Consignment auctions

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Abstract

This article investigates pollution permit consignment auctions. In this process firms obtain an initial endowment of permits that must be consigned to the auctioneer for sale. In the auction, firms bid for permits, obtain their equilibrium permit allocations, and receive revenue from their consigned permits. The main justifications for this auction are that it is politically attractive and generates clear price discovery. Yet we show this auction provides no clear price signal: in equilibrium, firms demand their own initial endowments and their payoffs are independent of the clearing price. Our results have policy implications for the California Cap-and-Trade Program.

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1 Introduction

Pollution markets are now a popular policy tool to regulate emissions. In these markets the regulator creates a fixed number of pollution permits that can be traded among firms: the process of permit exchange allows the pollution target to be met at the lowest possible cost. Although these markets are now prevalent in the regulatory landscape, a perpetual and controversial debate exists over how these permits are initially endowed to firms (e.g., Goulder et al., 1999; Cramton and Kerr, 2002; Goulder and Parry, 2008). Two broad approaches have been used: (i) free allocation of permits—also known as 'grandfathering'—and (ii) auctioning of permits. The free allocation of permits—usually based on a firm's historical emissions or output—has been advocated as a politically feasible approach to control pollution: firms generally prefer this process as they obtain rents and potential windfall profits. For auctioned permits, the main justifications include the ability to generate revenue as well as the establishment of a clear price signal that will enhance the functioning of the market. As these schemes have distinct relative merits, attention has focused on designing mechanisms that can

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incorporate the benefits of both approaches. One such mechanism is the so-called *consignment auction*.

A consignment auction combines aspects of free allocation and auctioning into one mechanism. In the first step, permits are initially endowed to firms based on a free allocation rule (such as endowments based on historical emissions). Firms must then consign their initial permit endowments to the auctioneer, who will sell the permits in a forthcoming auction. In the second step, firms submit their demand schedules and the clearing price is determined; firms pay the clearing price for their equilibrium allocation of permits and obtain revenue from their consigned permits sold at auction. Thus all auction revenue is diverted to the holders of the initial permit endowments. This novel initial allocation process has therefore been regarded as being both politically appealing and successful in generating clear price discovery (Hahn and Noll, 1982; Hahn, 1988; Burtraw et al., 2016).[1] As a consequence of such potential benefits, consignment auctions are now an increasingly popular mechanism: it is the main allocation process within the California Cap-and-Trade Program, which generates an annual auction revenue of over $3.8bn (CARB, 2016; Borenstein et al., 2015) and it has been advocated for use within the US Clear Power Plan (EPA, 2015; Burtraw et al., 2016).[2] Although, the *prima facie* benefits of consignment auctions appear to be substantial, there is, however, limited theoretical understanding of this process. Importantly, it is currently unclear how a consignment auction influences firms' bidding behavior and the associated equilibrium of the auction.

In this article we formally investigate the multi-unit uniform-price consignment auction and provide firms' bidding strategies as well as the equilibrium clearing price. We show that each firm has an incentive, in equilibrium, to submit a demand schedule equal to their initial permit endowment. Firms, then, are neither net buyers nor net sellers within the auction process and their payoffs are independent of the clearing price: the consignment auction does not induce price discovery. We find the clearing price may be higher compared a standard uniform-price auction and, as a result, aggregate pollution control costs may be larger for a consignment auction.

To investigate permit consignment auctions, we create a multi-unit uniform-price auction in which firms obtain auction revenue from their initial endowments. Uniform-price auctions have been extensively investigated (e.g., Wilson, 1979; Back and Zender, 1993; Wang and Zender, 2002; Ausubel et al., 2014). It is well known in the literature of multi-unit uniform-price auctions that bidders have incentives to shade their bids when they have more than one unit demand: so-called demand reduction. As there is a uniform price in the auction, each firm knows that their bids could affect the clearing price they pay for *all* units. Therefore by shading their bids they can reduce the auction clearing price and increase their payoffs. Back and

[1]We interpret clear price discovery here as a convergence towards the Walrasian clearing price of the auction. For experimental analysis comparing price discovery in alternative permit auction formats see Burtraw et al. (2010).

[2]Consignment sales are also a small proportion of the initial endowment process within the US Acid Rain Program (approximately 2.8% of the total annual allocation of permits) (Joskow et al., 1998; Ellerman et al., 2000). Note that the auction system is a double auction, where firms can voluntarily offer permits for sale within the auction; thus, the incentive structures are distinct from a consignment auction (Cason, 1993, 1995).

Zender (1993) show, as a result of demand reduction, that an equilibrium clearing price equal to the reserve price could become the Pareto dominant outcome for bidders. This is, in fact, bad news for the auctioneer since their expected payoff is no better than a case where all units are sold via a posted price equal to the reserve price. Although uniform-price auctions have been extended and analyzed in a variety of formats, there has been no analysis of a uniform-price auction with an initial consignment of goods. We denote this as a consignment auction and provide the first theoretical analysis.[3] As such, our framework corresponds to any sale of multi-unit goods, where the auctioneer directs the auction revenue to initially endowed bidders.

The origin of a permit consignment auction can be found within a discussion by Hahn and Noll (1982), where many of the potential benefits were proposed.[4] First, consignment auctions may facilitate the functioning of the market by requiring all permit holders to consign their permits to the auction; thus, there cannot be any permit 'hoarding' effects. Second, the process can increase transparency and fairness as all initial endowments, and the outcome of the auction, are common knowledge. Indeed, a regulator can take into account equity considerations by adjusting firms' initial permit endowments. Third, by requiring full consignment, there is the potential benefit of early and clear price discovery that can improve the process of the permit market. Importantly, this has been justified as a significant benefit of this process from the outset: "[t]he auction also guarantees that a quick price signal will emerge" (Hahn, 1988, p.47). More recently this argumentation has continued in relation to the recent US Clear Power Plan, where Burtraw et al. (2016, p.51) suggest that "consignment sales ensure that freely allocated allowances enter the market, they help facilitate liquidity and early price discovery". Although there exists many potential benefits of this novel auction format, we show in this article that an auction that consigns all permits may not provide a clear price signal: firms' payoffs are independent of the auction clearing price. This is in stark contrast to a standard uniform-price auction, where firms' payoffs are dependent on the clearing price and one would expect the clearing price to be equal to the reserve price. Thus we show that the clearing price is (weakly) higher under a consignment auction compared to a standard uniform-price auction with an identical reserve price.

In our basic model we assume that firms have a common marginal value for permits. Once permits have been initially endowed to firms, and consignment occurs, each firm submits a sealed-bid demand schedule. The regulator aggregates all demand schedules, determines the clearing price, and sells the permits to firms. Firms also receive revenue from the auctioneer as

[3]Although no theoretical analysis has been undertaken on consignment auctions, there are a limited number of experiments, which tests aspects of the consignment auction (Franciosi et al., 1993; Güler et al., 1994; Ledyard and Szakaly-Moore, 1994; Dormady and Healy, 2015). A large literature is also associated with double auctions, yet these auctions are distinct from what we consider here. In a double auction, firms can sell permits within the auction by placing an offer of permits. Normally, their choice to offer permits for sale (and their ask price) is decided by the firm and not a requirement of the regulatory process (as is the case within a consignment auction). A key distinction between a double auction and a consignment auction—and the driving force behind the peculiarities associated with a consignment auction—is that firms in the consignment auction are obligated to consign their initial permit endowment and receive the clearing price per permit.

[4]This has also been referred to as a 'revenue-neutral' or 'zero-revenue' auction.

a result of their consigned permits being sold at the clearing price. This framework showcases the underlying incentives for firms to bid in the auction as well as identifying the potential irrelevance of the clearing price in firms' equilibrium payoffs. We then develop our framework in two main directions: (i) providing the possibility of proportional consignment and (ii) relaxing the assumption on firms' marginal value of permits. First, we generalize our model to include a proportional consignment auction, where only a proportion of the total permits are initially endowed (and then consigned) to the auction. For a proportional consignment auction, we provide a model where, at the limits, our framework converges to either a full consignment auction or a standard uniform-price auction. In the proportional consignment auction, we continue to observe firms' demand schedules being dependent on their initial endowment but now the Pareto dominant clearing price is equal to the reserve price (similar to a standard uniform-price auction). Second, we extend our model to allow firms to have asymmetric marginal permit values. Extending our framework shows that for a large class of equilibria (i.e., sufficiently low clearing prices) our basic result is maintained. For sufficiently high clearing prices, the results are similar, but with the addition that firms with a low marginal valuation type have an incentive not to buy any units in the auction whereas firms with a high marginal type for permits increase their demand schedules. As a result, high-type firms are net buyers of permits and low-type firms are net sellers of permits.

The consignment auction is the main permit allocation mechanism within the California (Quebec) Cap-and-Trade Program.[5] California and Québec jointly run quarterly auctions in which around 75 million permits are sold with a reserve price of \$12.73 per permit (for 2016). Although permits are initially endowed to all Californian utilities, consignment rules differ depending of whether operators are either Investor Owned Utilities (IOUs) or Publicly Owned Utilities (POUs). Investor Owned Utilities have to consign all their endowments whereas Publicly Owned Utilities are allowed to retain a proportion of permits from consignment (they must be transferred into a compliance account). Over time the California auction has developed from a pure consignment auction to a proportional consignment auction, where a proportion of auctioned permits have not been endowed to firms: a proportion of auction revenue is thus generated. As we show in this article, moving towards a proportional consignment auction will generate price-discovery characteristics similar to a standard uniform-price auction. Consequently, a proportional consignment auction continues to experience demand reduction and has no inherent price-discovery advantage over a standard uniform-price auction.

A relatively small literature exists on environmental regulation when revenue is refunded to polluting firms (e.g., Gersbach and Requate, 2004; Sterner and Isaksson, 2006; Montero, 2008; Aidt, 2010; Cato, 2010; Fischer, 2011; Gersbach and Winkler, 2011). This literature has focused on how the rebate of tax (or auction) revenue to polluting firms can generate first-best incentives to reduce pollution. In particular, the literature on revenue refunding focuses on the *ex post* transfer of the generated revenue, and, as such, the regulator has flexibility in deciding the preferred rebate rule. For example, Gersbach and Requate (2004), find that—in

[5] Québec entered the joint auction in November 2014. Note also that only Californian entities (and not Québecian entities) can consign their initial endowments.

non-competitive product markets—there can exist an optimal tax-rebate system based on output shares. Montero (2008) proposes a new auction mechanism that allows the regulator to determine the number of pollution permits as well as firms' auction revenue rebates. He shows that making the rebate dependent on a firm's environmental damage can incentivize truthful bidding. Montero (2008), however, does not consider a consignment auction. In our article we provide a positive analysis of a consignment auction and show that there are inherent pitfalls associated with rebating revenue based on the *ex ante* consignment of permits. As we show in this article the presence of consigned permits has stark consequences for the auction and the resulting secondary market. In particular, for both common and asymmetric value auctions, a regulator cannot choose initial endowments to incite truthful bidding. Thus a consignment auction—as observed in current regulation—cannot induce truthful bidding and may also provide poor price signal information to the market.

Our contribution is twofold. First we add to the theory of auctions by developing a framework where we allow for the consignment of a multi-unit good. This, then, provides additional avenues by which we can consider an auctioneer's choices in designing an efficient and effective auction mechanism. Second, we advance the literature on permit consignment auctions and environmental regulation refunding. Not only we do provide the first theoretical approach to a consignment auction but our critical analysis advances the discussion on revenue rebating by providing insights into when—and under what circumstances—the use of rebating in environmental regulation is undesirable. This will provide permit auction designers with additional information about the use of consignment auctions.

The article is organized as follows. In Section 2 the basic model for a consignment auction is outlined. Section 3 provides the equilibrium outcome of the auction and compares the clearing price and associated abatement costs with a standard uniform-price auction. In Section 4 a proportional consignment auction is outline and Section 5 extends the basic model to incorporate asymmetric marginal valuations. Section 6 provides some concluding remarks.

2 The model

Consider a cap-and-trade market that consists of a set of firms $I = \{1, 2, .., n\}$. The regulator has a pollution target denoted by $Q \in \mathbb{R}_+$. In order to initially allocate the pollution permits, the regulator adopts a two-step consignment auction. In the first step, the regulator freely allocates all Q permits to firms, where each firm i receives an initial endowment of ω_i permits such that $\sum_{i \in I} \omega_i = Q$.[6] In the second step, firms participate in a uniform-price auction. The auctioneer generates an auction permit supply by requiring all firms to consign their free allocation to the auction. Thus in this consignment auction firms can bid for the total quantity available in

[6] This process—also known as 'grandfathering'—can, for example, be based on an individual firm's level of energy use, historical emissions, or some output-based criterion. Similar results would occur when a regulator separates permit allocation into both a free allocation of permits (that are not consigned) as well as permits used for a consignment auction: the level of auction supply would necessarily decrease and the marginal values for auctioned permits would be adjusted accordingly.

the auction Q and simultaneously receive revenue based on their initial endowment ω that is consigned to the auction. Suppose firms have a common marginal value v for permits.[7] We later relax this assumption and allow for asymmetric marginal values. The regulator does not know v, but has some prior information that it is distributed according to a distribution function $F(.)$ with support $[\underline{v}, \bar{v}]$. There is also a reserve price $r > 0$ in the auction and no bids less than the reserve price can be accepted.

The auction is a standard sealed-bid uniform-price auction, where each bidder i submits a non-increasing and left continuous demand function $x_i : [r, \infty) \to [0, Q]$. Given each firm's demand function, define the aggregate demand function for all firms as $D(p) = \sum_{i=1}^{n} x_i(p)$, which is also left continuous and non-increasing. Since the total quantity of permits available for sale in the auction is fixed to Q, the auction clearing price is defined as follows,

$$p^* = \max \left\{ p | D(p) \geq Q; p \geq r \right\}. \tag{1}$$

Thus the clearing price is defined at the intersection of the aggregate demand function and quantity Q, where the price is greater or equal to the reserve price. Otherwise, there will be excess supply and the clearing price will be set at the reserve price. If there is any discontinuity in the aggregate demand function at the point of intersection, the clearing price would remain at the point of discontinuity but there will be excess demand. In this particular case the excess demand will be rationed based on the following proportional rule

$$q_i(p^*) = x_i(p^*) - (D(p^*) - Q) \frac{\Delta x_i(p^*)}{\Delta D(p^*)}, \tag{2}$$

where $\Delta x_i(p^*)$ is the marginal demand of firm i at the clearing price p^* and $\Delta D(p^*)$ is the sum of marginal demands for all firms at the same price. Also $q_i(p^*)$ is the quantity that firm i receives given the clearing price and the submitted demand schedule. Note that if $D(p^*) \leq Q$ then $q_i(p^*) = x_i(p^*)$ and there would be no excess demand.

Thus for an auction clearing price p^*, firm i's payoff for receiving q_i quantities at the auction is,

$$\pi_i = (v - p^*) q_i + \omega_i p^*. \tag{3}$$

Equation (3) shows that a firm's payoff depends on their net benefit of purchasing permits as

[7]This v represents the avoided marginal costs of reducing pollution. To reduce pollution, firms modify or adopt new technological processes. When estimating marginal abatement cost curves, it has been found that a firm's long-run marginal abatement cost curve is globally convex and discontinuous—through changes in the cost of alternative (and more expensive) technologies—but within specific technology classes, the marginal cost is constant (e.g., Sweeney et al., 2008; Mckinsey and Company, 2016). Further, for a single auction, the permits issued are negligible relative to the full range of marginal abatement cost curves, as the entire curve usually reflects a much broader compliance period and long-term emissions target. It is intuitive, then, to focus our attention on the adoption of processes from within a specific technology class, where there is sufficiently large capacity to meet the current abatement requirements. In many auctions, such as the California auction, there are even legal (upper bound) restrictions on how many permits a firm can obtain from the auction, which would suggest a firm is unlikely to exhaust all opportunities from a specific technology class.

well as the revenue obtained from their consigned permits, $\omega_i p^*$. The objective of firm i is to submit a demand schedule x_i in order to maximize their payoff. Formally, firm i solves the following problem

$$\max_{x_i} \quad (v - p^*(x_i))x_i + \omega_i p^*(x_i), \tag{4}$$

subject to the condition that $x_i + Q_{-i} = Q$, where $Q_{-i} = \sum_{j \in I \setminus \{i\}} x_j$ is the aggregate demand of firms excluding firm i. Following Wilson (1979) and Kremer and Nyborg (2004), this optimization problem can be adjusted to consider the firm's choice of clearing price p^*. Formally, noting that $x_i = Q - Q_{-i}$ we can rewrite the above maximization problem as follows

$$\max_{p^*} \quad (v - p^*)(Q - Q_{-i}(p^*)) + \omega_i p^*. \tag{5}$$

In the above maximization problem the aim is to find the clearing price, which maximizes firm i's profit given the residual supply and its initial endowment. The residual supply is calculated based on the sum of demands submitted by $n-1$ other firms at the same clearing price. Such a clearing price would characterize the mutual best responses of all firms. Note also that within the auction process, if ω_i is larger than firm i's equilibrium permit allocation then firm i is a net seller, and similarly if ω_i is smaller than their equilibrium permit allocation then firm i is a net buyer.

3 Equilibrium of a consignment auction

Given our outlined model in the previous section, we now derive the equilibrium of the consignment auction. In particular, using (5) we can now derive the first-order condition for each firm and find the pure-strategy Nash equilibrium of the consignment auction.

Proposition 1. *When firms submit non-increasing weakly convex demand schedules, in any pure-strategy equilibrium of the consignment auction the quantity allocated to each firm must be equal to their own endowments, $q_i(p^*) = \omega_i$, for all $i \in I$. Further, the payoff of each firm becomes independent of the auction clearing price.*

Proof. Differentiate (5) with respect to p^* to find the first-order condition for all firms. For each firm $i \in I$ we have

$$(Q - Q_{-i}(p^*)) + (v - p^*)Q'_{-i}(p^*) - \omega_i = 0, \tag{6}$$

or,

$$x_i(p^*) + (v - p^*)Q'_{-i}(p^*) - \omega_i = 0. \tag{7}$$

The sum of endowments for all firms is $\sum_{i \in I} \omega_i = Q$ and also $\sum_{i \in I}(Q - Q_{-i}(p^*)) = Q$. Then the sum of first-order conditions for all firms yields

$$Q + (v - p^*)(n-1)\sum_{j \in I} x_j^{'}(p^*) - Q = 0. \tag{8}$$

Therefore, we have

$$(v - p^*)(n-1)\sum_{j\in I} x_j^{'}(p^*) = 0. \tag{9}$$

For this condition to be satisfied there are two possibilities. First, if $v = p^*$ then according to each firm i's first-order condition we have $x_i(p^*) = \omega_i$. Second, if $\sum_{j\in I} x_j^{'}(p^*) = 0$, then since demand schedules are non-increasing it must be the case that $x_1'(p^*) = ... = x_n'(p^*) = 0$, to guarantee the equality. Again from each firm i's first-order condition we have $x_i(p^*) = \omega_i$. Given the equilibrium demand schedule for each firm i at the clearing price, each firm receives a quantity equal to their initial endowment $q_i(p^*) = \omega_i$. Given the equilibrium quantity and using (5), each firm i's payoff is equal to $v\omega_i$ and thus independent of the clearing price. □

Proposition 1 illustrates the peculiarity of a consignment auction. Each firm in equilibrium bids for, and obtains, their own initial endowment ω_i; thus, there exists no net buyers or sellers of permits within the auction process. To observe the intuition for the result in Proposition 1 let us consider if there are any possible Pareto improvements when firms deviate from this strategy. Suppose firm i would like to sell a proportion of its endowed permits ρ to potential buyers and therefore receive a payoff of $\rho p^* \omega_i$ as well as the payoff for retaining a proportion $1-\rho$ of permits, $(1-\rho)v\omega_i$. For potential Pareto improvements to exist, it must be the case that firm i's payoff is larger than $v\omega_i$. Therefore, we must have

$$(1-\rho)v\omega_i + \rho p^* \omega_i > v\omega_i. \tag{10}$$

Rearranging and simplifying (10) reveals that firm i's payoff as a permit seller is larger if $p^* > v$. Therefore, for a seller to benefit from selling any units of permits, the price must be larger than its value. Yet this will never occur as no firm would buy a permit at a price greater than its value. Thus all firms bid and obtain their own initial endowments in equilibrium.

One of the major justifications for implementing a consignment auction is to enhance price discovery within a cap-and-trade market and provide transparent and immediate revelation of market information. Yet Proposition 1 suggests that, paradoxically, implementing a consignment auction—for the sole purpose of price discovery—may actually reduce any signal derived from the clearing price. As each firm's payoff is independent of the clearing price, the equilibrium bids may no longer accurately represent the underlying compliance motives of firms' pollution reduction activities. It follows that in a consignment auction there is no Pareto dominant clearing price.

Consignment auctions have also been justified on other grounds. For example, Hahn and Noll (1982) and Burtraw et al. (2016) have stressed the benefits of political acceptability and fairness within such a mechanism. Clearly, firms would prefer to obtain initial endowments and receive rents, yet this comes at a cost of reduced price discovery. This scheme, then, may be relatively easy to implement, but the auction may generate incorrect price signals that distort firms' abatement and technology choices within the cap-and-trade market.

An interesting policy implication is that the regulator's choice of initial endowment to firm i will have a direct impact on individual equilibrium bids. Indeed, providing additional initial endowments will increase firm i's equilibrium bid (and consequently reduce the bids of other firms for a given aggregate emissions cap). Note that although initial endowments may be distributed differently among the firms, the set of clearing prices remains the same; thus, the class of equilibria are invariant to changes in the initial endowments.

3.1 Consignment versus a standard uniform-price auction

To illicit a meaningful comparison of a consignment auction, it is worthwhile providing details of a standard multi-unit uniform-price auction. Standard uniform-price auctions have been frequently used to allocate pollution permits, such as in the European Union Emissions Trading Scheme (EU-ETS) and the early auctions of the US Regional Greenhouse Gas Initiative (RGGI). The standard uniform-price auction can be viewed as a special case of a consignment auction, where firms receive no initial endowments, $\omega_i = 0$ for all $i \in I$ in (3). It is well known from Back and Zender (1993) that in a standard multi-unit auction there exists a symmetric pure-strategy subgame-perfect equilibrium with clearing price $p^* \in [r, v]$, where each firm receives $\frac{Q}{n}$ permits.

As the equilibrium auction clearing price $p^* \in [r, v]$ and payoffs are decreasing in the clearing price, it follows that the Pareto dominant clearing price is $p^* = r$. That is, all firms increase their payoffs when their bids result in the lowest possible auction price. Indeed, this is observed quite frequently in permit auctions (Khezr and MacKenzie, 2016). To highlight the potential differences between a consignment and standard uniform-price auction, we consider the impact on the clearing price and thus the associated aggregated abatement costs. Noting that there is no Pareto dominant clearing price given the set of equilibria of a consignment auction, the following proposition is obtained.

Proposition 2. *The equilibrium auction clearing price in the consignment auction is (weakly) higher than the Pareto dominant clearing price in a standard uniform-price auction.*

Proof. As we mentioned, according to Back and Zender (1993), the set of equilibria for a uniform-price auction with a reserve price r is $p^* \in [r, v]$. Also the quantity each buyer receives is unchanged given any equilibrium price. Therefore, the Pareto dominant clearing price for buyers is r. However, as we show in Proposition 1 for the equilibrium of consignment auction any clearing price is equally likely and would not affect buyers' payoffs. Also according to rules of the auction, buyers are not allowed to submit bids lower than the reserve price. Thus the set of possible clearing prices in the consignment auction is any price greater or equal to the reserve price. □

Proposition 2 shows that we expect to observe prices in a consignment auction that are higher than a standard uniform-price auction. Although both auction formats appear not to provide accurate price discovery—the clearing price is likely to be the reserve price (standard uniform-price auction) or somewhere above the reserve price (consignment auction)—the consignment auction may be less well suited in signaling the true costs of compliance. For a standard uniform-price auction, the regulator, over time, can adjust the reserve price to something that they perceive as the correct clearing price. Yet under a consignment auction, the range of potential clearing prices is much larger and provides a greater level of inherent uncertainty over the clearing price: although the regulator can select the reserve price, the clearing price is determined spuriously as there exists no Pareto superior clearing price. Given the results of Proposition 2, a prediction of the potential aggregate abatement costs is immediate.

Corollary 1. *Aggregate abatement compliance costs resulting from a consignment auction are (weakly) higher than a standard uniform-price auction.*

Corollary 1 shows that, as prices are likely to be higher under a consignment auction, then for a similar emissions target Q, aggregate abatement costs must be higher at the clearing price. Thus to reduce pollution to a given level, a consignment auction will generate larger costs than a standard uniform-price auction. A main reason for using a consignment auction is that the revenue from permit sales are directed to firms, which results in a politically viable mechanism. Yet our results show that obtaining a politically viable mechanism comes at a direct trade-off of increased clearing prices, abatement costs, and, ultimately, unclear price discovery.

4 A proportional consignment auction

Up to this point we have assumed that the regulator endows all permits to firms. In this section we relax this assumption by allowing only a proportion of the total permits to be endowed to firms. Suppose the regulator decides to only initially endow a proportion of the total permits Q to firms, which must then be consigned to the auction: the remaining permits are simply auctioned. Therefore, the sum of endowments no longer equals Q. We investigate how the outcome of the auction may change. To do this, let the regulator allocate θ_i of permits to each firm i with $\sum_{i\in I}\theta_i < Q$. Define $\tilde{Q}$ as the residual permits that are auctioned (but not consigned) so that $\tilde{Q} = Q - \sum_1^n \theta_i$. We then have the following proposition.

Proposition 3. *There exist a class of linear equilibria with the following demand schedule for each firm i with endowment θ_i,*

$$x_i(p) = \theta_i + \frac{\tilde{Q}}{n} + \frac{(p^* - p)\tilde{Q}}{(v - p^*)n(n-1)}, \tag{11}$$

where the clearing price is $p^* \in [r, v]$ and for any equilibrium clearing price p^* firm i receives a quantity $q_i(p^*) = \theta_i + \frac{\tilde{Q}}{n}$.

Proof. We are looking for the linear equilibria with the following linear demand function,

$$x_i(p) = \alpha_i + \beta p. \tag{12}$$

The first-order condition for firm i becomes,

$$x_i(p^*) + (v - p^*)(n-1)\beta - \theta_i = 0, \tag{13}$$

where $x_i(p^*)$ is the demand submitted by firm i at the clearing price. Then the sum of the first-order conditions for all firms becomes,

$$Q - \sum_1^n \theta_i + (v - p^*)n(n-1)\beta = 0, \tag{14}$$

which can be simplified to

$$(v - p^*)n(n-1)\beta = -\tilde{Q}. \tag{15}$$

Rearranging for β, we have

$$\beta = -\frac{\tilde{Q}}{(v - p^*)n(n-1)}. \tag{16}$$

We are looking for an expression for demand schedules. Since the demand function is also satisfied at the clearing price p^*, we can substitute the demand function into the first-order condition to find an expression for α_i. Thus we have,

$$\alpha_i = \theta_i + \frac{\tilde{Q}}{n} + \frac{p^*\tilde{Q}}{(v - p^*)n(n-1)} \tag{17}$$

Using α_i and β gives the following demand function for each firm i

$$x_i(p) = \theta_i + \frac{\tilde{Q}}{n} + \frac{(p^* - p)\tilde{Q}}{(v - p^*)n(n-1)}. \tag{18}$$

Also at a price equal to p^* we have,

$$q_i(p^*) = x_i(p^*) = \theta_i + \frac{\tilde{Q}}{n}. \tag{19}$$

□

Proposition 3 presents a class of linear equilibria in which any price below v characterizes a pure-strategy subgame-perfect equilibrium with each firm i receiving a quantity equal to its initial endowment plus a similar share from the remaining quantity $\tilde{Q}$. The major implication of this result is that—for any auction clearing price—firms have identical permit holdings from the non-consigned permit allocation and this results in their payoff decreasing in the clearing price. As a consequence, there is an incentive for demand reduction: firms systematically bid below their true valuations to obtain a clearing price equal to the reserve price r. In this case a clearing price equal to the reserve price is Pareto optimum for all firms. This then suggests that a proportional consignment rule cannot solve the demand reduction problem of the uniform-price auction.

Note that this proportional consignment framework is general enough to incorporate both a pure consignment auction $\sum_{i\in I}\theta_i = Q$, a standard uniform-price auction $\sum_{i\in I}\theta_i = 0$, and any degree of consignment in between.[8] As the same clearing price is expected from both a proportional consignment auction and a standard uniform-price auction, this raises a number of questions regarding the main justifications for using such an approach. Recall that one main justification of a consignment auction is that it will facilitate price discovery (Hahn, 1988). Yet what we have shown is that a proportional consignment auction has the same price discovery characteristics as a standard—and arguably simpler—uniform-price auction: demand reduction is still present. Thus the only substantive justification for using a proportional consignment auction—relative to a uniform-price auction—is to provide rents to regulated firms in order to generate a politically appealing mechanism. Of course, there may also exist simpler and alternative *ex post* rebate schemes that could be used to distribute auction revenue (see, for example, Montero, 2008).

The results in Proposition 1 and 3 have some implications for the California Cap-and-Trade Program.[9] The California auction commenced with full consignment of permits then progressed towards partial consignment. This non-consigned proportion was initially very small—less than 20%—and then raised to more than half of the total permits. As Figure 1 shows, the left-hand panel details the clearing and reserve price for the California (and Québec) consignment auctions. As can be observed, the clearing price was initially very volatile and sometimes much larger than the reserve price.[10] However, once the proportion of non-consigned permits increased, the clearing price—as predicted—becomes very close to the reserve price, which is potential evidence of the existence of a demand reduction equilibrium. On the right-hand panel are the results from the US Regional Greenhouse Gas Initiative (RGGI) that uses a standard uniform-price auction.[11] Given our model this is a special case with $\sum_{i\in I}\theta_i = 0$, which results in the reserve price being the Pareto optimum outcome for the buyers. Thus, similar to the proportional consignment auction, RGGI clearing prices were close to—and in about 50% of cases, identical to—the reserve price.

5 Asymmetric valuations in consignment auctions

In this section we relax the assumption of common marginal value for firms and analyze the effect of this change on the equilibrium behavior of firms. Throughout this section we

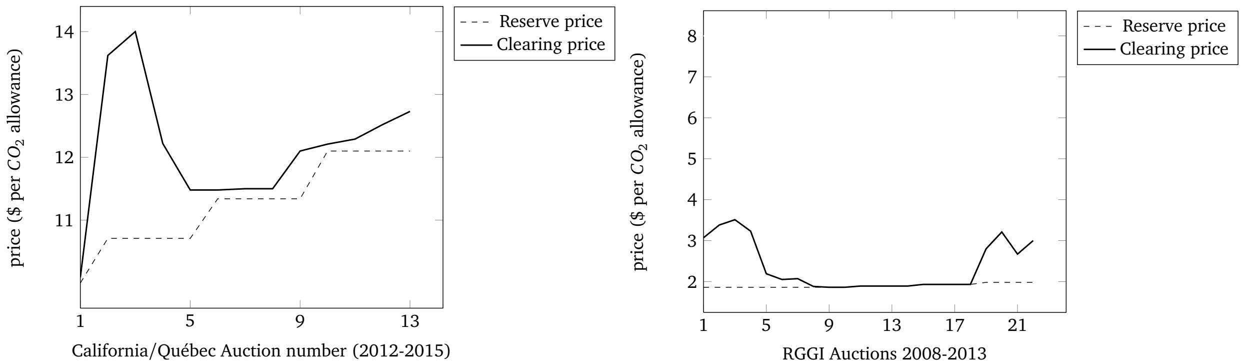
[8] When firms have the option not to consign all their initial allocation, our result still holds. In fact, a risk neutral firm is indifferent between consigning any unit into the auction and holding the permit because in equilibrium they will receive the consigned unit with certainty.

[9] In the Appendix we also analyze the inclusion of a cost containment permit reserve into the auction process. This is modeled in a similar fashion to the cost containment permit reserve implemented within in the California Cap-and-Trade Program.

[10] Note that there were 8 California auctions before a joint auction process included Québec. Further note that for both schemes, there appears to be 'learning' for the first number of auctions, then we observe more stability in the clearing prices.

[11] From Auction 23 onwards the introduction of the Cost Containment Reserve (CCR) altered the auction process: the system was no longer a standard uniform-price auction. As Khezr and MacKenzie (2016) show, the effect of a Cost Containment Reserve may place upward pressure on the clearing price.

Figure 1: Auction clearing prices AB32 and RGGI



Source: Adapted from: http://www.arb.ca.gov/cc/capandtrade/auction/results_summary.pdf and http://rggi.org/news/releases-archive

maintain the assumption that firms' values are unknown to the regulator but known to all firms. Suppose there are L firms with a value for any extra unit of permits equal to $v_l \in [\underline{v}, \bar{v})$ and the remaining H firms value each unit of permits $v_h \in (v_l, \bar{v}]$. Thus, the total number of firms is equal to the sum of L and H. Each firm i receives an initial allowance equal to ω_i which must be consigned to the auction. Then any low-type firm i has the following first-order condition:

$$x_i(p^*) + (v_l - p^*)Q'_{-i}(p^*) - \omega_i = 0, \tag{20}$$

and any high-type firm j has the following first-order condition:

$$x_j(p^*) + (v_h - p^*)Q'_{-j}(p^*) - \omega_j = 0. \tag{21}$$

We are now in a position to analyze the equilibria of the consignment auction with asymmetric marginal valuations.

Proposition 4. *When firms submit non-increasing weakly convex demand schedules, there exist a class of equilibria with equilibrium clearing prices $p^* \in [\underline{v}, v_l]$ such that all firms' equilibrium quantity would be equal to their initial endowments.*

Proof. Define Q_L as the sum of demands for low-type firms and Q_H as the sum of demands for high-type firms. Sum the first-order conditions in (20) and (21) for all firms. We have,

$$Q + (v_l - p^*)\big(LQ'_H + (L-1)Q'_L\big) + (v_h - p^*)\big(HQ'_L + (H-1)Q'_H\big) - Q = 0. \tag{22}$$

First, consider a clearing price $p^* \in [\underline{v}, v_l)$. Since both $v_l - p^*$ and $v_h - p^*$ become positive, and since the demands are non-increasing, then to satisfy all first-order conditions simultaneously we must have

$$Q'_H = Q'_L = 0 \tag{23}$$

in any equilibrium clearing price. This is not possible unless $x'_i = 0$ and $x'_j = 0$ for all $i \in L, j \in H$.

We can substitute this into the first-order conditions in (20) and (21) and we get

$$x_i(p^*) = \omega_i \quad \text{and} \quad x_j(p^*) = \omega_j \quad \forall i \in L, j \in H. \tag{24}$$

Second, we need to also consider a case with $p^* = v_l$. From (20) we directly get $x_i(p^*) = \omega_i$ for all is. From (22), since $v_h - p^*$ is positive we must have $x_i' = 0$ and $x_j' = 0$ for all $i \in L, j \in H$, which gives $x_j(p^*) = \omega_j$ for every high-type firms. It is easy to verify that as long as the demand schedules are weakly convex the second-order condition is satisfied for every firm's maximization problem. □

Proposition 4 shows that for low enough clearing prices, our framework with asymmetric marginal valuations results in an identical outcome to our base framework in Section 2; namely, all firms' equilibrium quantities are equal to their initial endowments prior to the auction. The result of Proposition 4 states that since the price is too low, even low-type firms would prefer to keep their endowments than selling them. In other words a low-type firm is not willing to sell its endowments at a price below v_l. In this case all firms generate positive equilibrium payoffs equal to their values multiplied by their endowments.

A classic advantage of a permit auction is that permits are allocated to firms with the highest values. Yet under a consignment auction, presented here, the equilibrium allocation of permits is determined by the regulator's (arbitrary) distribution of initial permit endowments; thus, it is likely that the equilibrium allocation of permits derived from the auction is allocatively inefficient. In the following proposition we consider another class of equilibria with larger clearing prices, where allocative efficiency is improved.

Proposition 5. *There exist a threshold price $v_l < \tilde{p} < v_h$ such that there is a class of linear equilibria with equilibrium clearing prices $p^* \in (\tilde{p}, v_h]$, where all low-type firms demand zero and all high-type firms demand permits such that their equilibrium quantity would become,*

$$q_j^H(p^*) = \omega_j + \frac{\Omega^L}{H}, \tag{25}$$

where Ω^L is equal to the sum of endowments of low-type firms.

Proof. Suppose all high-type firms follow a linear demand schedule similar to the one in Proposition 3 and all low-type firms submit zero demands. This case becomes similar to a proportional consignment because all low types are net sellers and the quantity available for high types exceed their endowments. If one wants to derive a similar demand schedule as the one in Proposition 3 for this case, it would be,

$$q_j(p) = \omega_j + \frac{\Omega^L}{H} + \frac{(p^* - p)\Omega^L}{(v_h - p^*)H(H-1)}, \tag{26}$$

where j represents a high-type firm and $\Omega^L = \sum_1^L \omega_i$.

Now consider a low-type firm that wants to deviate from this equilibrium and submit a positive demand x_l. We start with an arbitrary $v_l < p^* < v_h$ and show how the increase in price affects the deviating firm's payoff. Given the demand schedule by high-type firms, any positive quantity would raise the price. Also considering the fact that the aggregate demand is the sum of demands submitted by all firms, we have,

$$x_l = \frac{(p - p^*)\Omega^L}{(v_h - p^*)H(H-1)} H. \tag{27}$$

If firm i deviates and submits a demand function as in (27) then at a price $p > p^*$ it receives a quantity $x_l = q_l$. It is possible to rearrange (27) to obtain the following expression for price.

$$p = \frac{q_l(v_h - p^*)(H-1)}{\Omega^L} + p^*. \tag{28}$$

We know at any price p and quantity q_i, the payoff for a low-type firm is,

$$\pi_i = (v_l - p)q_i(p) + \omega_i p. \tag{29}$$

For a low-type firm to benefit from this deviation it must be the case that the marginal benefit of the increase in price becomes larger than the marginal loss. Otherwise, the low-type firm decides to sell all her endowments at p^* and demand zero at this price. Substituting from (28) into the second part of the firm's payoff we have,

$$\pi_i = (v_l - p)q_l + \omega_i \frac{q_l(v_h - p^*)(H-1)}{\Omega^L} + \omega_i p^*.$$

Also we know if firm i submits zero quantity her payoff is simply $\omega_i p^*$. Therefore, if the total change in firm i's payoff becomes negative, then it would not submit a positive quantity, that is, if the following inequality is satisfied firm i would not deviate:

$$(v_l - p)q_l + \omega_i \frac{q_l(v_h - p^*)(H-1)}{\Omega^L} < 0. \tag{30}$$

We can simply cancel q_l and write this inequality as follows.

$$\omega_i \frac{(v_h - p^*)(H-1)}{\Omega^L} < p - v_l. \tag{31}$$

Now substituting p from (28) and rearranging would result in,

$$\omega_i < q_l + \frac{\Omega(p^* - v_l)}{(v_h - p^*)(H-1)}. \tag{32}$$

If $p^* \to v_h$ the right hand side tends to infinity and the above would be satisfied. Also if p^* becomes equal to v_l then for any positive quantity less than ω_i the above is not satisfied. Since the right hand side is increasing in p^*, and because of continuity, there must exist a $\tilde{p} < v_h$ such that for any $p^* > \tilde{p}$ the above condition is satisfied and firm i has no incentive to deviate. $\square$

Proposition 5 proves that when the clearing price is sufficiently high, then there is an incentive for low-type firms to demand zero within the auction equilibrium and, consequently, become net sellers of permits. Intuitively, for high clearing prices, low-type firms would make a loss if they were to purchase permits, thus the best strategy is to demand zero and obtain the revenue $\tilde{p}\omega_i$. Interestingly, the reduced demand from the low types is replaced by increased demand from high-type firms, which now become net permit buyers. In particular, in equilibrium, high-type firms obtain their initial endowment plus an equal share of endowments from low-type firms.

6 Conclusion

This article investigates permit consignment auctions. A consignment auction is similar to a standard permit auction but with one key distinction: the pollution permits are initially endowed to firms and are required to be consigned to the auction supply prior to an auction. A consignment auction, therefore, provides each firm with a source of revenue from the sale of permits. Many potential advantages have been advocated. First, the process forces permits onto the market, which should discourage hoarding of permits and improve market exchange. Second, providing firms with the generated revenue is an important way to ensure the cap-and-trade market is politically acceptable. Third, using an auction may generate an immediate price signal, which can be used by the secondary market to cost effectively exchange permits: a classic advantage of using a permit auction.

In this article we consider firms' bidding strategies as well as the equilibrium clearing price within a consignment auction. Surprisingly we show that, in equilibrium, each firm will obtain exactly the same amount of permits as their initial endowment. This has major policy implications. As a result of firms' bidding strategies, firms' payoffs are independent of the clearing price. Thus the clearing price no longer reflects the true opportunity cost of holding a permit. As such, one would predict that the clearing prices in a consignment auction may be relatively higher than standard uniform-price auctions. We extend the basic model to include proportional consignment and analyze cases where firms have asymmetric marginal values for the permits.

Given our findings that a clear price signal may not be guaranteed, it is important to reassess the use of such an auction. For example, this mechanism is the major allocation mechanism for permits within the California Cap-and-trade Program. Thus it is important to consider if other forms of auction revenue rebating, such as rebating proposed by Montero (2008), can be used to satisfy the objectives of the regulator while simultaneously providing immediate and clear price discovery.

Appendix: a consignment auction with an allowance reserve

We make a small change in the basic model to introduce the possibility of allowance reserves. In particular, suppose the total emission target is $\bar{Q}$ and the regulator only advertises a fraction of the total cap in the initial consignment auction. Suppose the regulator decides to sell the remaining permits $\breve{Q}$ in a later auction with a fixed price p_e. This process exists in the California Cap-and-Trade Program, where six weeks after the initial consignment auction an additional fixed-price auction is allowed to distribute a fixed reserve of permits. $\breve{Q}$ acts as an allowance reserve, which is going to be used if there is enough demand in the market at the posted price. We investigate whether the introduction of an allowance reserve would affect the outcome of the initial consignment auction and also the bidding behavior of buyers when they face the second auction.

The design of the second auction is as follows. There are $\breve{Q}$ permits available at price p_e. Each firm submits a quantity demand $q_i \leq \breve{Q}$. If the total demand is less than $\breve{Q}$ then each firm receives its submitted quantity and pays $p_e q_i$. If the total demand is greater than $\breve{Q}$ then each firm receives a *pro rata* adjustment of its demand.

Proposition 6. *There is a unique pure-strategy Nash equilibrium in which each firm submits a demand equal to $\breve{Q}$ if $v \geq p_e$ and demands zero otherwise. Also the existence of the allowance reserve and the secondary auction has no effect on the bidding behavior of the initial consignment auction.*

Proof. First, it is not rational for a firm to submit a positive demand when the value for each unit is less than its price. So if $p_e > v$ then the best response for all firms is to simply demand zero. But when the price is below v—since all firms have a common marginal value—they demand as much as they can if there is no capacity constraint. Suppose all firms except firm i submit a demand equal to $\breve{Q}$. Suppose firm i submits a demand q_i below $\breve{Q}$. Then the allocation to firm i would be $\frac{q_i}{(n-1)\breve{Q}+q_i}Q$, which is increasing in q_i. In this case all other firms receive $\frac{\breve{Q}}{(n-1)\breve{Q}+q_i}Q$. Firm i's payoff is,

$$\pi_i = (v - p_e)\frac{q_i}{(n-1)\breve{Q} + q_i}Q, \tag{33}$$

which is also increasing in q_i. So firm i would benefit from increasing its submitted demand up to $\breve{Q}$, which is its best response. Second, for uniqueness, we show any other demand schedule below $\breve{Q}$ is not the best response for all firms and an increase in the demand would result in a higher payoff. Suppose all firms except firm i submit a demand $\dot{Q} < \breve{Q}$. If firm i increases their own demand to $\breve{Q}$, then they would receive a larger proportion of quantity and as long as $v \geq p_e$ the payoff to firm i would increase. Therefore, there is a clear incentive for firms to deviate from any equilibrium with a quantity below $\breve{Q}$. $\square$

Our predictions from Proposition 6 occur in the California scheme. Although the allowance reserve has been available from the outset of the scheme, the second allowance reserve auction has never run. The main reason, from Proposition 6, suggests that firms' marginal valuations must be below the trigger price p_e. This appears to be the case in California as the three-tiered fixed prices are set at \$45.20, \$50.86, \$56.51 per allowance and are substantially higher than the auction clearing price (of around \$12 per allowance).

References

Aidt, T. S. (2010). 'Green taxes: Refunding rules and lobbying', *Journal of Environmental Economics and Management*, **60**, pp. 31 – 43.

Ausubel, L. M., Cramton, P., Pycia, M., Rostek, M. and Weretka, M. (2014). 'Demand reduction and inefficiency in multi-unit auctions', *The Review of Economic Studies*, **81**, pp. 1366–1400.

Back, K. and Zender, J. F. (1993). 'Auctions of divisible goods: On the rationale for the treasury experiment', *The Review of Financial Studies*, **6**, pp. 733–764.

Borenstein, S., Bushnell, J., Wolak, F. A. and Zaragoza-Watkins, M. (2015). 'Expecting the unexpected: Emissions uncertainty and environmental market design', Working Paper 20999, National Bureau of Economic Research.

Burtraw, D., Goeree, J. K., Holt, C. A., Myers, E. C., Palmer, K. L. and Shobe, W. (2010). 'Price discovery in emissions permit auctions', Discussion Paper 10-32, Resource for the Future.

Burtraw, D., Linn, J., Palmer, K. L., Paul, A., McCormack, K. and Yin, H. (2016). 'Approaches to address potential CO_2 emissions leakage to new sources under the Clean Power Plan', report, Resources for the Future.

CARB (2016). 'California Cap-and-trade Program summary of auction settlement prices and results', http://www.arb.ca.gov/cc/capandtrade/auction/results_summary.pdf, California Air Resources Board.

Cason, T. N. (1993). 'Seller incentive properties of EPA's emission trading auction', *Journal of Environmental Economics and Management*, **25**, pp. 177 – 195.

Cason, T. N. (1995). 'An experimental investigation of the seller incentives in the EPA's emission trading auction', *The American Economic Review*, **85**, pp. 905–922.

Cato, S. (2010). 'Emission taxes and optimal refunding schemes with endogenous market structure', *Environmental and Resource Economics*, **46**, pp. 275–280.

Cramton, P. and Kerr, S. (2002). 'Tradeable carbon permit auctions: How and why to auction not grandfather', *Energy Policy*, **30**, pp. 333 – 345.

Dormady, N. and Healy, P. J. (2015). 'A laboratory investigation of the consignment mechanism in regional carbon auctions in the U.S.'. Mimeo.

Ellerman, A. D., Joskow, P. L., Schmalensee, R., Montero, J.-P. and Bailey, E. M. (2000). *Markets for clean air: The US Acid Rain Program*: Cambridge University Press (New York).

EPA (2015). 'Carbon pollution emission guidelines for existing stationary sources: Electric utility generating units', EPA, Federal Register 80 (205).

Fischer, C. (2011). 'Market power and output-based refunding of environmental policy revenues', *Resource and Energy Economics*, **33**, pp. 212 – 230.

Franciosi, R., Isaac, R. M., Pingry, D. E. and Reynolds, S. S. (1993). 'An experimental investigation of the Hahn-Noll revenue neutral auction for emissions licenses', *Journal of Environmental Economics and Management*, **24**, pp. 1–24.

Gersbach, H. and Requate, T. (2004). 'Emission taxes and optimal refunding schemes', *Journal of Public Economics*, **88**, pp. 713 – 725.

Gersbach, H. and Winkler, R. (2011). 'International emission permit markets with refunding', *European Economic Review*, **55**, pp. 759 – 773.

Goulder, L. H., Parry, I. W., Williams III, R. C. and Burtraw, D. (1999). 'The cost-effectiveness of alternative instruments for environmental protection in a second-best setting', *Journal of Public Economics*, **72**, pp. 329–360.

Goulder, L. H. and Parry, I. W. H. (2008). 'Instrument choice in environmental policy', *Review of Environmental Economics and Policy*, **2**, pp. 152–174.

Güler, K., Plott, C. R. and Vuong, Q. H. (1994). 'A study of zero-out auctions: testbed experiments of a process of allocating private rights to the use of public property', *Economic Theory*, **4**, pp. 67–104.

Hahn, R. W. (1988). 'Promoting efficiency and equity through institutional design', *Policy Sciences*, **21**, pp. 41–66.

Hahn, R. W. and Noll, R. G. (1982). 'Designing a market for tradable emissions permits', In W. Magat ed. *Reform of Environmental Regulation*: Cambridge, MA: Ballinger, pp. 119–146.

Joskow, P. L., Schmalensee, R. and Bailey, E. M. (1998). 'The market for sulfur dioxide emissions', *The American Economic Review*, **88**, pp. 669–685.

Khezr, P. and MacKenzie, I. A. (2016). 'Permit market auctions with allowance reserves', Working Paper 553, University of Queensland, School of Economics.

Kremer, I. and Nyborg, K. G. (2004). 'Underpricing and market power in uniform price auctions', *The Review of Financial Studies*, **17**, pp. 849–877.

Ledyard, J. O. and Szakaly-Moore, K. (1994). 'Designing organizations for trading pollution rights', *Journal of Economic Behavior & Organization*, **25**, pp. 167–196.

Mckinsey and Company (2016). ‘Greenhouse gas abatement cost curves’, , Mckinsey and Company. http://www.mckinsey.com/business-functions/sustainability-and-resource-productivity/our-insights/greenhouse-gas-abatement-cost-curves.

Montero, J.-P. (2008). ‘A simple auction mechanism for the optimal allocation of the commons’, *The American Economic Review*, **98**, pp. 496–518.

Sterner, T. and Isaksson, L. H. (2006). ‘Refunded emission payments theory, distribution of costs, and swedish experience of NO_x abatement’, *Ecological Economics*, **57**, pp. 93 – 106.

Sweeney, J., Weyant, J., Chan, T. T., Chowdhary, R., Gillingham, K., Guy, A., Houde, S., Lambie, A., Naga, R. P., Raybin, R., Sathe, A., Sudarshan, A., Westersund, J. and Zheng, A. (2008). ‘Analysis of measures to meet the requirements of California’s Assembly Bill 32’, Precourt Institute for Energy Efficiency, Standford University.

Wang, J. J. and Zender, J. F. (2002). ‘Auctioning divisible goods’, *Economic Theory*, **19**, pp. 673–705.

Wilson, R. (1979). ‘Auctions of shares’, *The Quarterly Journal of Economics*, pp. 675–689.