \beta_1(b) = \frac{b(1 - \mu)(1 - \gamma)}{\mu(1 - b) + (1 - \mu)(1 - \gamma)};$$

$$J_1(b) = \mu(1 - \alpha_t(b)) = \frac{\mu(1 - b)(1 - \mu)\gamma}{\mu b + (1 - \mu)\gamma} \text{ and } V_1(b) = \mu\beta_t(b) = \frac{\mu b(1 - \mu)(1 - \gamma)}{\mu(1 - b) + (1 - \mu)(1 - \gamma)}.$$

The four properties follow as a simple consequence of the above formulations. \square

Define the difference in payoffs between trading for and against the information as follows:

$$D_H(b, x, y) = -A(b, x, y) + J_{t-1}(A(b, x, y)) - B(b, x, y) - J_{t-1}(B(b, x, y)) + 2;$$

$$D_L(b, x, y) = B(b, x, y) + V_{t-1}(B(b, x, y)) + A(b, x, y) - V_{t-1}(A(b, x, y)).$$

For each $\theta \in \{H, L\}$, define x^θ and y^θ by:

$$x^\theta := \begin{cases} \min\{x : D_\theta(b, x, 1) = 0\} & \text{if } \{x : D_\theta(b, x, 1) = 0\} \neq \emptyset; \\ 1 & \text{otherwise,} \end{cases}$$

and

$$y^\theta := \begin{cases} \min\{y : D_\theta(b, 1, y) = 0\} & \text{if } \{y : D_\theta(b, 1, y) = 0\} \neq \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

Lemma I-2. *Suppose that C_{t-1} , M_{t-1} , SH_{t-1} and SL_{t-1} hold in period $t - 1$. If $0 < \bar{x}, \bar{y} \leq 1$, $\bar{x} + \bar{y} > 1$, $\theta \in \{H, L\}$ and $D_\theta(b, \bar{x}, \bar{y}) = 0$, then*

- *the payoff difference $D_\theta(b, x, \bar{y})$ is strictly decreasing as x increases for all $x \geq \bar{x}$;*
- *the payoff difference $D_\theta(b, \bar{x}, y)$ is strictly decreasing as y increases for all $y \geq \bar{y}$.*

Proof of Lemma I-2. By symmetry it suffices to prove that $D_L(b, x, \bar{y})$ and $D_H(b, x, \bar{y})$ are decreasing in x . First, consider $D_L(b, x, \bar{y})$. For a notational simplicity, in this proof, let $A(x) = A(b, x, \bar{y})$ and $B(x) = B(b, x, \bar{y})$. By Bayes' rule, $A(\bar{x}) > B(b, \bar{x}, \bar{y})$. Dividing the indifference condition for the type- L by $A(\bar{x}) - B(b, \bar{x}, \bar{y})$ gives

$$\frac{V_{t-1}(A(\bar{x})) - V_{t-1}(B(\bar{x}))}{A(\bar{x}) - B(\bar{x})} = \frac{A(\bar{x}) + B(\bar{x})}{A(\bar{x}) - B(\bar{x})} > 1.$$

This shows that $A(\bar{x}) > b_L$. By Bayes' rule, $A(x)$ is monotonically increasing in x , so for any $x \geq \bar{x}$ and $\Delta > 0$ we have $A(x), A(x + \Delta) > b_L$ and consequently

$$\begin{aligned} & (V_{t-1}(A(x + \Delta)) - A(x + \Delta)) - (V_{t-1}(A(x)) - A(x)) \\ &= (A(x + \Delta) - A(x)) \left(\frac{V_{t-1}(A(x + \Delta)) - V_{t-1}(A(x))}{A(x + \Delta) - A(x)} - 1 \right) > 0. \end{aligned}$$

That is, $V_{t-1}(A(x)) - A(x)$ is an increasing function of x . However, Bayes' rule and the monotonicity condition \mathbf{M}_{t-1} imply that $B(x) + V_{t-1}(B(x))$ is a decreasing function of x . As

$$D_L(b, x, \bar{y}) = B(x) + V_{t-1}(B(x)) + A(x) - V_{t-1}(A(x)),$$

the result follows. Second, consider $D_H(b, x, \bar{y})$. Similarly, dividing the indifference condition for the type- H by $A(\bar{x}) - B(\bar{x})$ gives

$$\frac{J_{t-1}(A(\bar{x})) - J_{t-1}(B(\bar{x}))}{A(\bar{x}) - B(\bar{x})} = \frac{A(\bar{x}) + B(\bar{x}) - 2}{A(\bar{x}) - B(\bar{x})} < -1.$$

This shows that $B(\bar{x}) < b_H$. Therefore, by using Proposition 1 and the same logic as the first case, we obtain the desired result. \square

For each $\theta \in \{H, L\}$, define $\tilde{y}_\theta : [0, 1] \rightarrow [0, 1]$ by $D_\theta(b, x, \tilde{y}_\theta(x)) = 0$ if $D_\theta(b, 1, 1) \leq 0$.

Lemma I-3. *For each $\theta \in \{H, L\}$, \tilde{y}_θ is continuous and is strictly decreasing in x .*

Proof of Lemma I-3. Suppose that \tilde{y}_θ is well defined. The continuity of D_θ indicates that \tilde{y}_θ is also continuous for each $\theta \in \{H, L\}$. Suppose that $x_1 > x_2$ and $\tilde{y}_L(x_1) \geq \tilde{y}_L(x_2)$. By Lemma I-2:

$$0 = D_L(b, x_1, \tilde{y}_L(x_1)) < D_L(b, x_2, \tilde{y}_L(x_1)) \leq D_L(b, x_2, \tilde{y}_L(x_2)),$$

which is a contradiction to $0 = D_L(b, x_2, \tilde{y}_L(x_2))$. By symmetry, the same holds for \tilde{y}_H . \square

Proposition I-2. *Suppose that \mathbf{C}_{t-1} , \mathbf{M}_{t-1} , \mathbf{SH}_{t-1} and \mathbf{SL}_{t-1} hold in period $t - 1$. The following holds:*

- (a) *if Regime HL arises at b , then $D_L(b, 1, 1) < 0$ and $D_H(b, 1, 1) < 0$;*
- (b) *if $D_L(b, 1, 1) \geq 0$ and $D_H(b, 1, 1) \geq 0$, then $\mathcal{E}_t(b) = \{(1, 1)\}$, so that only Regime \emptyset arises at b ;*
- (c) *if $D_L(b, 1, 1) < 0$ and $D_H(b, 1, 1) \geq 0$, then $\mathcal{E}_t(b)$ is a singleton whose unique element is in Regime L;*

(d) if $D_H(b, 1, 1) < 0$ and $D_L(b, 1, 1) \geq 0$, then $\mathcal{E}_t(b)$ is a singleton whose unique element is in Regime H ;

(e) if $D_H(b, 1, 1) < 0$ and $D_L(b, 1, 1) < 0$, then at most one element within Regime H is in $\mathcal{E}_t(b)$ and at most one element within Regime L is in $\mathcal{E}_t(b)$.

Proof of Proposition I-2.

Proof of (a). Suppose that Regime HL arises at b . Then, there must exist $(\bar{x}, \bar{y}) \in \mathcal{E}(b)$ with $\bar{x} < 1$ and $\bar{y} < 1$, such that $D_H(b, \bar{x}, \bar{y}) = 0$ and $D_L(b, \bar{x}, \bar{y}) = 0$. By Lemma I-2, we have:

$$0 = D_L(b, \bar{x}, \bar{y}) > D_L(b, \bar{x}, 1) > D_L(b, 1, 1). \quad (6)$$

By symmetry, we can also prove $0 > D_L(b, 1, 1)$. \square

Proof of (b). By (a) of this proposition, Regime HL does not arise. Aiming at a contradiction, suppose that Regime H arises. Then there exists an $\bar{x} < 1$ to satisfy $D_H(b, \bar{x}, 1) = 0$. Then, by Lemma I-2, we must have $D_H(b, 1, 1) < 0$, which contradicts our assumption. By symmetry, we can prove that Regime L does not arise. \square

Proof of (c). First, as $D_L(b, 1, 1) < 0$ and $D_H(b, 1, 1) \geq 0$, Regime \emptyset does not arise because taking an honest strategy is not optimal for the low type. Furthermore, by (a) of this proposition, Regime HL does not arise. Now suppose that Regime H arises. Then there exists an $\bar{x} < 1$ to satisfy $D_H(b, \bar{x}, 1) = 0$. Then, by Lemma I-2, we must have $D_H(b, 1, 1) < 0$, which contradicts our assumption.

Lemma I-3 indicates that there is no $y < 1$ to satisfy $D_H(b, 1, y) = 0$. As D_H is continuous in y , $D_H(b, 1, y) \geq 0$ must hold for all $y \in [0, 1]$. Notice that for any $\sigma \in [0, 1]$ and prior $b \in [0, 1]$, by Bayes' rule, both the bid and ask prices are equal to b . Thus, we have:

$$D_L(b, \sigma, 1 - \sigma) > 0 \quad \text{and} \quad D_H(b, \sigma, 1 - \sigma) > 0. \quad (7)$$

Now, as $D_L(b, 1, 1) < 0$ and $D_H(b, 1, 1) \geq 0$, (7) and Lemma I-3 imply that there exists a $\bar{y} < 1$ to satisfy

$$D_L(b, 1, \bar{y}) = 0 \quad \text{and} \quad D_H(b, 1, \bar{y}) \geq 0.$$

Therefore, we can see that Regime L arises. In addition, by Lemma I-2, there is only one \bar{y} to satisfy $D_L(b, 1, \bar{y}) = 0$. \square *Proof of (d).* Done symmetrically with (c) of this proposition. \square

Proof of (e). Suppose that there is one element in $\mathcal{E}_t(b)$ that belongs to Regime H . Then, by Lemma I-2, there is no other element in $\mathcal{E}_t(b)$ that also belongs to Regime H . Symmetrically the same holds for Regime L . \square

Lemma I-4. Suppose that C_{t-1} , M_{t-1} , SH_{t-1} and SL_{t-1} hold in period $t-1$. Furthermore, suppose that $D_H(b, 1, 1) < 0$, and $D_L(b, 1, 1) < 0$, while $D_H(b, x, y) = 0$ and $D_L(b, x, y) = 0$ do not intersect. Then

- either Regime H or Regime L arises, uniquely;
- if $x^L > x^H$, then $(\sigma_H, \sigma_L) = (x^H, 1)$, and if $x^L < x^H$, then $(\sigma_H, \sigma_L) = (1, y^L)$;
- bid and ask prices are monotonically increasing in b in period t .

Proof of Lemma I-4. Suppose that the two curves do not intersect. By symmetry and continuity, we can assume that

$$x^L > x^H \quad \text{and} \quad y^L > y^H. \quad (8)$$

Then, by setting $\bar{x} = 1$ and $\bar{y} = 1$ in Lemma I-2, we obtain:

$$\begin{aligned} D_L(b, x^H, 1) > 0 \quad & \text{and} \quad D_H(b, x^H, 1) = 0 \\ D_L(b, 1, y^L) = 0 \quad & \text{and} \quad D_H(b, 1, y^L) < 0. \end{aligned} \quad (9)$$

Therefore, we conclude that Regime H arises. Notice that Regime L does not arise because of the second line of (9), and Regime does not arise as the honest strategy is not optimal. Moreover, by Lemma I-2, there is no other x except for x^H to satisfy $D_H(b, x, 1) = 0$. This completes the proof and we can prove the result for the second case symmetrically.

Now, we will prove the last statement. When nobody manipulates, by the Bayes' rule we can show that bid and ask prices decrease as b decreases. Therefore, it remains to show that the result holds in Regime H and L . As the argument is symmetric, we only prove the result for Regime L . In Regime L , type- L 's indifference condition for σ is:

$$-\alpha_t(b) + V_{t-1}(\alpha_t(b)) = \beta_t(b) + V_{t-1}(\beta_t(b)).$$

Then

$$\partial_+ \alpha_t(b)(-1 + \partial_+ V_{t-1}(\alpha_t(b))) = \partial_+ \beta_t(b)(1 + \partial_+ V_{t-1}(\beta_t(b))). \quad (10)$$

By Proposition 1, $-1 + \partial_+ V_{t-1}(\alpha_t(b)) > 0$. By condition M_{t-1} , $(1 + \partial_+ V_{t-1}(\beta_t(b))) > 0$. Therefore (10) indicates that: $\partial_+ \alpha_t(b) > 0$ if and only if $\partial_+ \beta_t(b) > 0$. Let $\Delta = (1 - \mu)\gamma + \mu(1 - \sigma_{Lt}(b)) + \partial_+ \sigma_{Lt}(b)b(1 - b)$. Then, we have:

$$\partial_+ \alpha_t(b) = \frac{((1 - \mu)\gamma + \mu) \cdot \Delta}{(1 - \mu)\gamma + \mu b + \mu(1 - b)(1 - \sigma_{Lt}(b))} \quad \text{and} \quad \partial_+ \beta_t(b) = \frac{(1 - \mu)(1 - \gamma) \cdot (1 - \Delta)}{(1 - \mu)(1 - \gamma) + \mu(1 - b)\sigma_{Lt}(b)}.$$

Thus, if $\partial_+ \alpha_t(b) \leq 0$, we must have $\Delta < 0$, which implies $\partial_+ \beta_t(b) > 0$. We obtain a contradiction because (10) would not hold. \square

Define

$$A_{\tilde{\mu}}(b, x, y) = \frac{[(1 - \tilde{\mu})\gamma + \tilde{\mu}x]b}{(1 - \tilde{\mu})\gamma + \tilde{\mu}bx + \tilde{\mu}(1 - b)(1 - y)}$$

$$B_{\tilde{\mu}}(b, x, y) = \frac{[(1 - \tilde{\mu})(1 - \gamma) + \tilde{\mu}(1 - x)]b}{(1 - \tilde{\mu})(1 - \gamma) + \tilde{\mu}b(1 - x) + \tilde{\mu}(1 - b)y}.$$

Then compute that

$$A_{\tilde{\mu}}(b, x, y) - b = \frac{\tilde{\mu}b(1 - b)(x - 1 + y)}{\tilde{\mu}[bx + (1 - b)(1 - y)] + (1 - \tilde{\mu})\gamma}; \quad (11)$$

$$b - B_{\tilde{\mu}}(b, x, y) = \frac{\tilde{\mu}b(1 - b)(x - 1 + y)}{\tilde{\mu}[b(1 - x) + (1 - b)y] + (1 - \tilde{\mu})(1 - \gamma)}. \quad (12)$$

Lemma I-5. *Take $x \in (0, 1]$ and $y \in (0, 1]$ and suppose that $x > 1 - y$. Suppose that ∂_+x and ∂_+y are bounded (see (1) for the definition of ∂_+).*

(a) *For every ϵ_A and ϵ_B , there is a $\bar{\mu}_0$ such that every $\tilde{\mu} < \bar{\mu}_0$ satisfies $A_{\tilde{\mu}}(b, x, y) - b < \epsilon_A$ and $b - B_{\tilde{\mu}}(b, x, y) < \epsilon_B$ for all $b \in [0, 1]$;*

(b) *For every ϵ_A and ϵ_B , there is a $\bar{\mu}_1$ such that every $\tilde{\mu} < \bar{\mu}_1$ satisfies $|\partial_+A_{\tilde{\mu}}(b, x, y) - 1| < \epsilon_A$ and $|\partial_+B_{\tilde{\mu}}(b, x, y) - 1| < \epsilon_B$ for all $b \in [0, 1]$.*

Proof of Lemma I-5. As $x > 1 - y$, for every $\tilde{\mu} \in (0, 1)$,

$$\frac{b(1-b)(x-1+y)}{\max\{bx+(1-b)(1-y), \gamma\}} < \frac{b(1-b)(x-1+y)}{\tilde{\mu}[bx+(1-b)(1-y)]+(1-\tilde{\mu})\gamma} < \frac{b(1-b)(x-1+y)}{\min\{bx+(1-b)(1-y), \gamma\}}, \quad (13)$$

$$\frac{b(1-b)(x-1+y)}{\max\{bx+(1-b)(1-y), 1-\gamma\}} < \frac{b(1-b)(x-1+y)}{\tilde{\mu}[b(1-x)+(1-b)y]+(1-\tilde{\mu})(1-\gamma)} < \frac{b(1-b)(x-1+y)}{\min\{bx+(1-b)(1-y), 1-\gamma\}}.$$

By (11) and (12), we can see that $A_{\tilde{\mu}}(b, x, y) - b$ and $b - B_{\tilde{\mu}}(b, x, y)$ are smaller and greater than $\tilde{\mu}$ multiplied by the term, which is unaffected by $\tilde{\mu}$. Because $A_{\tilde{\mu}}(b, x, y) - b$ and $b - B_{\tilde{\mu}}(b, x, y)$ are continuous with respect to $\tilde{\mu} \in (0, 1)$, we obtain the first statement.

Second, note that:

$$\begin{aligned} \partial_+A_{\tilde{\mu}}(b, x, y) &= \frac{[(1-\tilde{\mu})\gamma + \tilde{\mu}(x+\partial_+x)]}{(1-\tilde{\mu})\gamma + \tilde{\mu}(bx+(1-b)(1-y))} - \frac{\tilde{\mu}b[(1-\tilde{\mu})\gamma + \tilde{\mu}x][x+y-1+b\partial_+x-(1-b)\partial_+y]}{[(1-\tilde{\mu})\gamma + \tilde{\mu}(bx+(1-b)(1-y))]^2}; \\ \partial_+B_{\tilde{\mu}}(b, x, y) &= \frac{[(1-\tilde{\mu})(1-\gamma) + \tilde{\mu}(1-x+\partial_+x)]}{(1-\tilde{\mu})(1-\gamma) + \tilde{\mu}(b(1-x)+(1-b)y)} - \frac{\tilde{\mu}b[(1-\tilde{\mu})(1-\gamma) + \tilde{\mu}(1-x)][1-x-y-b\partial_+x+(1-b)\partial_+y]}{[(1-\tilde{\mu})(1-\gamma) + \tilde{\mu}(b(1-x)+(1-b)y)]^2}. \end{aligned} \quad (14)$$

By symmetry, we only prove the result for $\partial_+A_{\tilde{\mu}}(b, x, y)$. By (14) we have:

$$\begin{aligned} \partial_+A_{\tilde{\mu}}(b, x, y) &= \frac{[(1-\tilde{\mu})\gamma]}{(1-\tilde{\mu})\gamma + \tilde{\mu}(bx+(1-b)(1-y))} \\ &\quad + \tilde{\mu} \cdot \left(\frac{[(x+\partial_+x)]}{(1-\tilde{\mu})\gamma + \tilde{\mu}(bx+(1-b)(1-y))} - \frac{b[(1-\tilde{\mu})\gamma + \tilde{\mu}x][x+y-1+b\partial_+x-(1-b)\partial_+y]}{[(1-\tilde{\mu})\gamma + \tilde{\mu}(bx+(1-b)(1-y))]^2} \right). \end{aligned} \quad (15)$$

Note that the second term in (15) is greater than:

$$\frac{[(x+\partial_+x)]}{\max\{\gamma, bx+(1-b)(1-y)\}} - \frac{b[\max\{\gamma, x\}][x+y-1+b\partial_+x-(1-b)\partial_+y]}{(\min\{\gamma, bx+(1-b)(1-y)\})^2}, \quad (16)$$

and smaller than

$$\frac{[(x+\partial_+x)]}{\min\{\gamma, bx+(1-b)(1-y)\}} - \frac{b[\min\{\gamma, x\}][x+y-1+b\partial_+x-(1-b)\partial_+y]}{(\max\{\gamma, bx+(1-b)(1-y)\})^2}, \quad (17)$$

For a sufficiently small μ , the first term in (15) is arbitrarily close to 1 and by applying a similar logic as that in the proof of the first statement to (16) and (17) which are unaffected by $\tilde{\mu}$, the second term in (15) is arbitrarily close to 0. Because $\partial_+A_{\tilde{\mu}}(b, x, y)$ is continuous with respect to $\tilde{\mu} \in (0, 1)$, we obtain the second statement. \square

Lemma I-6. Take $x \in (0, 1]$ and $y \in (0, 1]$ and suppose that $x > 1 - y$.

- If $r < 1$, for every ϵ , there is a $\bar{\mu}_2$ such that every $\tilde{\mu} < \bar{\mu}_2$ satisfies $\frac{A_{\tilde{\mu}}(b, x, y) - B_{\tilde{\mu}}(b, x, y)}{\tilde{\mu}^r} < \epsilon$.
- If $r > 1$, for every M , there is a $\bar{\mu}_3$ such that every $\tilde{\mu} < \bar{\mu}_3$ satisfies $\frac{A_{\tilde{\mu}}(b, x, y) - B_{\tilde{\mu}}(b, x, y)}{\tilde{\mu}^r} > M$.

Proof of Lemma I-6. By using (11) and (12), for every $\tilde{\mu} \in (0, 1)$, we can compute $\frac{A_{\tilde{\mu}}(b, x, y) - B_{\tilde{\mu}}(b, x, y)}{\tilde{\mu}^r}$, which equals

$$\tilde{\mu}^{1-r} \times \left(\frac{b(1-b)(x-1+y)}{\tilde{\mu}[bx+(1-b)(1-y)]+(1-\tilde{\mu})\gamma} + \frac{b(1-b)(x-1+y)}{\tilde{\mu}[b(1-x)+(1-b)y]+(1-\tilde{\mu})(1-\gamma)} \right). \quad (18)$$

By (18) and (13), we can see that $\frac{A_{\tilde{\mu}}(b, x, y) - B_{\tilde{\mu}}(b, x, y)}{\tilde{\mu}^r}$ is smaller and greater than $\tilde{\mu}^{1-r}$ multiplied by the sum of the two terms, which are unrelated to $\tilde{\mu}$. When $\tilde{\mu}$ is sufficiently small, $\tilde{\mu}^{1-r}$ is sufficiently large for $r > 1$, while it is sufficiently small for $r < 1$. Note that $\frac{A_{\tilde{\mu}}(b, x, y) - B_{\tilde{\mu}}(b, x, y)}{\tilde{\mu}^r}$ is continuous with respect to $\tilde{\mu} \in (0, 1)$. Thus, we obtain the desired results. \square

Proposition I-3. Suppose that there exists a pair of (\hat{x}, \hat{y}) to satisfy both of the indifference conditions.

If for every $x \in [x^H, 1] \cap [x^L, 1]$,

$$-\frac{(1 + \partial_- V_{t-1}(B(b, x, \tilde{y}_L(x))))}{(1 - \partial_+ V_{t-1}(A(b, x, \tilde{y}_L(x))))} > -\frac{(1 + \partial_- J_{t-1}(B(b, x, \tilde{y}_H(x))))}{(1 - \partial_+ J_{t-1}(A(b, x, \tilde{y}_H(x))))},$$

then $x^L > x^H$ and $y^L < y^H$, $(\sigma_{Ht}(b), \sigma_{Lt}(b)) \in \{(x^H, 1), (1, y^L), (\hat{x}, \hat{y})\}$, and Regime H, Regime L, and Regime HL arise in period t at b .

Proof of Proposition I-3. For each $\theta \in \{H, L\}$,

$$\frac{D_\theta(b, \bar{x}+\epsilon, \tilde{y}_\theta(\bar{x}+\epsilon)) - D_\theta(b, \bar{x}, \tilde{y}_\theta(\bar{x}))}{\epsilon} = \frac{D_\theta(b, \bar{x}+\epsilon, \tilde{y}_\theta(\bar{x}+\epsilon)) - D_\theta(b, \bar{x}, \tilde{y}_\theta(\bar{x}))}{\epsilon} + \frac{D_\theta(b, \bar{x}, \tilde{y}_\theta(\bar{x}+\epsilon)) - D_\theta(b, \bar{x}, \tilde{y}_\theta(\bar{x}))}{\tilde{y}_\theta(\bar{x}+\epsilon) - \tilde{y}_\theta(\bar{x})} \cdot \frac{\tilde{y}_\theta(\bar{x}+\epsilon) - \tilde{y}_\theta(\bar{x})}{\epsilon} = 0$$

Thus, we obtain:

$$\frac{\tilde{y}_\theta(\bar{x} + \epsilon) - \tilde{y}_\theta(\bar{x})}{\epsilon} = -\frac{D_\theta(b, \bar{x}+\epsilon, \tilde{y}_\theta(\bar{x}+\epsilon)) - D_\theta(b, \bar{x}, \tilde{y}_\theta(\bar{x}+\epsilon))}{\frac{D_\theta(b, \bar{x}, \tilde{y}_\theta(\bar{x}+\epsilon)) - D_\theta(b, \bar{x}, \tilde{y}_\theta(\bar{x}))}{\tilde{y}_\theta(\bar{x}+\epsilon) - \tilde{y}_\theta(\bar{x})}}. \quad (19)$$

where monotonicity of D_θ guarantees $\frac{D_\theta(b, \bar{x}, \tilde{y}_\theta(\bar{x} + \epsilon)) - D_\theta(b, \bar{x}, \tilde{y}_\theta(\bar{x}))}{\tilde{y}_\theta(\bar{x} + \epsilon) - \tilde{y}_\theta(\bar{x})}$ is nonzero.

Therefore, by (19) we have:

$$\partial_+ \tilde{y}_H(\bar{x}) = - \frac{\left[-\frac{dA(b, \bar{x}, \bar{y})}{d\bar{x}} (1 - \partial_+ J_{t-1}(A(b, \bar{x}, \bar{y}))) - \frac{dB(b, \bar{x}, \bar{y})}{d\bar{x}} (1 + \partial_- J_{t-1}(B(b, \bar{x}, \bar{y}))) \right]}{\left[-\frac{dA(b, \bar{x}, \bar{y})}{d\bar{y}} (1 - \partial_+ J_{t-1}(A(b, \bar{x}, \bar{y}))) - \frac{dB(b, \bar{x}, \bar{y})}{d\bar{y}} (1 + \partial_- J_{t-1}(B(b, \bar{x}, \bar{y}))) \right]} \quad (20)$$

and

$$\partial_+ \tilde{y}_L(\bar{x}) = - \frac{\left[\frac{dB(b, \bar{x}, \bar{y})}{d\bar{x}} (1 + \partial_- V_{t-1}(B(b, \bar{x}, \bar{y}))) + \frac{dA(b, \bar{x}, \bar{y})}{d\bar{x}} (1 - \partial_+ V_{t-1}(A(b, \bar{x}, \bar{y}))) \right]}{\left[\frac{dB(b, \bar{x}, \bar{y})}{d\bar{y}} (1 + \partial_- V_{t-1}(B(b, \bar{x}, \bar{y}))) + \frac{dA(b, \bar{x}, \bar{y})}{d\bar{y}} (1 - \partial_+ V_{t-1}(A(b, \bar{x}, \bar{y}))) \right]}. \quad (21)$$

Now,

$$\begin{aligned} \frac{dA(b, x, y)}{dx} &= \frac{\left[\frac{(1-\mu)}{\mu} \gamma + (1-y) \right] b(1-b)}{\left[\frac{(1-\mu)}{\mu} \gamma + bx + (1-b)(1-y) \right]^2} & \text{and} & \frac{dB(b, x, y)}{dx} &= -\frac{\left[\frac{(1-\mu)}{\mu} (1-\gamma) + y \right] b(1-b)}{\left[\frac{(1-\mu)}{\mu} (1-\gamma) + b(1-x) + (1-b)y \right]^2}; \\ \frac{dA(b, x, y)}{dy} &= \frac{\left[\frac{(1-\mu)}{\mu} \gamma + x \right] b(1-b)}{\left[\frac{(1-\mu)}{\mu} \gamma + bx + (1-b)(1-y) \right]^2} & \text{and} & \frac{dB(b, x, y)}{dy} &= -\frac{\left[\frac{(1-\mu)}{\mu} (1-\gamma) + (1-x) \right] b(1-b)}{\left[\frac{(1-\mu)}{\mu} (1-\gamma) + b(1-x) + (1-b)y \right]^2}. \end{aligned}$$

Notice that as $x > 1 - y$ in equilibrium, we have:

$$\begin{aligned} \frac{dA(b, x, y)}{dy} &= \frac{\left[\frac{(1-\mu)}{\mu} \gamma + x \right]}{\left[\frac{(1-\mu)}{\mu} \gamma + bx + (1-b)(1-y) \right]^2} > \frac{\left[\frac{(1-\mu)}{\mu} \gamma + (1-y) \right]}{\left[\frac{(1-\mu)}{\mu} \gamma + bx + (1-b)(1-y) \right]^2} = \frac{dA(b, x, y)}{dx}; \\ -\frac{dB(b, x, y)}{dy} &= \frac{\left[\frac{(1-\mu)}{\mu} (1-\gamma) + (1-x) \right]}{\left[\frac{(1-\mu)}{\mu} (1-\gamma) + bx + (1-b)y \right]^2} < \frac{\left[\frac{(1-\mu)}{\mu} (1-\gamma) + y \right]}{\left[\frac{(1-\mu)}{\mu} (1-\gamma) + bx + (1-b)y \right]^2} = -\frac{dB(b, x, y)}{dx}. \end{aligned} \quad (22)$$

Substituting the derivatives for A and B into (20) and (21), and using (22), we obtain $-\partial_+ \tilde{y}_H(\bar{x}) < -\partial_+ \tilde{y}_L(\bar{x})$ for any \bar{x} in the region, which implies $-\frac{y^H - \hat{y}}{1 - \hat{x}} < -\frac{y^L - \hat{y}}{1 - \hat{x}}$. Thus, we obtain $y^L < y^H$ and similarly we can obtain $x^L > x^H$. When Regime HL may arise, $D_H(b, 1, y^L) > 0$ and $D_L(b, x^H, 1) > 0$. So, Regime H and Regime L arise. \square

Lemma I-7. *Suppose that C_{t-1} , M_{t-1} , DH_{t-1} and DL_{t-1} hold in period $t - 1$. Then there are ϵ_H and ϵ_L such that the value functions $\{J_t, V_t\}$ satisfy C_t , M_t , DH_t , DL_t , SH_t and SL_t in period t .*

Proof.

Part C_t . To show that the value functions satisfy C_t , we will show

- (a) a period- t strategy σ_t is continuous and piecewise differentiable in b on $(0, 1)$;
- (b) ask and bid prices α_t and β_t are continuous and piecewise differentiable in b for all $b \in [0, 1]$;
- (c) the value functions $\{J_t, V_t\}$ satisfy C_t in period t .

First, we prove the continuity of the equilibrium strategies, the bid and ask prices and the value functions. Proposition I-2 ensures that the equilibrium strategy is unique within each regime in period t . Now, \mathcal{E}_t is a function of prior belief b . By the proof of Theorem 1, the equilibrium correspondence \mathcal{E}_t is upper semicontinuous. Therefore, we conclude that it is continuous within each regime. Now take a sequence $\{b^k\}$ that converges to b . Suppose that for a sufficiently small ϵ , Regime L arises at $b^k - \epsilon$ and Regime \emptyset arises at b . Take a sequence of the equilibrium strategy at each b^k in Regime L , which we denote by $\{\hat{\sigma}^k\}$. We assert that $\hat{\sigma}^k$ converges to the equilibrium strategy in Regime \emptyset as b^k goes to b . Suppose not, and then there is a distinct strategy $\hat{\sigma}$ with $\hat{\sigma}_L \neq 1$ and $\hat{\sigma}_H = 1$ at b . Then,

$$D_L(b, \hat{\sigma}_L, 1) = 0 \text{ and } D_L(b, 1, 1) \geq 0.$$

This is a contradiction, given Lemma I-2, as $\hat{\sigma}_L < 1$. By symmetry, we can prove that at the boundary belief where Regime H shifts to Regime \emptyset as b changes, the equilibrium strategy is continuous. Therefore, by Bayes' rule, the bid and ask prices are also continuous. As such, both value functions are a sum of the continuous functions in b ; i.e., the bid and ask prices, next-period value functions, and current-period value functions are continuous.

To prove that piecewise differentiability for the equilibrium strategies, the bid and ask prices and the value functions holds, note that by continuity, each function σ_{Ht} or σ_{Lt} does not have a jump. Therefore, for some interval, if they are not equal to one, the period- t equilibrium strategy σ_H or σ_L solves each of the indifference conditions.

Obviously, if they are constant at one, they are differentiable. By applying the implicit function theorem to the indifference conditions, σ_H or σ_L are piecewise differentiable in terms of b . Bid and ask prices are continuous and piecewise differentiable in terms of b or σ_H or σ_L . Therefore, we conclude that the bid and ask prices are piecewise differentiable. For the same reason as the proof for the continuity in (c), the result follows.

Part M_t. As the argument is symmetric, we only prove the case of the low type. Note that by Lemma I-4 and our induction hypothesis, the next-period value function and bid and ask prices are all monotonically increasing, starting at the origin. Thus, the summation of these functions is also monotonic. The case for J_t is proved similarly.

Part DH_t and DL_t. Note that

$$\begin{aligned} \partial_+ V_t(b) &= \mu \partial_+ \beta_t(b) + \mu \partial_+ \beta_t(b) \partial_+ V_{t-1}(\beta_t(b)) \\ &\quad + (1 - \mu) (\gamma \partial_+ \alpha_t(b) \partial_+ V_{t-1}(\alpha_t(b)) + (1 - \gamma) \partial_+ \beta_t(b) \partial_+ V_{t-1}(\beta_t(b))). \end{aligned}$$

When $x = \sigma_H$ and $y = \sigma_L$, $\alpha_t(b) = A_\mu(b, x, y)$ and $\beta_t(b) = B_\mu(b, x, y)$. By the first statement of this lemma, σ_t is piecewise-differentiable. Thus $\partial_+ \sigma_H$ and $\partial_+ \sigma_L$ are well defined. Thus $\partial_+ \alpha_t(b) =$

$\partial_+ A_\mu(b, x, y)$ and $\partial_+ \beta_t(b) = \partial_+ B_\mu(b, x, y)$. Our induction hypothesis \mathbf{DL}_{t-1} , together with (a) and (b) of Lemma I-5, completes the proof for $\partial_+ V_t(b)$. By symmetry, we can also prove the statement for $\partial_+ J_t(b)$.

Part SH_t and SL_t. If $\mu t \geq 1 + \delta_L(\mu)$, then by \mathbf{DL}_t , $\partial_+ V_t(b) > 1 + \delta_L(\mu) - \epsilon_L$ for every $b \in (0, 1)$. By choosing ϵ_L small enough, we obtain $\partial_+ V_t(b) \geq 1 + \delta_L(\mu)$ for every b in every period t . However, if $\mu t < 1 + \delta_L(\mu)$, then $\partial_+ V_t(b) < 1 + \delta_L(\mu) - \epsilon_L$ for every $b \in (0, 1)$. Then we obtain $\partial_+ V_t(b) < 1 + \delta_L(\mu)$ for every $b \in (0, 1)$. In this way, we obtain \mathbf{SL}_t . \square

Proof of Theorem 2. By Lemma I-5, because $J_1(b) = \mu(1 - A(b, 1, 1))$ and $V_1(b) = \mu B(b, 1, 1)$, for every ϵ_H and ϵ_L , there is a μ such that \mathbf{DH}_1 and \mathbf{DL}_1 hold. By Lemma I-7, when there is a unique equilibrium in period t , recursively the value functions in period t satisfy the properties for tame equilibrium and the monotonicity of prices is proved by Lemma I-4. Thus, we will show that the tame equilibrium satisfies the three statements.

Fix $r > 0$ and μ . Fix $t \in \{2, \dots, \lfloor \frac{1}{\mu^r} \rfloor\}$. Suppose that μ is small enough that \mathbf{C}_{t-1} , \mathbf{M}_{t-1} , \mathbf{DH}_{t-1} and \mathbf{DL}_{t-1} hold in period $t - 1$. We show the three claims.

Part 1: $r \in (0, 1]$. We show that Regime \emptyset arises and manipulation does not arise at any belief in period t . Proposition I-2's (a) indicates that if Regime \emptyset does not arise, then an honest strategy is not optimal for at least one type. For notational simplicity, we write:

$$\bar{A} := A(b, 1, 1) \quad \text{and} \quad \bar{B} := B(b, 1, 1).$$

Aiming to obtain a contradiction, by \mathbf{DL}_{t-1} suppose that there exists an arbitrarily small ϵ_A and ϵ_B for which the following holds:

$$\mu(t-1) (\bar{A} - \bar{B}) + \epsilon_A^L \bar{A} - \epsilon_B^L \bar{B} > (\bar{A} + \bar{B}). \quad (23)$$

As $t \in \{2, \dots, \lfloor \frac{1}{\mu^r} \rfloor\}$,

$$\mu \lfloor \frac{1}{\mu^r} \rfloor (\bar{A} - \bar{B}) + \epsilon_A^L \bar{A} - \epsilon_B^L \bar{B} > (\bar{A} + \bar{B}). \quad (24)$$

When $r \leq 1$, (24) does not hold as the left-hand side (LHS) is close to 0 by Lemma I-6 and the right-hand side (RHS) is strictly greater than 0 for $b \in (0, 1)$. This is a contradiction. By symmetry, we can also prove that $D_H(b, 1, 1) \geq 0$, and (a) in Proposition I-2 completes the first claim.

Part 2: $r \in (1, 2)$. We show that at some period $t \in \{\lfloor \frac{1}{\mu} \rfloor + 1, \dots, \lfloor \frac{1}{\mu^r} \rfloor\}$, Regime H arises at some belief b that is close to 1, and Regime L arises at some belief b that is close to 0. By (c) of Proposition I-2, Regime L arises if $D_L(b, 1, 1) < 0$ and $D_H(b, 1, 1) \geq 0$. For an arbitrarily small $\epsilon_{\bar{A}}$, $\epsilon_{\bar{B}}$, ϵ_A^L and ϵ_B^L , we can rewrite (23) as

$$\mu(t-1) (\bar{A} - \bar{B}) + \epsilon_A^L b(1 + \epsilon_{\bar{A}}) - \epsilon_B^L b(1 + \epsilon_{\bar{B}}) > b \cdot (2 + \epsilon_{\bar{A}} + \epsilon_{\bar{B}}).$$

Let $b = \mu$. The above inequality can be rewritten as

$$\mu(t-1)(\bar{A} - \bar{B}) + \epsilon_A^L \mu(1 + \epsilon_{\bar{A}}) - \epsilon_B^L \mu(1 + \epsilon_{\bar{B}}) > \mu \cdot (2 + \epsilon_{\bar{A}} + \epsilon_{\bar{B}}),$$

which indicates

$$(t-1)(\bar{A} - \bar{B}) > 2 + \epsilon_{\bar{A}} + \epsilon_{\bar{B}} - \epsilon_A^L(1 + \epsilon_{\bar{A}}) + \epsilon_B^L(1 + \epsilon_{\bar{B}}). \quad (25)$$

By the second statement of Lemma I-6, for $r > 1$, the LHS is large for $t \geq \frac{1}{\mu^r} + 1$, and the RHS is close to 2. Therefore, the above holds at some t , which gives us $D_L(b, 1, 1) < 0$. Now consider the type- H trader and

$$D_H(b, x, y) = 2 - A(b, x, y) - B(b, x, y) + J_{t-1}(A(b, x, y)) - J_{t-1}(B(b, x, y)), \quad (26)$$

can be separated into $J_{t-1}(B(b, x, y)) - J_{t-1}(A(b, x, y))$ and $2 - A(b, x, y) - B(b, x, y)$. Then because $\mu(t-1)(\bar{A} - \bar{B})$ approximates $J_{t-1}(B(b, x, y)) - J_{t-1}(A(b, x, y))$, when $r < 2$ and $t \leq \frac{1}{\mu^r}$,

$$\mu(t-1)(\bar{A} - \bar{B}) \leq \frac{\bar{A} - \bar{B}}{\mu^{r-1}} < 2,$$

where the last inequality is because of $r - 1 < 1$ and the first statement of Lemma I-6. Thus, (26) can be modified into the following relationship.

$$\mu(t-1)(\bar{A} - \bar{B}) + \epsilon_B^H \mu(1 + \epsilon_{\bar{B}}) - \epsilon_A^H \mu(1 + \epsilon_{\bar{A}}) < 2 - \mu \cdot (2 + \epsilon_{\bar{A}} + \epsilon_{\bar{B}}). \quad (27)$$

Therefore, $D_H(b, x, y) > 0$. Thus by (c) of Proposition I-2, we obtain the desired result. However, symmetrically we can prove that at $b = 1 - \mu$, $D_L(b, 1, 1) > 0$ and $D_H(b, 1, 1) < 0$.

Part 3: $r < 2$. We show that Regime HL never arises. Seeking a contradiction, suppose that in period t , there exist $\bar{\alpha}$ and $\bar{\beta}$ satisfying the two indifference conditions. Then, by substituting \mathbf{DH}_{t-1} and \mathbf{DL}_{t-1} into the indifference conditions, for sufficiently small ϵ_L 's and ϵ_H 's,

$$\begin{aligned} \mu(t-1)(\bar{\alpha} - \bar{\beta}) + \epsilon_L^{\bar{\alpha}} - \epsilon_L^{\bar{\beta}} &= \bar{\alpha} + \bar{\beta}; \\ -\mu(t-1)(\bar{\alpha} - \bar{\beta}) + \epsilon_H^{\bar{\alpha}} - \epsilon_H^{\bar{\beta}} &= \bar{\alpha} + \bar{\beta} - 2. \end{aligned} \quad (28)$$

Combining the two in (28) yields

$$\epsilon_L^{\bar{\alpha}} - \epsilon_L^{\bar{\beta}} + \epsilon_H^{\bar{\alpha}} - \epsilon_H^{\bar{\beta}} = 2(\bar{\alpha} + \bar{\beta}) - 2.$$

Thus, we obtain:

$$\mu(t-1)(\bar{\alpha} - \bar{\beta}) = 1 - \frac{\epsilon_L^{\bar{\alpha}} - \epsilon_L^{\bar{\beta}} + \epsilon_H^{\bar{\alpha}} - \epsilon_H^{\bar{\beta}}}{2} - (\epsilon_L^{\bar{\alpha}} - \epsilon_L^{\bar{\beta}}). \quad (29)$$

Note that as $t \in \{2, \dots, \lfloor \frac{1}{\mu^r} \rfloor\}$,

$$\mu \lfloor \frac{1}{\mu^r} \rfloor (\bar{\alpha} - \bar{\beta}) > \mu(t-1)(\bar{\alpha} - \bar{\beta}) \geq (\bar{\alpha} - \bar{\beta}). \quad (30)$$

By applying the squeeze theorem to (30) and Lemma I-6, the LHS of (29) is close to 0, which contradicts (29), which indicates that it is close to 1. \square

Proof of Proposition 2.

Part 1. When $r = 2$ and $\gamma = \frac{1}{2}$, as μ is sufficiently small, substituting $1 - r = -1$ to r in (18) in the proof of Lemma I-6, we obtain:

$$\frac{\alpha_t(b) - \beta_t(b)}{\mu^{r-1}} = b(1-b)(\sigma_H - 1 + \sigma_L) \cdot 4.$$

Applying a similar logic with (28) in the proof of Theorem 2, we show that the indifference conditions would not hold simultaneously for both types. Proposition I-2's (a) indicates that if Regime *HL* arises, $D_L(b, x, y) < 0$ and $D_H(b, x, y) < 0$ hold simultaneously. When μ is small, $A(b, x, y)$ and $B(b, x, y)$ are close to b . Thus by \mathbf{SH}_{t-1} and \mathbf{SL}_{t-1} , $D_L(b, x, y) < 0$ and $D_H(b, x, y) < 0$ can be modified into the following:

$$\begin{aligned} \frac{\alpha_t(b) - \beta_t(b)}{\mu^{r-1}} &= b(1-b)(\sigma_H - 1 + \sigma_L) \cdot 4 > 2b \\ -\frac{\alpha_t(b) - \beta_t(b)}{\mu^{r-1}} &= -b(1-b)(\sigma_H - 1 + \sigma_L) \cdot 4 < -2(1-b). \end{aligned}$$

Then $(1-b)(\sigma_H - 1 + \sigma_L) \cdot 2 > 1$ and $b(\sigma_H - 1 + \sigma_L) \cdot 2 > 1$. Adding them, we obtain $\sigma_H - 1 + \sigma_L > 1$, which is impossible. Thus, Regime *HL* does not arise. By Theorem 2, manipulation arises in equilibrium as $r \in (1, +\infty)$. This completes the proof.

Part 2. First, we show that if there exists a pair of (\hat{x}, \hat{y}) to satisfy both of the indifference conditions, Regime *H*, Regime *L*, and Regime *HL* arise in period t at b . By \mathbf{DH}_{t-1} and \mathbf{DL}_{t-1} , for some sufficiently small ϵ_V and ϵ_J ,

$$\begin{aligned} \partial_+ V_{t-1}(A(b, x, y)) &= \delta V \text{ and } \partial_+ V_{t-1}(B(b, x, y)) = \delta V + \epsilon_V; \\ \partial_+ J_{t-1}(A(b, x, y)) &= \delta J \text{ and } \partial_+ J_{t-1}(B(b, x, y)) = \delta J + \epsilon_J. \end{aligned}$$

By Proposition 1, \mathbf{DH}_{t-1} and \mathbf{DL}_{t-1} , $\delta V > 1$ and $\delta J < -1$. Then

$$-\frac{1 + \delta V + \epsilon_V}{1 - \delta V} > -\frac{1 + \delta J + \epsilon_J}{1 - \delta J},$$

because $\delta J < 0$ by \mathbf{C}_t , and so

$$\frac{\delta V + \epsilon_V + 1}{\delta V - 1} > 1 > \frac{-\delta J - \epsilon_J - 1}{-\delta J + 1}.$$

By Proposition I-3, this completes the proof of the claim.

Then it suffices to show that there exists a pair of (\hat{x}, \hat{y}) for which both of the indifference conditions hold. Note that it is not difficult to prove that when $\gamma = \frac{1}{2}$, $V_t(b) = J_t(1-b)$ and $J_t(b) = V_t(1-b)$ hold, and there is a symmetric equilibrium in the sense that when $\sigma \in \mathcal{E}_t(b)$, $\tilde{\sigma}_t(1-b)$ where $\tilde{\sigma}_L = 1 - \sigma_H$ (A formal proof is available upon request). When $\gamma = \frac{1}{2}$ and $b = \frac{1}{2}$, by symmetry with respect to $b = \frac{1}{2}$, because $A(b, \hat{x}, \hat{x}) - \frac{1}{2} = \frac{1}{2} - B(b, \hat{x}, \hat{x})$, for every $x \in [1, 0]$, we must have:

$$J_{t-1}(B(b, x, x)) - J_{t-1}(A(b, x, x)) = V_{t-1}(A(b, x, x)) - V_{t-1}(B(b, x, x)). \quad (31)$$

If there exists a pair $(\hat{x}, \hat{y}) = (\hat{x}, \hat{x})$ that satisfies the indifference condition for the type- L trader, then because $A(b, \hat{x}, \hat{x}) - \frac{1}{2} = \frac{1}{2} - B(b, \hat{x}, \hat{x})$, by (31), the indifference condition for the type- L yields the indifference condition for the type- H . Thus, it suffices to show that there exists a pair $(\hat{x}, \hat{y}) = (\hat{x}, \hat{x})$ that satisfies the indifference condition for the type- L . Then at $b = \frac{1}{2}$, by \mathbf{DL}_{t-1} , $\frac{V_{t-1}(A(b,1,1)) - V_{t-1}(B(b,1,1))}{A(b,1,1) - B(b,1,1)}$ is approximated by $\mu(t-1)$, when $r > 2$ and t is sufficiently large, we have $\mu(t-1) > \frac{1}{\mu}$, which yields:

$$\frac{V_{t-1}(A(b, 1, 1)) - V_{t-1}(B(b, 1, 1))}{A(b, 1, 1) - B(b, 1, 1)} > \frac{A(b, 1, 1) + B(b, 1, 1)}{A(b, 1, 1) - B(b, 1, 1)} = \frac{1}{\mu}.$$

Note that

$$\frac{A(b, x, x) + B(b, x, x)}{A(b, x, x) - B(b, x, x)} = \frac{1}{\mu(2x - 1)}$$

Note that when $x = \frac{1}{2}$, $\frac{A(b,x,x)+B(b,x,x)}{A(b,x,x)-B(b,x,x)} = +\infty$; thus, at $x = \frac{1}{2}$,

$$\frac{V_{t-1}(A(b, x, x)) - V_{t-1}(B(b, x, x))}{A(b, x, x) - B(b, x, x)} < \frac{A(b, x, x) + B(b, x, x)}{A(b, x, x) - B(b, x, x)}.$$

Therefore, there exists a pair $(\hat{x}, \hat{y}) = (\hat{x}, \hat{x})$ that satisfies the indifference condition for the type- L in the interval $(\frac{1}{2}, 1)$. This completes the proof. \square

Appendix II. The Computation Procedure

As we make use of an approximation, we set out a different notation to use for calibration. Bold-faced letters denote approximated variables in our simulation. For example, in the calibration, we denote the probability that the type- H trader buys in the high state at period t by \mathbf{h}_t and the probability that the type- L trader sells in the low state by \mathbf{l}_t . Moreover, let

$$\mathbf{H}_t = (1 - \mu)\gamma + \mu\mathbf{h}_t \quad \text{and} \quad \mathbf{L}_t = (1 - \mu)(1 - \gamma) + \mu\mathbf{l}_t.$$

Then, \mathbf{H}_t is the probability that a buy occurs in the high state in period t and \mathbf{L}_t is the probability that a sell occurs in the low state. We can write:

$$\alpha_t = \frac{\mathbf{H}_t b}{\mathbf{H}_t b + (1 - \mathbf{L}_t)(1 - b)} \quad \text{and} \quad \beta_t = \frac{(1 - \mathbf{H}_t)b}{(1 - \mathbf{H}_t)b + \mathbf{L}_t(1 - b)}.$$

When the type- L trader manipulates, we write the bid price as a function of the ask price and the probability that a buy will occur in the high state. Then, we obtain:

$$\beta_t = \frac{\alpha_t b(1 - \mathbf{H}_t)}{(\alpha_t - b\mathbf{H}_t)}. \quad (32)$$

In the computer program, we inspect each interval of b to check whether there is a pair of ask and bid prices that satisfies the following indifference condition for the type- L :

$$-\alpha_t + \mathbf{V}_{t-1}(\alpha_t) = \beta_t + \mathbf{V}_{t-1}(\beta_t), \quad (33)$$

where β_t satisfies (32). In our procedure, the new function \mathbf{V}_{t-1} is constructed through a linear interpolation from V_{t-1} , which is: for $\alpha_t \in [b_k, b_{k+1}]$,

$$\mathbf{V}_{t-1}(\alpha_t) = (\alpha_t - b_k) \frac{V_{t-1}(b_{k+1}) - V_{t-1}(b_k)}{(b_{k+1} - b_k)} + V_{t-1}(b_k),$$

and for $\beta_t \in [b_j, b_{j+1}]$,

$$\mathbf{V}_{t-1}(\beta) = (\beta_t - b_j) \frac{V_{t-1}(b_{j+1}) - V_{t-1}(b_j)}{(b_{j+1} - b_j)} + V_{t-1}(b_j).$$

Similarly, when the type- H trader manipulates, we write the ask price as a function of the bid price and the probability that a buy will occur in the low state. First, we solve \mathbf{H}_t as a function of the ask price α_t . Then we have:

$$\mathbf{H}_t = \frac{\alpha_t(1 - b)(1 - \mathbf{L}_t)}{(1 - \alpha_t)b}.$$

Then, we substitute \mathbf{H} into the bid price. Then, we obtain:

$$\beta_t = \frac{(b - \alpha_t) + \alpha_t \mathbf{L}_t(1 - b)}{(b - \alpha_t) + \mathbf{L}_t(1 - b)}.$$

We inspect each interval of b to check whether there is a pair of ask and bid prices that satisfies the following indifference condition for the type- H :

$$1 - \alpha_t + \mathbf{J}_{t-1}(\alpha_t) = \beta_t - 1 + \mathbf{J}_{t-1}(\beta_t). \quad (34)$$

$m_k^\theta := \frac{F_{t-1}^\theta(b_{k+1}) - F_{t-1}^\theta(b_k)}{b_{k+1} - b_k}$
$m_j^\theta := \frac{F_{t-1}^\theta(b_{j+1}) - F_{t-1}^\theta(b_j)}{(b_{j+1} - b_j)}$
$A_k^\theta := m_k^\theta - 1$
$B_j^\theta := m_j^\theta + 1$
$C^L := (b_j m_j^L - \mathbf{V}_{t-1}(b_j)) - (b_k m_k^L - \mathbf{V}_{t-1}(b_k))$
$C^H := (b_j m_j^H - \mathbf{J}_{t-1}(b_j)) - (b_k m_k^H - \mathbf{J}_{t-1}(b_k)) + 2$
$K(\theta) := -A_k^\theta + B_j^\theta - C^\theta$
$G(\theta, \mathbf{L}_t, b) := B_j^\theta [(1 - \mathbf{L}_t)(1 - b) - b] - A_k^\theta [(1 - \mathbf{L}_t)(1 - b) - 1] + C^\theta [1 - 2(1 - \mathbf{L}_t)(1 - b)]$
$N(\theta, \mathbf{L}_t, b) := B_j^\theta b - C^\theta [1 - (1 - \mathbf{L}_t)(1 - b)]$
$T_\theta := \mathbf{H}_t b \left(B_j^\theta - A_k^\theta - 2C^\theta \right) + \left(-B_j^\theta b + C^\theta \right)$
$M_\theta := \mathbf{H}_t b \left(-bB_j^\theta + A_k^\theta + C^\theta \right)$

Table 2: Summary of abbreviated notations

* Each θ belongs to $\{H, L\}$ and for each F^θ , $F^H = \mathbf{J}$ and $F^L = \mathbf{V}$.

By applying the method of linear interpolation, we construct a function \mathbf{J}_{t+1} that approximates J_{t-1} such that for $\alpha_t \in [b_k, b_{k+1}]$,

$$\mathbf{J}_{t-1}(\alpha) = (\alpha_t - b_k) \frac{J_{t-1}(b_{k+1}) - J_{t-1}(b_k)}{(b_{k+1} - b_k)} + J_{t-1}(b_k),$$

and for $\beta \in [b_j, b_{j+1}]$,

$$\mathbf{J}_{t-1}(\beta) = (\beta_t - b_j) \frac{J_{t-1}(b_{j+1}) - J_{t-1}(b_j)}{(b_{j+1} - b_j)} + J_{t-1}(b_j).$$

Finally, we obtain the following propositions. To keep the notation simple, we use some abbreviations, which are summarized in Table 2.

Proposition II-1. *When the type-L trader manipulates, an equilibrium ask price α_t solves:*

$$\alpha_t^2 A_k^L + \alpha_t (-b\mathbf{H}_t A_k^L + b(\mathbf{H}_t - 1)B_j^L + C^L) - C^L b\mathbf{H}_t = 0,$$

subject to $\mathbf{H}_t = (1 - \mu)\gamma + \mu$ and $\mathbf{L}_t = \frac{\alpha_t(1-b) - b(1-\alpha_t)\mathbf{H}_t}{\alpha_t(1-b)} \leq (1 - \mu)(1 - \gamma) + \mu$.

Proof of Proposition II-1. From (33),

$$-\alpha_t + (\alpha_t - b_k)m_k^L + \mathbf{V}_{t-1}(b_k) = \frac{\alpha_t b(-1 + \mathbf{H}_t)}{(-\alpha_t + b\mathbf{H}_t)} + \left(\frac{\alpha_t b(-1 + \mathbf{H}_t)}{(-\alpha_t + b\mathbf{H}_t)} - b_j \right) m_j^L + \mathbf{V}_{t-1}(b_j).$$

Reorganizing terms, we can obtain the desired equation. □

Proposition II-2. *When the type-H trader manipulates, then an equilibrium ask price α_t solves:*

$$\alpha_t^2 A_k^H + X \alpha_t + Y = 0,$$

where

$$\begin{aligned} X &= 1 + b + 2\mathbf{L}_t(1 - b) - [b_k + b + \mathbf{L}_t(1 - b)] m_k^H + [b_j - 1 + \mathbf{L}_t(1 - b)] m_j^H + \\ &\quad + \mathbf{J}_{t-1}(b_k) - \mathbf{J}_{t-1}(b_j); \\ Y &= -\mathbf{L}_t(1 - b) + [b + \mathbf{L}_t(1 - b)] [b_k m_k^H - 1 + \mathbf{J}_{t-1}(b_j) - \mathbf{J}_{t-1}(b_k)] + \\ &\quad + [b(1 - b_j) - b_j \mathbf{L}_t(1 - b)] m_j^H, \end{aligned}$$

subject to $\mathbf{L}_t = (1 - \mu)(1 - \gamma) + \mu$ and $\mathbf{H}_t = \frac{\alpha_t(1 - \mathbf{L}_t)(1 - b)}{b(1 - \alpha_t)} \leq (1 - \mu)\gamma + \mu$.

Proof of Proposition II-2. From (34),

$$\begin{aligned} 1 - \alpha_t + (\alpha_t - b_k) m_k^H + \mathbf{J}_{t-1}(b_k) &= \\ &= \frac{(b - \alpha_t) + \alpha_t \mathbf{L}_t(1 - b)}{(b - \alpha_t) + \mathbf{L}_t(1 - b)} - 1 + \left(\frac{(b - \alpha_t) + \alpha_t \mathbf{L}_t(1 - b)}{(b - \alpha_t) + \mathbf{L}_t(1 - b)} - b_j \right) m_j^H + \mathbf{J}_{t-1}(b_j). \end{aligned}$$

Similarly to Proposition II-1, we can obtain the desired equation by calculation. \square

Lastly, we consider the case where both types manipulate. To compute the equilibrium in this case, we simultaneously solve the two equations (33) and (34). Then, (33) and (34) can be rewritten as: for each $\theta \in \{H, L\}$, Let $\mathbf{x}_H = \mathbf{H}_t b$ and $\mathbf{x}_L = (1 - \mathbf{L}_t)(1 - b)$. Then, for each $\theta \in \{H, L\}$, the indifference conditions can be rewritten as:

$$\begin{aligned} [-A_k^\theta + B_j^\theta - C^\theta] \mathbf{x}_H^2 + \mathbf{x}_H (-b B_j^\theta + A_k^\theta + C^\theta) + \\ + \mathbf{x}_L [\mathbf{x}_H (B_j^\theta - A_k^\theta - 2C^\theta) + (-B_j^\theta b + C^\theta)] - C^\theta \mathbf{x}_L^2 = 0. \end{aligned} \quad (35)$$

Proposition II-3. *When both types manipulate, $\mathbf{x}_H = \mathbf{H}_t b$ and $\mathbf{x}_L = (1 - \mathbf{L}_t)(1 - b)$ satisfy (35) for each $\theta \in \{H, L\}$. Moreover, \mathbf{H}_t and \mathbf{L}_t must satisfy:*

$$(1 - \mu)\gamma < \mathbf{H}_t < (1 - \mu)\gamma + \mu\gamma \quad \text{and} \quad (1 - \mu)(1 - \gamma) < \mathbf{L}_t < (1 - \mu)(1 - \gamma) + \mu.$$

Simultaneously solving for \mathbf{H}_t and \mathbf{L}_t is somehow tricky as the procedure has to find a two-dimensional fixed point. First, we identify a pair of strategies that derives bid and ask prices so as to make both types indifferent for each belief. Given \mathbf{H}_t , we can find an interval for \mathbf{L}_t where (34) holds and given \mathbf{L}_t , (33) holds. By using this point as an initial point, we use the Newton–Raphson method to obtain the solution to the above equations. We denote the LHS of (35) by $f_\theta(\mathbf{x}_H, \mathbf{x}_L)$. Then, keeping all coefficients fixed, we obtain:

$$\begin{aligned} \frac{df_\theta}{d\mathbf{x}_H} &= [-A_k^\theta + B_j^\theta - C^\theta] 2\mathbf{x}_H + (-b B_j^\theta + A_k^\theta + C^\theta) + \mathbf{x}_L (B_j^\theta - A_k^\theta - 2C^\theta) \\ \frac{df_\theta}{d\mathbf{x}_L} &= [\mathbf{x}_H (B_j^\theta - A_k^\theta - 2C^\theta) + (-B_j^\theta b + C^\theta)] - 2C^\theta \mathbf{x}_L. \end{aligned}$$

Let $\mathbf{x} = \begin{pmatrix} \mathbf{x}_H \\ \mathbf{x}_L \end{pmatrix}$, $f(\mathbf{x}) = \begin{pmatrix} f_H(\mathbf{x}) \\ f_L(\mathbf{x}) \end{pmatrix}$ and $\mathcal{J} = \begin{pmatrix} \frac{df_H}{d\mathbf{x}_H} & \frac{df_H}{d\mathbf{x}_L} \\ \frac{df_L}{d\mathbf{x}_H} & \frac{df_L}{d\mathbf{x}_L} \end{pmatrix}$. By the Newton–Raphson method,

$$f(\mathbf{x} + \delta\mathbf{x}) = f(\mathbf{x}) + \mathcal{J}\delta\mathbf{x}.$$

Assuming $f(\mathbf{x} + \delta\mathbf{x}) \approx 0$ yields:

$$\delta\mathbf{x} = -\mathcal{J}^{-1}f(\mathbf{x}). \quad (36)$$

We obtain a convergent point \mathbf{x}^* by using (36).

Proof of Proposition II-3. Let $\mathbf{H}_t b + (1 - \mathbf{L}_t)(1 - b) = \mathbf{P}$. Then, from (33),

$$\begin{aligned} & (m_k^L - 1)(1 - \mathbf{P})\mathbf{H}_t b - b_k \mathbf{P}(1 - \mathbf{P})m_k^L + \mathbf{P}(1 - \mathbf{P})\mathbf{V}_{t-1}(b_k) \\ & = (m_j^L + 1)\mathbf{P}(1 - \mathbf{H}_t)b - \mathbf{P}(1 - \mathbf{P})b_j m_j^L + \mathbf{P}(1 - \mathbf{P})\mathbf{V}_{t-1}(b_j). \end{aligned}$$

Then,

$$(A_k^L \mathbf{H} - B_j^L \mathbf{P}) b - \mathbf{P}\mathbf{H}_t b (A_k^L - B_j^L) + \mathbf{P}(1 - \mathbf{P})C^L = 0.$$

Reorganizing terms, resubstituting $\mathbf{P} = [\mathbf{H}_t b + (1 - \mathbf{L}_t)(1 - b)]$, and again reorganizing terms, we obtain:

$$\begin{aligned} & \left[-A_k^L + B_j^L - C^L \right] \mathbf{H}_t^2 b^2 \\ & + \mathbf{H}_t b \left[B_j^L [(1 - \mathbf{L}_t)(1 - b) - b] - A_k^L [(1 - \mathbf{L}_t)(1 - b) - 1] + C^L [1 - 2(1 - \mathbf{L}_t)(1 - b)] \right] \\ & - (1 - \mathbf{L}_t)(1 - b) \left[B_j^L b - C^L [1 - (1 - \mathbf{L}_t)(1 - b)] \right] = 0. \end{aligned}$$

By symmetry, we obtain:

$$\begin{aligned} & \left[-A_k^H + B_j^H - C^H \right] \mathbf{H}_t^2 b^2 \\ & + \mathbf{H}_t b \left[B_j^H [(1 - \mathbf{L}_t)(1 - b) - b] - A_k^H [(1 - \mathbf{L}_t)(1 - b) - 1] + C^H [1 - 2(1 - \mathbf{L}_t)(1 - b)] \right] \\ & - (1 - \mathbf{L}_t)(1 - b) \left[B_j^H b - C^H [1 - (1 - \mathbf{L}_t)(1 - b)] \right] = 0. \end{aligned}$$

□

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