

A Nash Equilibrium in Electoral Competition Models*

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Abstract

Since the introduction of better-reply security by Reny (1999), the literature studying the existence of a pure strategy Nash equilibrium (PSNE) in discontinuous games has grown substantially. In this paper, we introduce a weak notion of better-reply security, which is applicable to both quasiconcave and nonquasiconcave games. Our conditions for *feeble better-reply security* are simple, easy to verify and particularly useful in electoral competition games. We also provide necessary and sufficient conditions for the existence of a PSNE in a canonical electoral competition game. Finally, this paper demonstrates why a PSNE fails to exist when a particular type of discontinuity exists in a model.

Key Words: Noncooperative games, discontinuous payoffs, pure strategy Nash equilibrium, existence of equilibrium, better-reply security, electoral competitions.

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1 Introduction

Many important games such as Hotelling's model, various auction games and Bertrand competition have discontinuous payoffs. Since the introduction of *better-reply security* by Reny (1999), the literature studying the conditions for the existence of a pure strategy Nash equilibrium (PSNE) has grown significantly.¹

In this paper, we introduce a weak notion of better-reply security, which is applicable to both quasiconcave and nonquasiconcave games. Our conditions for *feeble better-reply security* are simple, easy to verify and particularly useful in electoral competition games. We also provide necessary and sufficient conditions for the existence of a PSNE in these games. Finally, by using the notion of feeble better-reply security, we demonstrate why a PSNE fails to exist when a particular type of discontinuity exists in a model.

In the literature, Carmona (2011) proposes the condition of *weak better-reply security*, and establishes an existence result which can also imply theorems in Reny (1999) and Barelli and Soza (2009).² On the other hand, McLennan, Monteiro and Tourky (2011) weaken the hypotheses of Reny (1999)'s existence theorem and propose the *MR-security* condition, which is also applicable to nonquasiconcave games.

However, with regard to applicability, as pointed out by Reny (2013)³, some of the hypotheses in previous studies are difficult to verify. For instance, MR-security by McLennan, Monteiro and Tourky (2011) requires us to construct an operator that restricts the set of strategies for which payoff security is satisfied, or weak better-reply security by Carmona (2011) requires us to construct an upper hemicontinuous correspondence.⁴ In the view that in a political competition model, quasiconcavity of payoffs is usually not assumed, and MR-security by McLennan, Monteiro and Tourky (2011) is difficult to apply, we develop the concept of *feeble better-reply security* from weak better-reply security in Carmona (2011) and MR-security in McLennan, Monteiro and Tourky (2011).⁵ Our feeble better-reply security is equivalent to

¹Among many, the recent study of Bich and Laraki (2013) uses the ideas of approximation games and a sharing equilibrium in Simon and Zame (1990) to show the existence of an approximate equilibrium in the class of quasiconcave games with discontinuous payoffs. Barelli, Govindan and Wilson (2014) study a type of Colonel Blotto game to demonstrate the existence result. Prokopovych (2013) proposes a new form of the better-reply security condition, the *strong single deviation property*.

²Reny (1999) also provides sufficient conditions, namely *payoff security* and *reciprocal upper semicontinuity*, for quasiconcave and compact games to be better-reply secure so that the game has a PSNE. Bagh and Jofre (2006) generalize the condition of reciprocal upper semicontinuity by allowing for the payoffs of all players to drop down at some point as long as the payoff of one player jumps up, relative to his old payoff, somewhere in his strategy set. Carmona (2009) establishes the conditions of *weak upper semicontinuity* and *weak payoff security*, which together are sufficient for the existence of a PSNE.

³See footnote 2 in Reny (2013).

⁴Moreover, weak better-reply security is only applicable for quasiconcave games.

⁵Judging from page 1661 in McLennan, Monteiro and Tourky (2011), it appears that they are unaware of the analogs of the two conditions of Carmona (2009) in their setting. We believe that our feeble better-reply

Carmona (2011)'s weak better-reply security for quasiconcave games.

Our proofs using feeble better-reply security are brief. As mentioned above, proving the PSNE existence by using the MR-security condition of McLennan, Monteiro and Tourky (2011) requires a construction of a restriction operator.⁶ Our feeble better-reply security does not require constructing such a restriction operator. In other words, although MR-security is more general than feeble better-reply security, feeble better-reply security is easier for verification and application. We show this using a model of electoral competition, in which the players' expected payoff functions do not always satisfy continuity or quasiconcavity.

We build an electoral competition model on the basis of the model in Roemer (1997), which is a one-dimensional two-party model with uncertainty about the median voter's bliss point. In Roemer's model, the parties are interested solely in the policy after the election. We extend his model in the sense that the parties are also interested in winning the election. Feeble better-reply security allows us to provide a simple proof of the existence in the model, which have been increasingly important in political science literature (Ball, 1999, Aragonés and Palfrey, 2005, Saporiti, 2008, Bernhardt, Duggan and Squintani, 2009, Hummel, 2013, Drouvelis, Saporiti and Vriend, 2014, Takayama, 2014). Further, as a corollary to our main theorem, we prove the existence of an equilibrium in Roemer's original model.

We also provide necessary and sufficient conditions for the existence of a PSNE by applying the notion of feeble better-reply security. This concept allows us to demonstrate why a PSNE fails to exist when a particular type of discontinuity exists in a model. These necessary and sufficient conditions are indeed a general version of the ones proposed in Drouvelis, Saporiti and Vriend (2014) and the ones shown by using the Hotelling model of price competition in d'Aspremont, Gabszewicz and Thisse (1979) and Dasgupta and Maskin (1986).

The organization of this paper is as follows. The second section defines the concept of feeble better-reply security, and then provides the main theorem. The third section provides the existence results in the electoral competition model. The last section concludes with a discussion of the relationship between feeble better-reply security and other existence conditions in the literature. Finally, the appendix includes an illustrative example and figures to describe the existence results using the electoral competition model.

2 The Theorems

Consider a noncooperative normal form game

$$G = (X_1, \dots, X_N, u_1, \dots, u_N),$$

security is an analog of weak better-reply security of Carmona (2011) in their setting, which the two conditions in Carmona (2009) are equivalent to.

⁶The proofs for the model using MR-security are available upon request.

where, for each $i \in \{1, \dots, N\}$, player i 's strategy set X_i is a nonempty compact convex subset of a metrizable locally convex topological vector space, and player i 's payoff function u_i is a bounded function from the set of strategy profiles $X = \prod_{i=1}^N X_i$ to \mathbb{R} .

2.1 The Main Theorem

For each $i \in N$, define a function, $\underline{u}_i : X \rightarrow \mathbb{R}$, by $\underline{u}_i(d_i, x_{-i}) = \liminf_{x'_{-i} \rightarrow x_{-i}} u_i(d_i, x'_{-i})$. This is the payoff that strategy d_i can almost guarantee to player i if his opponents play any strategies close enough to x_{-i} . In order to define feeble better-reply security, for each strategy profile that is not a PSNE, we examine if such a strategy profile satisfies the following two conditions. The first condition is a version of payoff security, similar to the better-reply security of Reny (1999). In the second condition, because we do not impose the condition of quasiconcavity in our existence theorem, we set a restriction on the convex hull of the upper contour set of a given player. Finally, we apply the fixed point principle stated in McLennan, Monteiro and Tourky (2011).

For $\alpha \in \mathbb{R}$, we define

$$\begin{aligned} B_i^\alpha(x) &= \{y_i \in X_i : \underline{u}_i(y_i, x_{-i}) \geq \alpha\}; \text{ and} \\ C_i^\alpha(x) &= \text{con } B_i^\alpha(x), \end{aligned}$$

where $\text{con } Z$ is the convex hull of the set Z .

Definition 1 (Feeble better-reply security). For a set $Z \subset X$, a game G is *feebly better-reply secure on Z* if there is some $\alpha^N = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ such that the following conditions hold:

1. for each $i \in N$ and any $z \in Z$, $B_i^{\alpha_i}(z)$ is nonempty; and
2. for any $z \in Z$, there is some player $i \in N$ such that $z_i \notin C_i^{\alpha_i}(z)$.

The game G is *feebly better-reply secure at $x \in X$* if there is some neighborhood U_x of x such that it is feebly better-reply secure on U_x . Then, the game G is *feebly better-reply secure* if it is feebly better-reply secure at each $x \in X$ which is not a PSNE.

Let $\Gamma = \{(x, u) \in X \times \mathbb{R}^N : u(x) = u\}$ be the graph of the game's vector payoff function, and let $\bar{\Gamma}$ be its closure. We now review the definitions for better-reply security introduced in Reny (1999) and B -security developed by McLennan, Monteiro and Tourky (2011).

Definition. Player i can *secure* a payoff of $\alpha_i \in \mathbb{R}$ on $Z \subset X$ if there exists some $y_i \in X_i$ such that $u_i(y_i, x'_{-i}) \geq \alpha_i$ for any $x'_{-i} \in Z_{-i}$.

Moreover, player i can *secure* α_i at $x \in X$ if he can secure α_i on some neighborhood of x .

Definition (Better-reply security). A game G is *better-reply secure* if whenever $(x, \alpha^N) \in \bar{\Gamma}$ and x is not a PSNE, there is some player i and some $\varepsilon > 0$ such that he can secure $\alpha_i + \varepsilon$ at x .

McLennan, Monteiro and Tourky (2011) show that better-reply security is equivalent to the following condition.

Definition (*B*-security). A game G is *B-secure* on $Z \in X$ if there is some $\alpha^N \in \mathbb{R}^N$ and $\varepsilon > 0$ such that the following conditions hold:

1. every player i can secure $\alpha_i + \varepsilon$ on Z ; and
2. for each $z \in Z$, there exists some player i satisfying $u_i(z) < \alpha_i - \varepsilon$.

The game G is *B-secure* at $x \in X$ if it is *B-secure* on some neighborhood of x .

In comparison with the second condition of *B*-security, the second condition of feeble better-reply security has additional strength. However, when G is quasiconcave, $C_i^{\alpha_i}(z)$ is replaced by $B_i^{\alpha_i}(z)$, and it then becomes weaker. Moreover, as mentioned by McLennan, Monteiro and Tourky (2011), the ε in *B*-security plays an important role and makes it harder to satisfy. As a result, feeble better-reply security is weaker than *B*-security in general. This also implies that feeble better-reply security is more general than better-reply security. Next is an example that is not better-reply secure but is feebly better-reply secure.

Example 1 (Page 1647 in McLennan, Monteiro and Tourky (2011)). There is one player with a strategy set $X_1 = [0, 1]$. His payoff, $u_1 : X_1 \rightarrow \mathbb{R}$, is given by

$$u_1(x_1) = \begin{cases} 0 & \text{if } x_1 = 0, \\ (x_1 - \frac{1}{2})^2 & \text{if } 0 < x_1 \leq 1. \end{cases}$$

McLennan, Monteiro and Tourky (2011) show that this example is not *B*-secure, and hence, not better-reply secure. However, it is feebly better-reply secure. To see that, set $\alpha_1 = \frac{1}{4}$. Notice that $x_1 = 1$ is the only maximizer of u_1 at $u_1(1) = \frac{1}{4}$. At each $x_1 \in [0, 1)$, it is feebly better-reply secure. Particularly, at $x_1 = 0$, because $1 \in B_1^{\frac{1}{4}}(0)$, $B_1^{\frac{1}{4}}(0) \neq \emptyset$ and thus the first condition of feeble better-reply security holds. To see the second condition of feeble better-reply security also holds at $x_1 = 0$, note that $0 \notin C_1^{\frac{1}{4}}(0)$. Finally, because u_1 is continuous in other points, by setting $\alpha_1^{x_1} = u_1(x_1) + \varepsilon$ for $\varepsilon > 0$ sufficiently small, each $B_1^{\alpha_1^{x_1}}(x_1)$ is not empty but x_1 itself does not belong to $C_1^{\alpha_1^{x_1}}(x_1)$. Thus, this example is also applicable to the concept of feebly better-reply security.

Now, we state our first main result.

Theorem 1. If a game G is feebly better-reply secure, then it has a pure strategy Nash equilibrium.

To prove Theorem 1, we use the following fixed point theorem which is proved in McLennan, Monteiro and Tourky (2011).

Lemma (Lemma 7.1 in McLennan, Monteiro and Tourky (2011)). Let X be a nonempty convex compact subset of a topological vector space and let $P : X \rightrightarrows X$ be a set-valued mapping. If there is a finite closed cover H_1, \dots, H_J of X such that $\bigcap_{z \in H_j} P(z) \neq \emptyset$ for each $j = 1, \dots, J$, then there exists $x^* \in X$ such that $x^* \in \text{con } P(x^*)$.

Lemma 1. Suppose that U_1, \dots, U_l are subsets of X and, for each $h = 1, \dots, l$, a game G is feebly better-reply secure on U_h . Then, G is feebly better-reply secure on any nonempty $U \subset \bigcap_{h=1}^l U_h$.

Proof. For each $h = 1, \dots, l$ and $i \in N$, there is some $\alpha_i^h \in \mathbb{R}$ such that $B_i^{\alpha_i^h}(z) \neq \emptyset$ for any $z \in U_h$ holds. For each i , let $\alpha_i = \max_h \alpha_i^h$, and let h_i be such that $\alpha_i^{h_i} = \alpha_i$. Also, let $\alpha = (\alpha_1, \dots, \alpha_N)$. By definition of feeble better-reply security, for each i , there is some $y_i \in X_i$ such that $y_i \in B_i^{\alpha_i^{h_i}}(z)$ for any $z \in U_{h_i}$. Then, for any $z \in U$, $z \in U_{h_i}$ for each i . Thus, we have $y_i \in B_i^{\alpha_i}(z)$ and so $B_i^{\alpha_i}(z) \neq \emptyset$ for any $z \in U$.

For each $h = 1, \dots, l$ and any $z \in U_h$, there is some player $i \in N$ such that $z_i \notin C_i^{\alpha_i^h}(z)$. Then, because $\alpha_i \geq \alpha_i^h$, we have $z_i \notin C_i^{\alpha_i}(z)$ for any $z \in U_h$. This implies that, for any $z \in U$, there is some player i with $z_i \notin C_i^{\alpha_i}(z)$. \square

Proof of Theorem 1. Aiming at a contradiction, suppose that G is feebly better-reply secure and has no PSNE. Because X is a compact subset of metrizable topological vector space, it is covered by closed subsets H_1, \dots, H_l such that, for each $h = 1, \dots, l$, there is some $\alpha^h \in \mathbb{R}^N$ and an open neighborhood U_h of H_h such that G is feebly better-reply secure on U_h .

For each x , let $U_x = \bigcap_{h:x \in H_h} U_h$. Define a function $\psi : X \rightarrow \mathbb{R}^N$ by

$$\psi(x) = \left(\max_{h:x \in H_h} \alpha_1^h, \dots, \max_{h:x \in H_h} \alpha_N^h \right).$$

Then, from the proof in Lemma 1, G is feebly better-reply secure on U_x . As a result, it is feebly better-reply secure at x .

For each $i \in N$, let $P_i(x) = B_i^{\psi_i(x)}(x)$, and $P : X \rightrightarrows X$ by $P(x) = P_1(x) \times \dots \times P_N(x)$. The remaining task is to show that P satisfies the hypotheses of Lemma 7.1 in McLennan, Monteiro and Tourky (2011). Then, there is some $x^* \in X$ such that $x^* \in \text{con } P(x^*)$, i.e. $x \in C_i^{\psi_i(x^*)}(x^*)$, for all i . However, we hypothesized that G is feebly better-reply secure at x^* . Therefore, $x^* \notin C_i^{\psi_i(x^*)}(x^*)$ for some i , contradicting the result in Lemma 7.1 in McLennan, Monteiro and Tourky (2011).

Choose a particular $x \in X$. Because ψ is upper semicontinuous and takes finitely many values, there is a closed subset $Z_x \subset U_x$ of x such that $\psi_i(z) \leq \psi_i(x)$ for all i and $z \in Z_x$. By Lemma 1, since G is feebly better-reply secure on Z_x , we have

$$\bigcap_{z \in Z_x} P_i(z) = \bigcap_{z \in Z_x} B_i^{\psi_i(z)}(z) \supset \bigcap_{z \in Z_x} B_i^{\psi_i(x)}(z) \neq \emptyset,$$

for all i . Hence,

$$\begin{aligned}\bigcap_{z \in Z_x} P(z) &= \bigcap_{z \in Z_x} P_1(z) \times \cdots \times P_N(z) \\ &= \bigcap_{z \in Z_x} P_1(z) \times \cdots \times \bigcap_{z \in Z_x} P_N(z) \neq \emptyset.\end{aligned}$$

Because X is compact, it is covered by some finite collection of Z_x , so that G satisfies Lemma 7.1 of McLennan, Monteiro and Tourky (2011). \square

Recently, Reny (2013) provides some generalizations of the equilibrium existence results in Reny (1999), McLennan, Monteiro and Tourky (2011) and Barelli and Meneghel (2013). Reny (2013) proposes the concept of *point security with respect to a subset of players* by using ordinal preference relations. Reny (2013) expresses his conditions by using its local nature in order to overcome practical difficulties in proving conditions proposed in previous studies. Our feeble better-reply security comes from a similar idea, although our aim is to use it in electoral competition games to obtain simpler proofs, but not to propose a more general condition. In the next section, we provide the simple proofs by using the concept.

In the next section, we introduce an electoral competition model, and players have a continuous but not quasiconcave payoff function when they are only interested in the policy outcome. The following example shows that when a payoff is continuous but not quasiconcave, the violation of feeble better-reply security may lead to the nonexistence of a PSNE.

Example 2. There are two players with strategy sets $X_i = [-1, 1]$, $i = 1, 2$. The payoff of player i , $u_i : X_1 \times X_2 \rightarrow \mathbb{R}$, is given by

$$u_1(x_1, x_2) = |x_1 - x_2|, \quad \text{and} \quad u_2(x_1, x_2) = |x_1 + x_2|.$$

Then, the best response correspondence for each player i , $BR_i(x_{-i}) : X_{-i} \rightrightarrows X_i$, is

$$BR_1(x_2) = \begin{cases} 1 & \text{if } x_2 < 0, \\ \{-1, 1\} & \text{if } x_2 = 0, \\ -1 & \text{if } x_2 > 0, \end{cases} \quad BR_2(x_1) = \begin{cases} -1 & \text{if } x_1 < 0, \\ \{-1, 1\} & \text{if } x_1 = 0, \\ 1 & \text{if } x_1 > 0. \end{cases}$$

Hence, nonexistence of a PSNE is evident in the lack of an intersection in the two best response correspondences. Moreover, there is no value of α^N that satisfies both of the conditions of feeble better-reply security. To see this, consider the strategy profile $x = (x_1, x_2) = (0, 0)$. It is obvious that x is not a PSNE. In this profile, given an opponent's strategy $x_{-i} = 0$, each player i maximizes his payoff at $x_i = -1$ and $x_i = 1$. Hence, in order to satisfy the second condition of feeble better-reply security, we need $\alpha_i > \max_{x_i \in X_i} u_i(x_i, 0)$. However, this violates the first condition of feeble better-reply security.

3 The Existence Proofs in Electoral Competition Models

3.1 The Model

Consider an economy with two goods: a *private good* and a *public good*. In this economy, there is a continuum of voters, and all of them have the same quasilinear preferences over the two goods, which are represented by a *direct utility function*, $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, given by

$$u(b, g) = b + \varphi(g), \quad (1)$$

where $\varphi(g)$ is increasing and strictly concave, b is the individual voter's consumption level of the private good, and g is the per capita value of the public good. In this model, *income*, h , is the only characteristic that determines preferences over the policies. We assume h is distributed according to a probability measure on H with a mean of \bar{h} . As the public good is financed entirely by tax revenue, the political issue is a *flat tax rate*. Then, we define a voter's *indirect utility function*, $v : H \times [0, 1] \rightarrow \mathbb{R}$, by

$$v(h, t) = (1 - t)h + \varphi(t\bar{h}). \quad (2)$$

Note that $v(h, t)$ is continuous and strictly concave in t . Hence, $v(h, t)$ is single-peaked in t . By using (2), we define the monotonic bliss point of a voter whose income is h , $\theta_h \in [0, 1]$, as

$$\theta_h = \operatorname{argmax}_{t \in [0, 1]} v(h, t). \quad (3)$$

We assume that there are two political parties, *Party L* and *Party R*. Following Roemer (1997), each party wants to maximize the utility of some voter whose income is h_L or h_R with $h_L < h_R$, respectively. Furthermore, we assume that parties have mixed motivations in the sense that they are interested in winning the election as well as in ideology, i.e., the policy implemented after the election. Let $k^i \geq 0$ be the intrinsic value that party $i = L, R$ places on holding office. We assume that the values of k^i are common knowledge.

The randomly distributed median voter's income has a cumulative distribution function G such that we assume it has no mass-point and its support is continuous in $[h_L, h_R]$. As a consequence of (3) and the assumption of the distribution of the median voter's income, for each party $i = L, R$, there is a single ideal policy θ_{h_i} , and $\theta_{h_R} < \theta_{h_m} < \theta_{h_L}$, where θ_{h_m} is the median voter's bliss point. For simplicity, we write $\theta_{h_R} = \theta_R$, $\theta_{h_m} = \theta_m$ and $\theta_{h_L} = \theta_L$, respectively.

In the game, each party i independently and simultaneously announces a policy $x_i \in X_i = [0, 1]$. Then, each voter casts their vote for the party whose announced policy gives the highest utility according to the indirect utility function defined by (2). When the distribution of h_m does not have a mass-point, the probability of tie is zero, unless $x_L = x_R$. Finally, parties implement their announced policies if they win the election.

Given a pair of announced policies (x_L, x_R) , let $\pi : X \rightarrow [0, 1]$ be the probability that L wins the election. From (2), $\pi(x_L, x_R)$ is indeed

$$\pi(x_L, x_R) = \Pr[v(h_m, x_L) \geq v(h_m, x_R)]. \quad (4)$$

For a given m , he strictly prefers L to R if and only if

$$(x_R - x_L)h_m > \varphi(x_R \bar{h}) - \varphi(x_L \bar{h}).$$

Define $\sigma : X \rightarrow \mathbb{R}$ by:

$$\sigma(x_L, x_R) = \begin{cases} 0 & \text{if } \frac{\varphi(x_R \bar{h}) - \varphi(x_L \bar{h})}{x_R - x_L} \leq h_L \\ \frac{\varphi(x_R \bar{h}) - \varphi(x_L \bar{h})}{x_R - x_L} & \text{if } \frac{\varphi(x_R \bar{h}) - \varphi(x_L \bar{h})}{x_R - x_L} \in (h_L, h_R) \\ 1 & \text{if } \frac{\varphi(x_R \bar{h}) - \varphi(x_L \bar{h})}{x_R - x_L} \geq h_R. \end{cases} \quad (5)$$

Then, σ sets a cut-off wealth level at which the probability of L winning the election becomes either zero or one. Then, by using the cumulative distribution function of the median voter's income, we can write $\pi(x_L, x_R)$ as

$$\pi(x_L, x_R) = \begin{cases} 1 - G(\sigma(x_L, x_R)) & \text{if } x_L < x_R \\ G(\sigma(x_L, x_R)) & \text{if } x_L > x_R. \end{cases} \quad (6)$$

Further, assume that $\pi(x_L, x_R) = p$ for some $p \in [0, 1]$ when $x_L = x_R$. Unlike Roemer (1997) who applies an equal-sharing rule $p = 1/2$ when the two parties announce an identical policy, we assume that the winning probability for L is some constant p in $[0, 1]$ at the event of tie.

By the probability of winning function, $\pi(x_L, x_R)$, and the indirect utility function, $v(h, x_i)$, we define an objective function, $E\Pi_i : X \times \mathbb{R} \rightarrow \mathbb{R}$, of the party $i = L, R$ by

$$\begin{aligned} E\Pi_L(x_L, x_R, k^L) &= \pi(x_L, x_R)(v(h_L, x_L) + k^L) + (1 - \pi(x_L, x_R))v(h_L, x_R) \text{ and} \\ E\Pi_R(x_L, x_R, k^R) &= \pi(x_L, x_R)v(h_R, x_L) + (1 - \pi(x_L, x_R))(v(h_R, x_R) + k^R). \end{aligned} \quad (7)$$

Throughout, we let $-i = L$ or R , when $i = R$ or L . Following Roemer (1997), we also assume a decreasing hazard rate.⁷ Let $\sigma'_i(x_i, x_{-i})$ denote the first derivative of σ with respect to x_i for each $i = L, R$.

Assumption 1. Let σ and G be twice differentiable. The hazard rate $\frac{\sigma'_i(x_i, x_{-i})G'_i(\sigma(x_i, x_{-i}))}{G(\sigma(x_i, x_{-i}))}$ is decreasing in x_i .

⁷As we show in Lemma 2, $\sigma(x_i, x_{-i})$ is decreasing in x_i for any given x_{-i} . Then, for instance, if G is linear (the underlying distribution is uniform), $\frac{-G'_i(\sigma(x_i, x_{-i}))}{G(\sigma(x_i, x_{-i}))}$ is decreasing in x_i . Thus, σ with decreasing $|\sigma'_i(x_i, x_{-i})|$ is enough to satisfy this assumption.

Assumption 1 is a relatively common assumption in the literature of electoral competition models (e.g. Ortuño-Ortín (1997), Roemer (1997), Ortuño-Ortín (2002), Bernhardt, Duggan and Squintani (2009), Hummel (2013) and Takayama (2014)) and in the literature of auction theory (e.g. Branco (1997) and Bhattacharya, Goel, Gollapudi and Munagala (2010)), it assumes a more general version of monotone hazard rates. We denote this electoral competition model as \mathcal{G} .

To state our main theorem, let

$$\widehat{x}_L(x_R) = \operatorname{argmax}_{x_L \in X_L} \text{E}\Pi_L(x_L, x_R, k^L) \quad \text{and} \quad \widehat{x}_R(x_L) = \operatorname{argmax}_{x_R \in X_R} \text{E}\Pi_R(x_L, x_R, k^R).$$

Condition (Conditions on Discontinuities). Each intrinsic value k^i , for $i = L, R$, in \mathcal{G} respectively satisfies the following conditions: if $x_L = x_R$, then

$$\begin{aligned} \text{E}\Pi_L(\widehat{x}_L(x_R), x_R, k^L) &\geq \limsup_{\tilde{x}_L \rightarrow (x_R)^-} \text{E}\Pi_L(\tilde{x}_L, x_R, k^L); \quad \text{and} \\ \text{E}\Pi_R(x_L, \widehat{x}_R(x_L), k^R) &\geq \limsup_{\tilde{x}_R \rightarrow (x_L)^+} \text{E}\Pi_R(x_L, \tilde{x}_R, k^R). \end{aligned}$$

In the appendix, we present an illustrative example, which shows that a larger k^L or k^R induces a larger discontinuity in the objective function. Now, we state our main theorem in this model.

Theorem 2. If Conditions on Discontinuities hold, an electoral competition model \mathcal{G} has a PSNE.

Notice that when $k^L = k^R = 0$, the objective functions are continuous. Therefore, Conditions on Discontinuities hold. This case is the model proposed by Roemer (1997), which is a special case of the game \mathcal{G} . As a corollary to our main theorem, we obtain the following result.

Corollary 1. When $k^L = k^R = 0$, an electoral competition model \mathcal{G} has a PSNE.

3.2 Preliminary Results

Proposition 1. Under Assumption 1, the set of maximizers of $\text{E}\Pi_i(x_i, x_{-i}, k^i)$ in response to x_{-i} is convex.

In order to prove Proposition 1, we observe how Assumption 1 affects the shape of the objective functions. Moreover, because the arguments for L and R are analogous, we focus on the situation for L . In what follows, we will prove Proposition 1 by studying two cases: $x_L > \bar{x}_R$ and $x_L < \bar{x}_R$. First, we study the case of $x_L > \bar{x}_R$.

Proposition 2. Let $\bar{x}_R \in [0, 1)$ be an arbitrary policy. When $x_L > \bar{x}_R$, $\text{E}\Pi_L(x_L, \bar{x}_R, k^L)$ is quasiconcave in x_L .

The proof of Proposition 2 rests on two results. We show that, for a given $\bar{x}_R \in [0, 1)$, $\text{E}\Pi_L(x_L, \bar{x}_R, k^L)$ is single-peaked, constant or monotonically decreasing. In preparation, notice that because the median voter's income is located between h_L and h_R , and $v(h, t)$ is strictly concave in t , when $x_L > \bar{x}_R$, there is a unique policy which satisfies $v(h_i, x_{-i}) = v(h_i, x_i)$ for each $i = L, R$. Thus, define $I_i(x_{-i})$ by x_i to satisfy $v(h_i, x_{-i}) = v(h_i, x_i)$ when $x_L > \bar{x}_R$, and 0 otherwise.

Now, by taking the first derivative of $\sigma(x_L, \bar{x}_R)$, we obtain the following lemma.

Lemma 2. $\sigma(x_L, \bar{x}_R)$ is decreasing in x_L .

Next, we claim that $\pi(x_L, \bar{x}_R)$ is either zero or one in certain intervals.

Lemma 3. Suppose $x_L > \bar{x}_R$ in $[0, 1]$. If $x_L \geq I_L(\bar{x}_R)$, $\pi(x_L, \bar{x}_R) = 0$. On the other hand, when $x_L \leq I_R(\bar{x}_R)$, $\pi(x_L, \bar{x}_R) = 1$. Finally, if $\bar{x}_R \geq \theta_L$, $\pi(x_L, \bar{x}_R) = 0$.

Proof. First consider a case where $x_L \geq I_L(\bar{x}_R)$. Then, a voter whose bliss point is θ_L is indifferent between L and R only if $x_L = I_L(\bar{x}_R)$. Otherwise, he strictly prefers R . Then, because by assumption $\theta_m \in (\theta_R, \theta_L)$, the median voter strictly prefers R . Hence, if $x_L \geq I_L(\bar{x}_R)$, $\pi(x_L, \bar{x}_R) = 0$. Symmetrically, we can prove that when $x_L \leq I_R(\bar{x}_R)$, $\pi(x_L, \bar{x}_R) = 1$. Finally, suppose $\bar{x}_R \geq \theta_L$. This indicates $x_L > \bar{x}_R \geq \theta_L$. By $\theta_m \in (\theta_R, \theta_L)$, the median voter strictly prefers R . Hence, if $\bar{x}_R \geq \theta_L$, $\pi(x_L, \bar{x}_R) = 0$. \square

Proof of Proposition 2. First of all, as $x_L > \bar{x}_R$, we can write the L 's objective function by

$$\text{E}\Pi_L(x_L, \bar{x}_R, k^L) = G(\sigma(x_L, \bar{x}_R))(v(h_L, x_L) - v(h_L, \bar{x}_R) + k^L) + v(h_L, \bar{x}_R).$$

Then, the first derivative of $\text{E}\Pi_L(x_L, \bar{x}_R, k^L)$ with respect to x_L , $\text{E}\Pi'_L(x_L, \bar{x}_R, k^L)$, is

$$\sigma'_L(x_L, \bar{x}_R)G'_L(\sigma(x_L, \bar{x}_R))(v(h_L, x_L) - v(h_L, \bar{x}_R) + k^L) + G(\sigma(x_L, \bar{x}_R))v'(h_L, x_L). \quad (8)$$

When $\theta_L \leq \bar{x}_R$, by Lemma 3, $\text{E}\Pi_L(x_L, \bar{x}_R, k^L)$ is constant at $v(h_L, \bar{x}_R)$ and the result is immediate. Thus, we only consider the following two cases.

Case 1: $\theta_R \leq \bar{x}_R < \theta_L$. To start with, we consider the case of $I_R(\bar{x}_R) < x_L \leq I_L(\bar{x}_R)$. By definition of I_i , $I_R(\bar{x}_R) \leq \bar{x}_R$. First, suppose that when x_L is close enough to \bar{x}_R , $\text{E}\Pi_L(x_L, \bar{x}_R, k^L)$ is strictly increasing. Notice that this only depends on the magnitude of k^L and there is only one cut-off of k^L to make it increasing or decreasing. In this case, we consider the case where k^L is smaller than this cut-off so that the first term in (8) is also small, while the other case is considered in the second part of this proof. Thus, there must be a unique point $x'_L \in (\bar{x}_R, I_L(\bar{x}_R))$ for which (8) is equal to 0. Then,

$$\frac{\sigma'_L(x'_L, \bar{x}_R)G'_L(\sigma(x'_L, \bar{x}_R))}{G(\sigma(x'_L, \bar{x}_R))} = -\frac{v'(h_L, x'_L)}{v(h_L, x'_L) - v(h_L, \bar{x}_R) + k^L}. \quad (9)$$

Because $v(h, x_L)$ is strictly concave and increasing for $x_L < \theta_L$, $v'(h_L, x'_L) > v'(h_L, x_L)$ and $v(h_L, x_L) - v(h_L, \bar{x}_R) > v(h_L, x'_L) - v(h_L, \bar{x}_R) > 0$ for any $x_L \in (x'_L, \theta_L)$. Noting that the RHS is negative, the RHS is increasing. On the other hand, Assumption 1 and Lemma 2 indicate that for any $x_L > x'_L$, the LHS is decreasing. Thus, x'_L is unique when $x'_L \in (\bar{x}_R, \theta_L)$. Moreover, by Lemma 2, since $G(\sigma(x_L, \bar{x}_R))$ is decreasing in x_L , and $v(h_L, x_L)$ is decreasing for $x_L \geq \theta_L$, $\text{E}\Pi_L(x'_L, \bar{x}_R, k^L)$ is maximized at θ_L for any $x'_L \in [\theta_L, I_L(\bar{x}_R))$. Overall, there is a unique point $x'_L \in (\bar{x}_R, I_L(\bar{x}_R))$ for which (8) is equal to 0.

Second, instead, suppose that when x_L is close enough to \bar{x}_R , $\text{E}\Pi_L(x_L, \bar{x}_R, k^L)$ is decreasing. Then, for any $x'_L \in (\bar{x}_R, I_L(\bar{x}_R))$, we must have

$$\frac{\sigma'_L(x'_L, \bar{x}_R)G'_L(\sigma(x'_L, \bar{x}_R))}{G(\sigma(x'_L, \bar{x}_R))} \leq -\frac{v'(h_L, x'_L)}{v(h_L, x'_L) - v(h_L, \bar{x}_R) + k^L}.$$

However, as is mentioned above, since the RHS is increasing while the LHS is decreasing, the above inequality holds for any $x'_L \in (\bar{x}_R, \theta_L)$. Then, for any $x'_L \in [\theta_L, I_L(\bar{x}_R))$, $\text{E}\Pi_L(x'_L, \bar{x}_R, k^L)$ is maximized at θ_L . Thus, $\text{E}\Pi_L(x_L, \bar{x}_R, k^L)$ is monotonically decreasing in x_L .

Finally, when $I_L(\bar{x}_R) \leq x_L$, by Lemma 3, $\text{E}\Pi_L(x_L, \bar{x}_R, k^L)$ is constant at $v(h_L, \bar{x}_R)$, which completes the proof in this case.

Case 2: $\bar{x}_R < \theta_R$.

Case 2-1: $I_R(\bar{x}_R) < \theta_L$. Notice that when $I_R(\bar{x}_R) < x_L < I_L(\bar{x}_R)$, by the same argument with the one in Case 1, $\text{E}\Pi_L(x_L, \bar{x}_R, k^L)$ is single-peaked. When $\bar{x}_R < x_L \leq I_R(\bar{x}_R)$, by Lemma 3, $\text{E}\Pi_L(x_L, \bar{x}_R, k^L)$ is equal to $v(h_L, x_L) + k^L$, which is maximized at $I_R(\bar{x}_R)$. When $I_L(\bar{x}_R) \leq x_L$, by the same logic as in the third case of Case 1, we can complete the proof in this case.

Case 2-2: $I_R(\bar{x}_R) \geq \theta_L$. Because $\sigma'_L(x_L, \bar{x}_R) < 0$ by Lemma 2 and $v(h_L, x_L)$ is strictly decreasing in x_L when $\theta_L \leq I_R(\bar{x}_R) < x_L$, by the first derivative, $\text{E}\Pi_L(x_L, \bar{x}_R, k^L)$ is strictly decreasing in x_L for $I_R(\bar{x}_R) < x_L < I_L(\bar{x}_R)$. When $x_L \leq I_R(\bar{x}_R)$, by Lemma 3, $\text{E}\Pi_L(x_L, \bar{x}_R, k^L)$ is equal to $v(h_L, x_L) + k^L$, which is single-peaked at θ_L . When $I_L(\bar{x}_R) \leq x_L$, by the same logic with the third case of Case 1, we can complete the proof in this case. \square

Next, we analyze the shape of the objective function when $x_L < \bar{x}_R$.

Lemma 4. Let $\bar{x}_R \in (0, 1]$ be an arbitrary policy, and take $x_L < \bar{x}_R$. When $\theta_L \geq \bar{x}_R$, then $\text{E}\Pi_L(x_L, \bar{x}_R, k^L) \leq \limsup_{\tilde{x}_L \rightarrow (\bar{x}_R)^-} \text{E}\Pi_L(\tilde{x}_L, \bar{x}_R, k^L)$.

Proof. When $\theta_L \geq \bar{x}_R$, $v(h_L, x_L)$ is increasing in x_L , and $v(h_L, x_L) \leq v(h_L, \bar{x}_R)$. Also by Lemma 2, $\pi(x_L, \bar{x}_R) < \pi(\bar{x}_R - \varepsilon, \bar{x}_R)$, where $\varepsilon > 0$ is sufficiently small. Then we complete the proof by substituting them into (7). \square

Lemma 5. Let $\bar{x}_R \in (0, 1]$ be an arbitrary policy, and take $x_L < \bar{x}_R$. When $\theta_L < \bar{x}_R$, then there is a unique policy $x_L^* \in [I_L(\bar{x}_R), \bar{x}_R)$ which maximizes $\text{E}\Pi_L(x_L, \bar{x}_R, k^L)$.

Proof. Suppose $\theta_L < \bar{x}_R$. First, for any $x_L \in [I_L(\bar{x}_R), \bar{x}_R)$, we have $\pi(x_L, \bar{x}_R) = 1$, implying $\text{E}\Pi_L(x_L, \bar{x}_R, k^L) = v(h_L, x_L) + k^L$. As we assume $v(h_L, x_L)$ is single-peaked at θ_L , there is a unique policy $x_L^* = \theta_L \in [I_L(\bar{x}_R), \bar{x}_R)$ that maximizes $\text{E}\Pi_L(x_L, \bar{x}_R, k^L)$. Second, by Lemma 2, for any $x_L \leq I_L(\bar{x}_R)$, L can maximize their payoff by choosing a policy $x_L = I_L(\bar{x}_R)$. As the objective functions are continuous when $x_L \neq \bar{x}_R$, $\text{E}\Pi_L(x_L, \bar{x}_R, k^L) \leq v(h_L, I_L(\bar{x}_R)) + pk^L = v(h_L, \bar{x}_R) + pk^L$ for any $x_L < I_L(\bar{x}_R)$. Overall, we obtain the desired result. \square

Proof of Proposition 1. There are two cases to consider.

Case 1: $\bar{x}_R \leq \theta_L$. Maximizers of $\text{E}\Pi_L(x_L, \bar{x}_R, k^L)$ are greater than or equal to \bar{x}_R by Lemma 4 and Conditions on Discontinuities. Then, Proposition 2 completes our proof.

Case 2: $\theta_L < \bar{x}_R$. In this case, a maximizer of $\text{E}\Pi_L(x_L, \bar{x}_R, k^L)$ is less than \bar{x}_R . Then, from Lemma 5, the objective function is single-peaked, and hence we obtain the desired result. \square

3.3 The Existence Proof

Proof of Theorem 2. Let $x = (x_L, x_R) \in X$ be a strategy profile that is not a PSNE. Except in the case of $x_R = x_L$, the objective functions are continuous. First, consider the case of $x_R \neq x_L$. Because the objective functions are continuous in this case, $\underline{u}_i(d_i, x_{-i}) = u_i(d_i, x_{-i}) = \text{E}\Pi_i(d_i, x_{-i}, k^i)$. Furthermore, for each i , set $\alpha_i = f_i(x_{-i}) = \sup_{d_i \in X_i} \text{E}\Pi_i(d_i, x_{-i}, k^i)$. Then, in response to an opponent player's strategy, each player i has a best response to it, and hence the first condition of feeble better-reply security is satisfied.

Let $d : X \rightarrow \mathbb{R}_+$ be the Euclidean distance on X , and let $U_x(\delta) = \{y \in X : d(y, x) < \delta\}$ be an open neighborhood about x with radius $\delta > 0$. Because (x_L, x_R) is not a PSNE, there is some player i whose $\text{E}\Pi_i(x_i, x_{-i}, k^i)$ is not maximized in response to x_{-i} . Then, because we set $\alpha_i = f_i(x_{-i}) = \sup_{d_i \in X_i} \text{E}\Pi_i(d_i, x_{-i}, k^i)$, by Proposition 1, the set of maximizers of $\text{E}\Pi_i(x_i, x_{-i}, k^i)$ is convex, which is indeed $C_i^{\alpha_i}(x)$. Then, it implies that $x_i \notin C_i^{\alpha_i}(x)$. Then, by continuity of the objective functions, we can choose a sufficiently small $\delta > 0$ such that, for any $(z_L, z_R) \in U_x(\delta)$, $z_i \notin C_i^{\alpha_i}(z)$ holds.

Second, consider the case of $x_R = x_L$. Again, for each i , we set $\alpha_i = f_i(x_{-i}) = \sup_{d_i \in X_i} \text{E}\Pi_i(d_i, x_{-i}, k^i)$. Without loss of generality, let $i = L$. Take a sufficiently small δ . We consider a δ -neighborhood of (x_L, x_R) with $x_L = x_R$. Then, by Conditions on Discontinuities, we have

$$\text{E}\Pi_L(\widehat{x}_L(x_R), x_R, k^L) \geq \limsup_{\tilde{x}_L \rightarrow (x_R)^-} \text{E}\Pi_L(\tilde{x}_L, x_R, k^L).$$

Therefore, the first condition of feeble better-reply security holds. For the second condition of feeble better-reply security, there are two cases to consider. First, for a given x_R , if maximizers of $\text{E}\Pi_L(x_L, x_R, k^L)$ are less than or equal to x_R , we can apply the same argument above. Second, if maximizers of $\text{E}\Pi_L(x_L, x_R, k^L)$ are greater than x_R , then because, for any x'_L such

that $x'_L < x_R$, the payoff is strictly less than $f_L(x_R)$. Therefore, by choosing a sufficiently small δ , the second condition of feeble better-reply security holds. As a result, \mathcal{G} is feebly better-reply secure. Therefore, by Theorem 1, \mathcal{G} has a PSNE. \square

As Ball (1999) points out, generally a party wants to choose a policy between their opponent's policy and their own ideal policy.⁸ However, as an office-motivation becomes stronger, the party obtains an incentive for “undercutting” behavior such that they would be better off by choosing a policy infinitesimally close to that of their opponent. Then, if one party is relatively more office-motivated while the other is more policy-motivated, the former party is willing to choose the same position as the policy-motivated party, forcing the latter to randomize their announced policy in order not to be predictable. Consequently, a PSNE may not exist.⁹ One extreme example is when $k^L \rightarrow \infty$ and $k^R = 0$.¹⁰ The existence of a PSNE depends on how different k^L and k^R are relative to the electoral uncertainty.¹¹

3.4 Necessary Conditions

Here, we show that Conditions on Discontinuities are necessary for the existence of a PSNE. If Conditions on Discontinuities do not hold, then there is a party that can increase the payoff by choosing a strategy that is infinitesimally close to that of the opponent. However, such an undercutting behavior decreases the payoff of the opponent. Therefore, the opponent is willing to randomize their announced policy in order not to be predictable. As a result, a PSNE does not exist. Drouvelis, Saporiti and Vriend (2014, p.92) provide analogous conditions for the existence of a PSNE. In fact, their conditions are a special case on the following result because they specify the distribution of the median voter's bliss point in their model.

Theorem 3. If an electoral competition model \mathcal{G} has a PSNE, Conditions on Discontinuities hold.

Proof. Suppose there is a PSNE denoted by (x_L^*, x_R^*) . Then, by definition of a PSNE, we have

$$\begin{aligned} \text{E}\Pi_L(x_L^*, x_R^*, k^L) &= \max_{x'_L \in X_L} \text{E}\Pi_L(x'_L, x_R^*, k^L) \quad \text{and} \\ \text{E}\Pi_R(x_L^*, x_R^*, k^R) &= \max_{x'_R \in X_R} \text{E}\Pi_R(x_L^*, x'_R, k^R). \end{aligned} \tag{10}$$

As the arguments for L and R are analogous, without loss of generality, we clarify the condition for L . Set $x_L^* = \hat{x}_L(x_R^*)$. Then, from (10), as (x_L^*, x_R^*) is a PSNE, it must satisfy

$$\text{E}\Pi_L(\hat{x}_L(x_R^*), x_R^*, k^L) = \text{E}\Pi_L(x_L^*, x_R^*, k^L) \geq \limsup_{\tilde{x}_L \rightarrow (x_R^*)^-} \text{E}\Pi_L(\tilde{x}_L, x_R^*, k^L).$$

⁸This result is not so hard to prove in our setting. The proof is available upon request.

⁹However, it is often shown that there exists a mixed strategy equilibrium.

¹⁰See Saporiti (2008) for more details.

¹¹An illustrative example is found in the appendix.

Otherwise, L profitably deviates to a strategy that is infinitesimally smaller than x_R^* , contradicting the hypothesis that (x_L^*, x_R^*) is a PSNE. \square

Another interesting example where an undercutting behavior plays an important role to assure the existence of a PSNE is the Hotelling model of price competition (d'Aspremont, Gabszewicz and Thisse, 1979, Dasgupta and Maskin, 1986). It has been shown that in the Hotelling model of price competition, when jumps of payoffs are within a certain range, there is a PSNE, while otherwise, it fails to exist. One can show that similar to our example of an electoral competition model here, it fails to be feeble better-reply secure, when jumps become large.

Let $(x_L^*, x_R^*) \in X$ be a PSNE of \mathcal{G} . Notice that in response to x_{-i} , x_i which yields the winning probability of zero to party i is strictly dominated by other policies that yield a strictly positive winning probability. Hence, such a policy cannot be an equilibrium strategy. Symmetrically, a policy x_i that yields the winning probability of one also cannot be supported as an equilibrium. Thus any strategy that yields a zero or one winning probability would not constitute a Nash equilibrium. The following proposition states this result.

Proposition 3. If \mathcal{G} has a PSNE, (x_L^*, x_R^*) , in response to x_{-i} , for each $i = L, R$, x_i^* yields a winning probability in $(0, 1)$.

The tie-breaking rule relates to the situation where the two parties announce an identical policy at the equilibrium. In particular, by Proposition 3, p must be in a certain interval, so that we can assure the existence of a PSNE.

Theorem 4. If \mathcal{G} has a PSNE, (x_L^*, x_R^*) , such that $x_L^* = x_R^* = x^*$, for any $x_L \in X_L$ and any $x_R \in X_R$, we must have

$$\begin{aligned} (p - \pi(x_L, x^*))k^L &\geq \pi(x_L, x^*)(v(h_L, x_L) - v(h_L, x^*)) && \text{and} \\ (\pi(x^*, x_R) - p)k^R &\geq (1 - \pi(x^*, x_R))(v(h_R, x_R) - v(h_R, x^*)). \end{aligned}$$

Proof. Suppose there is a PSNE denoted by (x^*, x^*) . Then, by definition of PSNE, we have

$$\begin{aligned} \text{E}\Pi_L(x^*, x^*, k^L) &= \max_{x'_L \in X_L} \text{E}\Pi_L(x'_L, x^*, k^L) \quad \text{and} \\ \text{E}\Pi_R(x^*, x^*, k^R) &= \max_{x'_R \in X_R} \text{E}\Pi_R(x^*, x'_R, k^R). \end{aligned}$$

As the arguments for L and R are analogous, without loss of generality, we clarify the condition for L . Then, because (x^*, x^*) is a PSNE, for any $x_L \in X_L$,

$$\begin{aligned} \text{E}\Pi_L(x^*, x^*, k^L) &\geq \text{E}\Pi_L(x_L, x^*, k^L) \\ v(h_L, x^*) + pk^L &\geq \pi(x_L, x^*)(v(h_L, x_L) - v(h_L, x^*)) + \pi(x_L, x^*)k^L + v(h_L, x^*) \\ (p - \pi(x_L, x^*))k^L &\geq \pi(x_L, x^*)(v(h_L, x_L) - v(h_L, x^*)). \end{aligned}$$

\square

4 Discussion with Related Results

Among the recent equilibrium existence results in the literature, McLennan, Monteiro and Tourky (2011) and Barelli and Meneghel (2013) study games that may not be quasiconcave. It is not difficult to show that when players' strategy spaces are metric and locally convex, feeble better-reply security is a sufficient condition for MR-security in McLennan, Monteiro and Tourky (2011).¹² In relation to the generalized payoff security in Barelli and Soza (2009), Carmona (2011) proposes the concept of weak better-reply security and states that the conditions of weak better-reply security and generalized better-reply security are equivalent when players' strategy spaces are metric and locally convex, and the game is quasiconcave. Barelli and Meneghel (2013) propose continuous security which is the most general condition before Reny (2013), and generalized better-reply security is sufficient for continuous security.

The first condition of feeble better-reply security is comparable to the one of weak better-reply security, if a game G is quasiconcave and thus our existence theorem also allows us to obtain the results of Bagh and Jofre (2006) and Carmona (2009). Let f_i be a real-valued function on X_{-i} such that $f_i(x_{-i}) < \sup_{d_i \in X_i} u_i(d_i, x_{-i})$ for all $x_{-i} \in X_{-i}$ and all i . Then, Lemma 1 of Carmona (2011) shows that if the game is quasiconcave and compact, and satisfies a generalized payoff security, then there exists an open neighborhood V_x of x and an upper hemicontinuous correspondence φ_i with nonempty, closed and convex values such that

$$\varphi_i(z) \subseteq \bar{B}'_i(z) = \{y_i \in X_i : \underline{u}_i(y_i, z_{-i}) > f_i(z_{-i})\}$$

for each i and for each $z \in V_x$.

With regard to applicability in the literature of electoral competition models, the conditions of feeble better-reply security are easier to verify than those of weak better-reply security in Carmona (2011) in two points. The first is that, in electoral competition models, the players' expected payoff functions do not always satisfy quasiconcavity, and our feeble better-reply security is still applicable in such models while weak better-reply security is not. The second is that weak better-reply security requires us to construct an upper hemicontinuous correspondence φ_i . However, our condition does not require us to obtain a correspondence.

¹²Example 2.1 in Tian (2013) is an example for MR-security but not for feeble better-reply security.

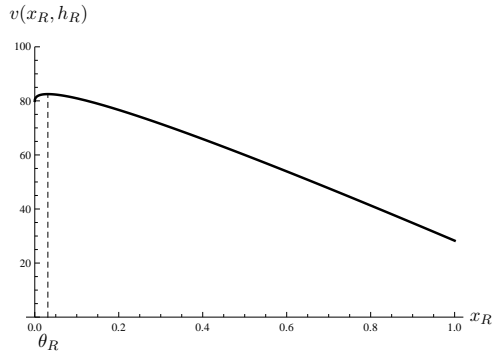
Appendix: Illustrative Example

The following figures in Figures 1 show our illustrative example, where \bar{h} is set at 50 and $\varphi(t\bar{h}) = 4\sqrt{t\bar{h}}$. We set the policy space $[0, 1]$. The cumulative distribution function G is assumed to be linear so that the underlying distribution for h is uniform. The first two figures show the indirect utility for $h_R = 80$ and $h_L = 20$, respectively. Figure 1c shows the winning probability $\pi(x_L, \bar{x}_R)$ where \bar{x}_R is assumed to be 0.1.

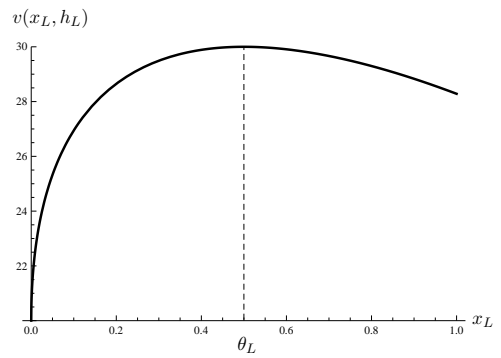
The last three figures present the objective function for party L . The first among the three figures, Figure 1d, is for Roemer's model, which is a special case of the game \mathcal{G} . As described, there is no discontinuity in this objective function. The next two figures are for a game with mixed motivations. In the first case, k^L is assumed to be 1. The objective function in this case is presented in Figure 1e, which satisfies the Conditions on Discontinuities. In our calculation, θ_L is 0.5 while θ_R is 0.03125. The maximizer of the objective function, $\hat{x}_L(\bar{x}_R)$, in this case is 0.2359. The expected payoff at this point is obviously larger than the level of $\limsup_{\tilde{x}_L \rightarrow (\bar{x}_R)^-} \text{E}\Pi_L(\tilde{x}_L, \bar{x}_R, k^L)$ (which is the level set by the white circle in the figure). However, this is not the case in Figure 1f. In this case, $k^L = 2$ is assumed. As shown, $\limsup_{\tilde{x}_L \rightarrow (\bar{x}_R)^-} \text{E}\Pi_L(\tilde{x}_L, \bar{x}_R, k^L)$ is larger than the value at any other points in X_L . As such, this does not satisfy the Conditions on Discontinuities. Then, as described in the main body of the paper, an undercutting behavior by party L arises for x_L sufficiently close to \bar{x}_R . Then, a PSNE fails to exist.

Figure 1: Illustrative Example

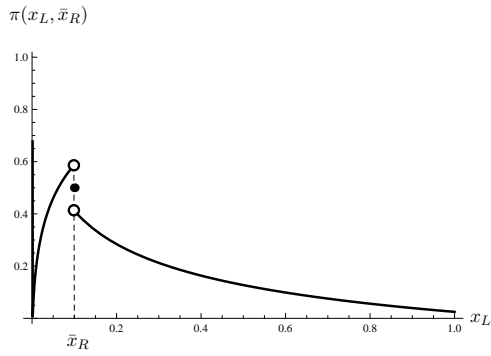
(a) Indirect Utility for $h_R = 80$



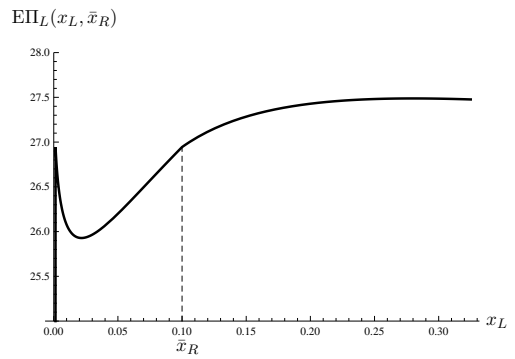
(b) Indirect Utility for $h_L = 20$



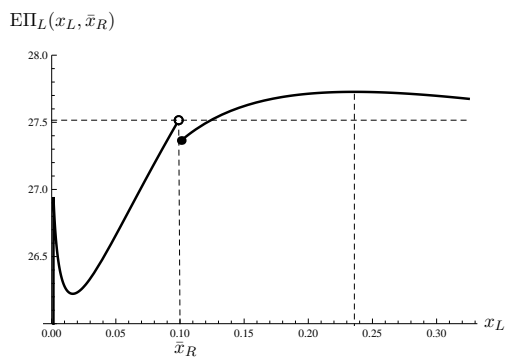
(c) Winning Probability



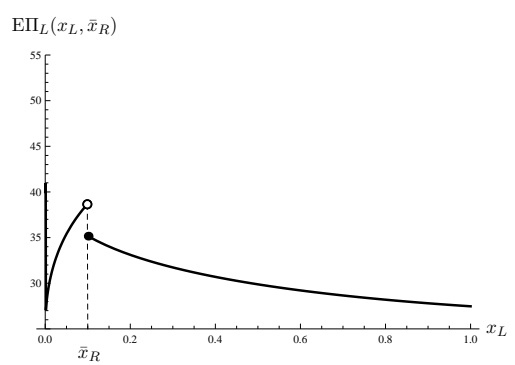
(d) L 's Objective Function when $k^L = 0$



(e) L 's Objective Function when $k^L = 1$



(f) L 's Objective Function when $k^L = 2$



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