Corrigendum to: "Discounted Stochastic Games with No Stationary Nash Equilibrium: Two Examples"*

Yehuda John Levy[†] Andrew McLennan[‡]

July 17, 2014

Abstract

Levy (2013) presents examples of discounted stochastic games that do not have stationary equilibria. The second named author has pointed out that one of these examples is incorrect. In addition to describing the details of this error, this note presents a new example by the first named author that succeeds in demonstrating that discounted stochastic games with absolutely continuous transitions can fail to have stationary equilibria.

1 Introduction

The paper "Discounted Stochastic Games with No Stationary Nash Equilibrium: Two Examples" (Levy (2013)) presents two constructions of discounted stochastic games with continuous state spaces that do not possess stationary equilibria. One is in the class of stochastic games with deterministic transitions, while the other is in the class of games in which all transitions are absolutely continuous with respect to a fixed measure.

The construction of the game from the second class in (Levy (2013)) is accomplished in three steps. 1. An example presented in Kohlberg and Mertens (1986) has a circular set of Nash equilibria, and a map from the set of equilibria to small perturbations of the game is constructed. 2. An intricate construction concerning realising piece-wise linear functions on the square as outcomes of strategic games is presented. 3. These techniques are combined to define the desired stochastic game.

*The authors are grateful to Abraham Neyman, for uncountably many proofreads, suggestions, and corrections. The authors also with to thank Wei He and Yeneng Sun for their feedback.

[†]Nuffield College, University of Oxford, UK, OX1 1NF.

[‡]School of Economics, The University of Queensland, St Lucia Qld 4072

The second named author has pointed out a serious error in Step 1 of the construction. Roughly, it is desired that each point in the set of equilibria be distant from the set of equilibria of the associated perturbed game, but for the given construction this is not the case. Section 4.6 of Levy (2013) describes conditions on a "base" strategic game (instead of the game from Kohlberg and Mertens (1986)) that would allow the construction to be carried out. However, topological obstructions preclude the existence of such a game; an accompanying note (McLennan (2014)) gives a theorem (which is stated in Section 5) that implies this. (We remark that Proposition 4.3 of Levy (2013), which is the core of Step 2 mentioned above, is correct, and both it and the techniques used to prove it will hopefully prove useful in future work.)

In addition to describing the details of the error, we present a new example, due to the first named author, of a discounted stochastic games with absolutely continuous transitions that has no stationary equilibria, thereby confirming a negative answer to the question of existence of stationary equilibria for such games. The new example resembles the earlier erroneous construction, but is significantly shorter and simpler. It retains the mechanism that transmits information around a circle, and it continues to use the "base game" taken from Kohlberg and Mertens (1986), but the relevant aspects of this game (as summarized in Lemma 3.2) are seemingly quite different from before. The underlying topological "engine" of the previous example has been replaced by a phenomenon drawn from measure theory, namely the existence of functions exhibiting a type of erratic behaviour that is typical, e.g., for the path of a Brownian motion.

The new example also has the following properties, which were claimed for the erroneous example of Levy (2013), neither of which holds for the example with deterministic transitions presented in that paper:

(1) Stationary equilibria do not exist for any discount factor.

(2) The state space is compact, payoffs are continuous, and transitions are norm-continuous.

The stochastic game model is recalled in Section 2. The new examples are presented in Section 3. The proof that these games possess no stationary equilibria is in Section 4. Section 5 gives details of the error in Levy (2013).

2 Stochastic Game Model

A stochastic game $\Gamma = \langle \Omega, \mathcal{P}, I, r, q \rangle$ with a continuum of states and finitely many actions has the following components:

- A standard Borel space[1] Ω of *states*.
- A nonempty finite set $\mathcal{P}$ of *players*.

[1] That is, a space that is homeomorphic to a Borel subset of a complete, metrizable space.

- $I = \prod_{\ell \in \mathcal{P}} I^\ell$, where each I^ℓ is a nonempty finite set of *actions* for ℓ.
- A bounded Borel-measurable *stage payoff function* $r : \Omega \times I \to \mathbb{R}^{\mathcal{P}}$.
- A Borel-measurable[2] *transition function* $q : \Omega \times I \to \Delta(\Omega)$.

Definition 2.1. *An Absolutely Continuous (A.C.) stochastic game is a stochastic game $\Gamma = \langle \Omega, \mathcal{P}, I, r, q \rangle$ for which there is a $\nu \in \Delta(\Omega)$ such that for all $z \in \Omega$ and $a \in I$, $q(z, a)$ is absolutely continuous with respect to ν.*

A *stationary strategy* for player ℓ is a Borel-measurable mapping $\sigma^\ell : \Omega \to \Delta(I^\ell)$. Let Σ_0^ℓ denote the set of stationary strategies for player ℓ, and let $\Sigma_0 = \prod_{\ell \in \mathcal{P}} \Sigma_0^\ell$ be the set of stationary strategy profiles. Together with the transition function and an initial state z, a stationary strategy profile σ induces a probability measure P_z^σ on the space $H^\infty := (\Omega \times I)^{\mathbb{N}}$ of infinite histories in a canonical way (see Levy (2013) or, e.g., Bertsekas and Shreve (1996)). For discount factor $\beta \in (0, 1)$,

$$\gamma_\sigma(z) := E_z^\sigma\big(\sum_{n=1}^{\infty} \beta^{n-1} r(z_n, a_n)\big)$$

is the expected payoff vector under σ in the game starting from state $z = z_1$. A profile $\sigma \in \Sigma_0$ is a *stationary equilibrium* of Γ for discount factor β if

$$\gamma_\sigma^\ell(z) \geq \gamma_{(\tau, \sigma^{-\ell})}^\ell(z)$$

for all $z \in \Omega$, $\ell \in \mathcal{P}$, and $\tau \in \Sigma_0^\ell$. We will present examples of A.C. stochastic games that do not possess stationary Nash equilibria for any discount factor.

For $z \in \Omega$ and $a \in I$ let

$$X_\sigma(z, a) := r(z, a) + \beta \int_\Omega \gamma_\sigma \, dq(z, a).$$

Note that $\gamma_\sigma(z) = X_\sigma(z, \sigma(z))$. We recall the following classical dynamical programming criterion for a stationary equilibrium, which is called the *one-shot deviation principle.*

Proposition 2.2. *A profile $\sigma \in \Sigma_0$ of stationary strategies is a stationary equilibrium for β if and only if, for all $z \in \Omega$, $\sigma(z)$ is a Nash equilibrium of the game $X_\sigma(z, \cdot)$.*

3 The Example

3.1 Notations

Recall that $\langle \cdot, \cdot \rangle$ denotes the inner product of vectors. In addition the following notational conventions will be used:

[2]Where $\Delta(\Omega)$, the space of regular Borel probability measures on Ω, possesses the Borel structure induced from the topology of narrow convergence.

- Throughout $\|\cdot\|$ denotes the L_∞ norm. That is, for a vector or bounded real-valued function f, $||f|| = \sup|f|$, where the supremum is taken over the set of indices or the domain of f.
- If p is a mixed action over an action space A and $a \in A$, then $p[a]$ denotes the probability that p chooses a.
- In connection with a tuple c indexed by the elements of some set $T \subset \mathcal{P}$ of players, if $\ell_1, \ldots, \ell_k \in T$, then $c^{\ell_1,\ldots,\ell_k}$ will denote $(c^{\ell_1}, \ldots, c^{\ell_k})$.

3.2 The Stochastic Game

Our construction has three phases: a) selecting four perturbations of a "base" game; b) specification of a rescaled version of the stage game; c) the stochastic game itself.

The strategic form game below is given in Appendix B of Kohlberg and Mertens (1986). Inspection reveals that there are six pure equilibria. When these are arranged in a circle, as shown in the right hand diagram, it is easy to verify that mixtures of any two "adjacent" pure equilibria are mixed equilibria. For each agent and each pair of pure strategies for that agent, such mixtures evidently encompass all the equilibria in which that agent mixes strictly over those two pure strategies. There are no equilibria in which both agents mix over all three pure strategies (for each agent R is weakly dominated) so there are no other equilibria. Thus the set of Nash equilibria is homeomorphic to a circle.

The Game G

$A\backslash B$	L	M	R
L	$1, 1$	$0, -1$	$-1, 1$
M	$-1, 0$	$0, 0$	$-1, 0$
R	$1, -1$	$0, -1$	$-2, -2$

Table 3.1a

Equilibria of G

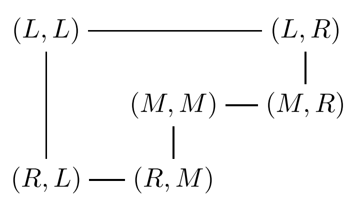


Figure 3.1b

Figure 3.1

Lemma 3.1. *For each pure Nash equilibrium of G there are games arbitrarily close to G for which that is the unique Nash equilibrium.*

Proof. First, observe that for each player L is weakly dominant, so by slightly increasing each player's payoff from playing L, (L, L) becomes the unique equilibrium. Second, for each player R is weakly dominant if the other player does not play R, so a pure equilibrium in which one player plays R becomes unique if that player's payoff from R is increased, the other player's payoff from R is decreased, and the other player's payoffs from L and M are perturbed appropriately. To make (M, M) the unique equilibrium, lower the payoffs Player B

receives from R, then lower the payoffs that Player A receives from L, then increase the payoffs both players receive from M. $\square$

Our base game is the game G above expanded to include players C and D who have trivial one element sets of pure strategies and who receive payoffs as shown below. Abusing notation slightly, we regard the pure and mixed strategy profiles of this game as pairs $a = (a^A, a^B)$ and $x = (x^A, x^B)$.

$A\backslash B$	L	M	R
L	$(1,1)$	$(0,0)$	$(1,-1)$
M	$(0,0)$	$(-1,-1)$	$(-1,-1)$
R	$(-1,1)$	$(-1,-1)$	$(0,0)$

Figure 3.2: The Payoffs to C, D ($G^{C,D}$)

The next result states the properties of G related to C and D that figure in the subsequent analysis. At least superficially, it seems to not matter that the set of Nash equilibria of G is circular. However, we do not know what topological restrictions are entailed by the following conditions.

Lemma 3.2.

(a) For each $(j,k) \in \{-1,1\}^2$, any neighborhood of G contains a game $G_{j,k}$ whose unique Nash equilibrium x satisfies $G^{C,D}(x) = (j,k)$.

(b) For any equilibrium x of G, $||G^{C,D}(x)|| = 1$.

Proof. Here (a) follows from the Lemma 3.1, and (b) is verified by inspection. $\square$

In view of (b) and the fact that $G^{C,D}(a) \in [-1,1]^2$ for all pure strategy profiles a of G, the upper semicontinuity of the Nash equilibrium correspondence implies that there is an $\eta_0 > 0$ such that

$$\tfrac{7}{8} \leq ||G^{C,D}(x)|| \leq 1$$

whenever x is an equilibrium of a game G' such that $||G' - G|| \leq \eta_0$. For each $(j,k) \in \{-1,1\}^2$ we fix a perturbation $G_{j,k}$ of G such that $\|G_{j,k} - G\| \leq \eta_0$ and the unique Nash equilibrium x of $G_{j,k}$ satisfies $G^{C,D}(x) = (j,k)$. (The payoffs $G^{C,D}_{j,k}$ play no role in what follows.) For each $z = (z^E, z^F) \in [-1,1]^2$ let G_z be the convex combination of the $(G_{j,k})$ given by

$$G_z = \sum_{(j,k)\in\{-1,1\}^2} \frac{(1 + j \cdot z^E)(1 + k \cdot z^F)}{4} G_{j,k}.$$

We turn to the rescaled version of the stage game. The set of players is $\mathcal{P} = \{A, B, C, C', D, D', E, F\}$. Players A and B have pure strategies L, M, and R as above, but in this game players C and D have pure strategies 0 and 1. Players C' and D' also have pure strategies 0 and 1, and players E and F have pure strategies -1 and 1. Pure and mixed strategy profiles will be denoted by

$$a = (a^A, a^B, a^C, a^{C'}, a^D, a^{D'}, a^E, a^F) \text{ and } x = (x^A, x^B, x^C, x^{C'}, x^D, x^{D'}, x^E, x^F).$$

The payoffs depend on a parameter $\varrho \in (-\frac{1}{2}, \frac{1}{2})$. For such a ρ let $v_E^\varrho = (1, -\varrho)$ and $v_F^\varrho = (\varrho, 1)$. The payoffs in the game $g(\varrho, \cdot)$ are:

$$\begin{aligned}
g^A(\varrho, a) &= G^A_{a^E, a^F}(a^A, a^B), \\
g^B(\varrho, a) &= G^B_{a^E, a^F}(a^A, a^B), \\
g^C(\varrho, a) &= \begin{cases} -G^C(a^A, a^B) - \frac{1}{16}, & a^C = a^{C'}, \\ -G^C(a^A, a^B) + \frac{1}{16}, & a^C \neq a^{C'}, \end{cases} \\
g^{C'}(\varrho, a) &= -g^C(\varrho, a), \\
g^D(\varrho, a) &= \begin{cases} -G^D(a^A, a^B) - \frac{1}{16}, & a^D = a^{D'}, \\ -G^D(a^A, a^B) + \frac{1}{16}, & a^D \neq a^{D'}, \end{cases} \\
g^{D'}(\varrho, a) &= -g^D(\varrho, a), \\
g^E(\varrho, a) &= a^E \cdot \langle v_E^\varrho, (2a^C - 1, 2a^D - 1) \rangle, \\
g^F(\varrho, a) &= a^F \cdot \langle v_F^\varrho, (2a^C - 1, 2a^D - 1) \rangle.
\end{aligned}$$

We now specify the stochastic game. Let δ_1 be the Dirac measure at 1. For $t \in [0, 1]$ let U_t be the uniform distribution on $[t, 1]$, identifying U_1 with δ_1. For $a \in I$ let

$$h(a) = a^C + (1 - a^{C'}) + a^D + (1 - a^{D'}), \tag{3.1}$$

and for $t \in [0, 1]$ let

$$\tilde{q}(t, a) = (1 - t) \cdot \tfrac{1}{64} h(a).$$

Let $\varrho : [0, 1] \to (-\frac{1}{2}, \frac{1}{2})$ be a measurable function. The stochastic game $\tilde{\Gamma}$ is as follows:

- The state space Ω is $[0, 1]$, with the Borel σ-algebra.
- The set of players and their action spaces are the same as in $g(\varrho, \cdot)$.
- The stage payoff function is $r(t, a) = (1 - t) g(\varrho(t), a)$.
- The transition function is $q(t, a) = \tilde{q}(t, a) U_t + (1 - \tilde{q}(t, a)) \delta_1$.

The game $\tilde{\Gamma}$ has the following features. First, 1 is an absorbing state, with payoff 0 for all players. The transitions from state t are mixtures of two types: U_t, which distributes uniformly in $[t, 1]$, or quitting to 1. As such, the game progresses towards the right. Note in particular that for all $t \in [0, 1]$ and all $a \in I$, $q(t, a)$ is absolutely continuous w.r.t. $\frac{1}{2}(U_0 + \delta_1)$. Thus $\tilde{\Gamma}$ is A.C.

The transitions are controlled by C, C', D, and D', so in each period the other players will only be concerned with maximizing their stage payoffs. The stage game payoff to C' is the negation of the stage game payoff to C, and the stage game payoff to D' is the negation of the stage game payoff to D. Consequently C and C' will have opposite views concerning the desirability of transitioning to 1, as will D and D'. Leaving aside the components of the stage game payoffs for C and C' that depend only on the behavior of A and B, the conflict between C and C' at time t is a zero sum 2×2 game that will be shown to always have a unique mixed equilibrium. The payoffs of this game, and thus also the equilibrium, depend on the relative importance of expected future payoffs and the constant terms of the current stage payoffs. The conflict between D and D' is similar, but the equilibrium will usually be different because the expected future payoffs will differ due to the disparate impact of A and B's behavior on C and C' versus D and D'.

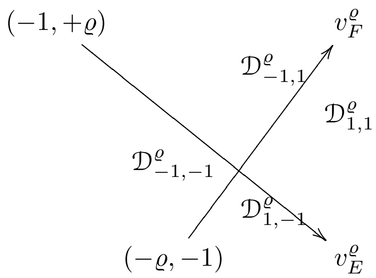
Players E and F try to guess the signs of the inner products

$$\langle v_E^\varrho, (2a^C - 1, 2a^D - 1)\rangle \quad \text{and} \quad \langle v_F^\varrho, (2a^C - 1, 2a^D - 1)\rangle$$

respectively. For $\varrho \in (-\frac{1}{2}, \frac{1}{2})$ and $j, k \in \{\pm 1\}$, let

$$\mathcal{D}_{j,k}^\varrho := \{\omega = (\omega^C, \omega^D) \in \mathbb{R}^2 \mid j \cdot \langle v_E^\varrho, \omega\rangle > 0 \text{ and } k \cdot \langle v_F^\varrho, \omega\rangle > 0\}.$$

Observe that v_E^ϱ and v_F^ϱ are orthogonal, so the $\mathcal{D}_{j,k}^\varrho$ are just the open quadrants of the plane under a certain rotation.



Set $\mathcal{D}^\varrho = \bigcup_{j,k=\pm 1} \mathcal{D}_{j,k}^\varrho$. We will see that in any stationary equilibrium of $\tilde{\Gamma}$, for almost all t the expectation of $(2a^C - 1, 2a^D - 1)$ lies in $\mathcal{D}^\varrho$, so that E and F play pure strategies, and consequently A and B are playing one of the perturbations $G_{j,k}$ of G. In this sense the behavior of A and B is well controlled.

We now state the main step in the argument:

Proposition 3.3. *Suppose that $\varrho : [0,1] \to (-\frac{1}{2}, \frac{1}{2})$ is a measurable function such that if $\pi : [0,1] \to \mathbb{R}$ is differentiable a.e., then $\pi(t) \neq \varrho(t)$ a.e. Then for any $\beta \in (0,1)$, the game $\tilde{\Gamma}$ does not possess a stationary equilibrium for discount factor β.*

To pass from this to the existence of a game without a stationary equilibrium it remains to show that there is a measurable function ϱ that disagrees a.e. with any a.e. differentiable function. It turns out that it is not enough to require that ϱ be nowhere differentiable, but there is a stronger condition that works. Let λ denote Lesbesgue measure.

Definition 3.4. *If $E \subseteq \mathbb{R}$ is Lesbesgue measurable, $f : E \to \mathbb{R}$ is Lebesgue measurable, $x \in E$, and $L \in \mathbb{R}$, then f is* approximately differentiable *at x with* approximate derivative L *if, for all $\varepsilon > 0$,*

$$\frac{1}{2\delta}\lambda\Big((x-\delta, x+\delta) \cap \big\{y \in E : \Big|\frac{f(y)-f(x)}{y-x} - L\Big| < \varepsilon\big\}\Big) \to 1$$

as $\delta \to 0$.

Clearly, if f is differentiable at x with $f'(x) = L$, then f is approximately differentiable at x with approximate derivative L. The following is included in Theorem 3.3 of (Saks, 1937, Sec VII.3)[3]:

Lemma 3.5. *If $f, g : [0,1] \to \mathbb{R}$ are Lesbesgue measurable, f is approximately differentiable a.e., g is approximately differentiable almost nowhere, and $E = \{\, x : f(x) = g(x) \,\}$, then $\lambda(E) = 0$.*

Berman (1970) shows that, with probability one, the path of a Brownian motion is nowhere approximately differentiable. It is well known that, with probability one, the path of a Brownian motion is continuous. The existence of almost nowhere approximately differentiable functions is also shown more directly in Jarník (1934); see also Preiss and Zajïcek (2000) and the references within. Consequently:

Theorem 3.1. *There exists stochastic games of the form $\tilde{\Gamma}$, for continuous ϱ, such that for each $\beta \in (0,1)$, $\tilde{\Gamma}(\beta)$ does not possess a stationary equilibrium.*

Note that the stage payoffs of $\tilde{\Gamma}$ depend in a continuous manner on the state when ϱ is continuous.

4 Proof of Proposition 3.3

Let $\varrho : [0,1] \to (-\frac{1}{2}, \frac{1}{2})$ satisfy the hypothesis of Proposition 3.3. Fix a discount factor $\beta \in (0,1)$. By way of contradiction, we suppose that σ is a stationary equilibrium of $\tilde{\Gamma}(\beta)$. We first introduce a new function, along with its most basic properties, after which our analysis has two phases.

For each $t \in [0,1]$ let

$$\omega(t) = (\omega^C(t), \omega^{C'}(t), \omega^D(t), \omega^{D'}(t)) = \tfrac{1}{8}\beta \int_0^1 \gamma_\sigma^{C,C',D,D'}(s)\, dU_t(s).$$

Lemma 4.1. *For all $t \in [0,1]$, $\omega^{C'}(t) = -\omega^C(t)$ and $\omega^{D'}(t) = -\omega^D(t)$.*

[3] For the sake of self containment we sketch the proof. A point $x \in E$ is a *density point* of E if $\lim_{\delta \to 0} \frac{1}{2\delta}\lambda(E \cap (x-\delta, x+\delta)) = 1$. Let x be a density point at which f is approximately differentiable. Then $f|_E$ is approximately differentiable at x because f is, and g is approximately differentiable at x because $g|_E$ is. By the Lesbesgue density theorem (which is a special case of the Lesbesque differentiation theorem (e.g. Federer (1969) Thm. 2.9.8)) almost every $x \in E$ is a density point.

Proof. For any t, $\gamma^C(t)$ and $\gamma^{C'}(t)$ are the expectations of random variables, each of which is the negation of the other, and similarly for $\gamma^D(t)$ and $\gamma^{D'}(t)$. $\square$

Lemma 4.2. *For all t, $||\omega(t)|| < \frac{1}{4}$.*

Proof. The probability of the game continuing (i.e., of the game *not* going to the absorbing state 1) is never greater than $\frac{1}{16}$, and $||r^{C,C',D,D'}|| \leq \frac{17}{16}$, so

$$||\omega(t)|| \leq \tfrac{1}{8} \int_t^1 ||\gamma_\sigma^{C,C',D,D'}(s)||\,ds \leq \tfrac{1}{8} \sum_{j=0}^{\infty} \tfrac{1}{16^j} \cdot ||r^{C,C',D,D'}|| = \tfrac{1}{8} \cdot \tfrac{16}{15} \cdot \tfrac{17}{16} < \tfrac{1}{4}.$$

$\square$

4.1 Equilibrium in a Stage

Fix a particular t. For $a \in I$, $\varrho \in (-\frac{1}{2}, \frac{1}{2})$, and $\omega = (\omega^C, \omega^{C'}, \omega^D, \omega^{D'}) \in \mathbb{R}^4$ let $g_\omega(\varrho, \cdot)$ be the perturbation of $g(\varrho, \cdot)$ given by

$$g_\omega^\ell(\varrho, a) = \begin{cases} g^\ell(\varrho, a), & \ell = A, D, E, F, \\ g^\ell(\varrho, a) + \frac{1}{8} h(a) \omega^\ell, & \ell = C, C', D, D'. \end{cases}$$

where recall that $h(\cdot)$ is defined in (3.1). Since $\gamma_\sigma(1) \equiv 0$, for $\ell = C, C', D, D'$ the definition of the transitions gives

$$\begin{aligned} X_\sigma^\ell(t, a) &= r^\ell(t, a) + \tilde{q}(t, a) \cdot \beta \int_0^1 \gamma_\sigma^\ell(s)\,dU_t(s) \\ &= (1-t)\Big(g^\ell(\varrho(t), a) + \tfrac{1}{64} h(a) \beta \int_0^1 \gamma_\sigma^\ell(s)\,dU_t(s)\Big) \qquad (4.1) \\ &= (1-t)\Big(g^\ell(\varrho(t), a) + \tfrac{1}{8} h(a) \omega^\ell(t)\Big) = g_{\omega(t)}^\ell(\varrho(t), a). \end{aligned}$$

For $\ell = A, B, E, F$ the difference between $X_\sigma^\ell(t, a)$ and $(1-t) g_{\omega(t)}^\ell(\varrho(t), a)$ is the expected future payoff, which is unaffected by a^ℓ. Therefore the one shot deviation principle implies that:

Lemma 4.3. *In the game $X_\sigma(t, \cdot) - (1-t) g_{\omega(t)}(\varrho(t), \cdot)$ no player has any affect on her own payoff. Consequently $\sigma(t)$ is an equilibrium of $g_{\omega(t)}(\rho(t), \cdot)$.*

For the sake of more compact notation we write x in place of $\sigma(t)$ and ϱ and ω in place of $\varrho(t)$ and $\omega(t)$ in the remainder of this subsection, which extracts the relevant consequences of x being an equilibrium of $g_\omega(\varrho, \cdot)$. Let

$$y = (y^C, y^D) = (2x^C[1] - 1, 2x^D[1] - 1)$$

and

$$z = (z^E, z^F) = (2x^E[1] - 1, 2x^F[1] - 1).$$

Equilibrium analysis for $g_\omega(\varrho, \cdot)$ has the following consequences:

Lemma 4.4.

(a) $x^{A,B}$ *is an equilibrium of* G_z*;*

(b) $x^C[1] = \frac{1}{2} + \frac{1}{2}\omega^C$ *and* $x^{C'}[1] = \frac{1}{2} - \frac{1}{2}\omega^{C'}$*;*

(c) $x^D[1] = \frac{1}{2} + \frac{1}{2}\omega^D$ *and* $x^{D'}[1] = \frac{1}{2} - \frac{1}{2}\omega^{D'}$*;*

(d) $z^E = 1\,(-1)$ *if* $\langle v_E^\varrho, y\rangle > (<)\, 0$*;*

(e) $z^F = 1\,(-1)$ *if* $\langle v_F^\varrho, y\rangle > (<)\, 0$*;*

(f) If $\omega^{C,D} \in \mathcal{D}_{j,k}^\varrho$*, then* $z = (j,k)$*.*

(g) If $\omega^{C,D} \in \mathcal{D}_{j,k}^\varrho$*, then* $G^{C,D}(x) = (j,k)$*.*

Proof. Here (a) follows from $g_\omega^{A,B}(\varrho,\cdot) = g^{A,B}(\varrho,\cdot)$. Observe that $g_\omega^{C,C'}(\varrho,x)$ is the sum of

$$(-G^C(x^A,x^B), G^C(x^A,x^B)) + \tfrac{1}{8}(x^D[1] + x^{D'}[0])\omega^{C,C'}$$

and the payoffs resulting from applying $x^{C,C'}$ to the bimatrix game

$C\backslash C'$	1	0
1	$-1+4\omega^C, 1+4\omega^{C'}$	$1+2\omega^C, -1+2\omega^{C'}$
0	$1+2\omega^C, -1+2\omega^{C'}$	$-1, 1$

.

Only the last term is affected by $x^{C,C'}$, so $x^{C,C'}$ must be an equilibrium of the bimatrix game. When $|\omega^C|, |\omega^{C'}| < 1$ (as per Lemma 4.2) this bimatrix game has a unique equilibrium. Now direct calculation shows that $x^C[1] = \frac{1}{2} - \frac{1}{2}\omega^{C'}$ and $x^{C'}[1] = \frac{1}{2} + \frac{1}{2}\omega^C$. Recognizing that $\omega^{C'} = -\omega^C$ (Lemma 4.1) gives (b), and (c) follows symmetrically. Substituting into the definition of y gives $y = \omega$, after which (d), (e), and (f) follow from $g_\omega^{E,F}(\varrho,\cdot) = g^{E,F}(\varrho,\cdot)$, while (g) follows from (a) and (f) and the choice of the games $(G_{j,k})$. $\square$

From the specification of the payoffs we have $||g^{C,D}(\varrho,x) - (-G^{C,D}(x))|| \le \frac{1}{16}$. Since $\|G_z - G\| < \eta_0$ we have $\frac{7}{8} \le ||G^{C,D}(x)|| \le 1$, which yields the following result, and if $\omega^{C,D} \in \mathcal{D}^\varrho$, then (g) yields (a) of Lemma 4.6 below.

Lemma 4.5. $\frac{13}{16} \le ||g^{C,D}(\varrho,x)|| \le \frac{17}{16}$.

Lemma 4.6. *If* $\omega^{C,D} \in \mathcal{D}^\varrho$*, then:*

(a) $\frac{15}{16} \le |g^C(\varrho,x)| \le \frac{17}{16}$ *and* $\frac{15}{16} \le |g^D(\varrho,x)| \le \frac{17}{16}$*;*

(b) if $|\omega^C| \ge \frac{1}{2}|\omega^D|$*, then* $g^C(\varrho,x)\cdot\omega^C < 0$*;*

(c) if $|\omega^D| \ge \frac{1}{2}|\omega^C|$*, then* $g^D(\varrho,x)\cdot\omega^D < 0$*.*

Proof. By symmetry it suffices to prove (b). As $\omega \ne 0$, $|\omega^C| \ge \frac{1}{2}|\omega^D|$ implies $\omega^C \ne 0$. Since $|\varrho| < \frac{1}{2}$, if $\omega^D \ne 0$ then $|\varrho\cdot\omega^D| < \frac{1}{2}|\omega^D| \le |\omega^C|$, so

$$\mathrm{sign}(\langle v_E^\varrho, \omega^{C,D}\rangle) = \mathrm{sign}(\omega^C - \varrho\omega^D) = \mathrm{sign}(\omega^C).$$

If $\omega^{C,D} \in \mathcal{D}_{j,k}^\varrho$, then $\mathrm{sign}(\langle v_E^\varrho, \omega^{C,D}\rangle) = j = G^C(x)$ (Lemma 4.4(g)) and $|g^C(\varrho,x) - (-G^C(x))| \le \frac{1}{16}$. $\square$

4.2 Equilibrium Over Time

For $t \in [0,1]$ let

$$V^\ell(t) = \gamma_\sigma^\ell(t) \quad \text{and} \quad W^\ell(t) = \int_t^1 V^\ell(s)\,ds.$$

Clearly $W(1) = 0$, and since 1 is an absorbing state with payoff 0, $V(1) = 0$. Since V is measurable and bounded, W is Lipschitz. Rademacher's theorem (e.g. Federer (1969) Thm. 3.1.6) implies that W is a.e. differentiable. Let

$$J(t) = \tfrac{1}{2}(W^C(t))^2 + \tfrac{1}{2}(W^D(t))^2.$$

Proposition 4.7. *$J' > 0$ a.e.*

Since J is absolutely continuous, the fundamental theorem of calculus holds, so $0 = J(1) > J(0) \geq 0$. This contradiction proves Proposition 3.3, so it remains only to prove Proposition 4.7. Two lemmas prepare the main argument.

Lemma 4.8. *For all $t \in [0,1)$:*

(a) $||V^{C,D}(t) - r^{C,D}(t,\sigma(t))|| \leq \frac{1}{8}(1-t)$.

(b) $\frac{11}{16}(1-t) \leq ||V^{C,D}(t)|| \leq \frac{19}{16}(1-t)$.

Proof. Since

$$V^{C,D}(t) = \gamma_\sigma^{C,D}(t) = X_\sigma^{C,D}(t,\sigma(t)) = (1-t)g_{\omega(t)}^{C,D}(\varrho(t),\sigma(t)),$$

(a) follows from $\|\omega(t)\| < \frac{1}{4}$ and $\frac{1}{8}h(a) \leq \frac{1}{2}$. To obtain (b), combine (a) and the result of multiplying the inequality of Lemma 4.5 by $1-t$. □

Lemma 4.9. *For a.e. $t \in [0,1)$:*

(a) $W^{C,D}(t) \neq 0$.

(b) $\omega^{C,D}(t) \in \mathcal{D}^{\varrho(t)}$.

(c) If $|W^C(t)| \geq \frac{1}{2}|W^D(t)|$, *then* $V^C(t) \cdot W^C(t) \leq -\frac{13}{16}(1-t)|W^C(t)|$.

(d) If $|W^D(t)| \geq \frac{1}{2}|W^C(t)|$, *then* $V^D(t) \cdot W^D(t) \leq -\frac{13}{16}(1-t)|W^D(t)|$.

Proof.

(a) This follows Lemma 4.8 since $\frac{dW}{dt} = -V$ a.e.

(b) Define $\eta : [0,1] \to \mathbb{R}^2$ by $\eta(t) = \frac{\omega^{C,D}(t)}{||\omega^{C,D}(t)||}$. This is a.e. defined and a.e. differentiable, since both the denominator and numerator are Lipshitz, hence a.e. differentiable, and the latter is a.e. non-zero by (a). (Observe that $\omega^{C,D}(t)$ is proportional to $W^{C,D}(t)$.) Clearly, $\eta(t) \in \mathcal{D}^{\varrho(t)}$ if and only if $\omega^{C,D}(t) \in \mathcal{D}^{\varrho(t)}$. For a.e. t, the requirement $\eta(t) \notin \mathcal{D}^{\varrho(t)}$ is equivalent (because $||\eta(\cdot)|| \equiv 1$ and $||\varrho|| < \frac{1}{2}$) to $\eta(t) \in \{\pm(\varrho(t),1), \pm(1,-\varrho(t))\}$. Due to the assumed irregularity of $\varrho(\cdot)$, $\eta^C(t) \neq \pm\varrho(t)$ and $\eta^D(t) \neq \pm\varrho(t)$ for almost all t.

(c) Since $W^{C,D}(t)$ is a positive scalar multiple of $\omega^{C,D}(t)$, it suffices to prove the claim with $\omega^{C,D}(t)$ in place of $W^{C,D}(t)$. In view of (b) we may assume that $\omega^{C,D}(t) \in \mathcal{D}^{\varrho(t)}$, so $r^C(t, \sigma(t)) \cdot \omega^C(t) < 0$, $\frac{13}{16}(1-t) \le |r^C(t, \sigma(t))|$, and $|r^C(t, \sigma(t)) - V^C(t)| \le \frac{1}{8}(1-t)$. (These inequalities follow from Lemma 4.6(b), Lemma 4.6(a), and Lemma 4.8(a) respectively). Collectively these facts imply the claim.

(d) By symmetry, the proof of (c) also establishes (d).

□

Proof of Proposition 4.7. Let $t \in [0, 1)$ be such that all the properties of Lemma 4.9 hold. To simplify notation we drop the argument t. The chain rule gives $J' = -W^C \cdot V^C - W^D \cdot V^D$, so it suffices to show that $W^C \cdot V^C + W^D \cdot V^D < 0$. Either

$$|W^C| \ge \tfrac{1}{2}|W^D| \quad \text{and hence} \quad V^C \cdot W^C \le -\tfrac{13}{16}(1-t)|W^C|$$

or

$$|W^D| \ge \tfrac{1}{2}|W^C| \quad \text{and hence} \quad V^D \cdot W^D \le -\tfrac{13}{16}(1-t)|W^D|.$$

If both hold, then

$$V^C \cdot W^C + V^D \cdot W^D \le -\tfrac{13}{16}(1-t) \cdot (|W^C| + |W^D|) < 0.$$

(The final inequality is from Lemma 4.9(a).) Therefore we may suppose that one of these holds, say the first without loss of generality, and the other does not, so $|W^D| < \frac{1}{2}|W^C|$. Since $|V^D| \le \frac{19}{16}(1-t)$ (Lemma 4.8),

$$\begin{aligned} V^C \cdot W^C + W^D \cdot V^D &\le -\tfrac{13}{16}(1-t)|W^C| + |W^D| \cdot |V^D| \\ &\le (1-t) \cdot (-\tfrac{13}{16}|W^C| + \tfrac{19}{16}|W^D|) \\ &< (1-t) \cdot (-\tfrac{13}{16}|W^C| + \tfrac{19}{16} \cdot \tfrac{1}{2}|W^C|) \\ &= -\tfrac{7}{32}(1-t)|W^C| < 0. \end{aligned}$$

□

5 Description of the Error

The error in Levy (2013) has the following description. Part (iv) of the Proposition 4.1 on page 1991 is not correct. On page 1990 there is the following game with ε positive and small:

$G_Z(\frac{1}{2}) =$

$A\backslash B$	L	M	R
L	1 - ε,1 - ε	-ε,-1	-1, 1 - ε
M	-1,-ε	ε, ε	-1 + ε, -ε
R	1 - ε, -1	-ε, -1 + ε	-2,-2

(When $\varepsilon = 0$ this is the Kohlberg-Mertens game in Section 3.) As the paper points out, this has the pure equilibria (L, L) and (M, M) and the mixed equilibrium

$$\left(\tfrac{2\varepsilon}{2+\varepsilon}L + \tfrac{2-\varepsilon}{2+\varepsilon}M, \tfrac{2\varepsilon}{2+\varepsilon}L + \tfrac{2-\varepsilon}{2+\varepsilon}M\right).$$

The problem, which concerns (iv) of Proposition 4.1, arises from the fact that these are not the only equilibria. Indeed, in no equilibrium can both players use the strategy R with positive probability due to dominance. However, there are equilibria in which one player uses R. Specifically, the entire set of equilibria is $NE := \{(M, M)\} \cup P_1 \cup P_2$ where

$$\begin{aligned} P_1 = \operatorname{con}\{L, \tfrac{\varepsilon}{2}L + (1 - \tfrac{\varepsilon}{2})R\} \times \{L\} \\ \cup \{\tfrac{\varepsilon}{2}L + (1 - \tfrac{\varepsilon}{2})R\} \times \operatorname{con}\{L, \tfrac{2\varepsilon}{2+\varepsilon}L + \tfrac{2-\varepsilon}{2+\varepsilon}M\} \\ \cup \operatorname{con}\{\tfrac{\varepsilon}{2}L + (1 - \tfrac{\varepsilon}{2})R, \tfrac{2\varepsilon}{2+\varepsilon}L + \tfrac{2-\varepsilon}{2+\varepsilon}M\} \times \{\tfrac{2\varepsilon}{2+\varepsilon}L + \tfrac{2-\varepsilon}{2+\varepsilon}M\} \end{aligned}$$

(here con denotes the convex hull) and P_2 is the image of this under transposition of the players.

In Proposition 4.1 $E_{x\otimes y}[\vartheta]$ is a bilinear $\mathbb{R}^2$-valued function of (x, y), and (iv) of Proposition 4.1 holds if and only if the first component of $E_{x\otimes y}[\vartheta]$ is close to 1 for all $(x, y) \in NE$ but for $\left(\tfrac{\varepsilon}{2}L+(1-\tfrac{\varepsilon}{2})R, L\right)$ and $\left(\tfrac{\varepsilon}{2}L+(1-\tfrac{\varepsilon}{2})R, \tfrac{2\varepsilon}{2+\varepsilon}L+\tfrac{2-\varepsilon}{2+\varepsilon}M\right)$ this is quite far from being the case.

As we mentioned at the outset, Section 4.6 of Levy (2013) specifies conditions on a game (which would have the role played by the Kohlberg-Mertens game in the overall construction) that would allow the construction to succeed, but Corollary 5.1 below implies that they cannot hold.

McLennan (2014) gives the following theorem.

Theorem 5.1. *Let X be a compact convex subset of a locally convex topological space, let $U \subset X$ be open with $\overline{U}$ compact, let $F : \overline{U} \to X$ be an upper semicontinuous convex valued correspondence with no fixed points in $\overline{U} \setminus U$, let P be a compact absolute neighborhood retract, and let $\rho : \overline{U} \to P$ be a continuous function. If the fixed point index of F is not zero, then there is a neighborhood V of F in the (suitably topologized) space of upper semicontinuous convex valued correspondences from $\overline{U}$ to X such that for any continuous function $g : P \to V$ there is a $p \in P$ and a fixed point x of $g(p)$ such that $\rho(x) = p$.*

To obtain the following result as a consequence of this, let X be the set of mixed strategy profiles of G, let F be its best reply correspondence, and for $e \in P$, let $g(e)$ be the best response correspondence of $h(e)$.

Corollary 5.1. *If G is a finite strategic form game,* NE *is its set of Nash equilibria, P is a compact subset of* NE *that is an absolute neighborhood retract*[4]*, U is a neighborhood of* NE *in the space of mixed strategy profiles, and $\rho : U \to P$ is a retraction, then there is a neighborhood W of G in the space of games (for the given strategic form) such that for any continuous $h : P \to W$ there is some $e \in P$ such that $\rho^{-1}(e)$ contains a Nash equilibrium of $h(e)$.*

[4]It is always the case that NE is itself an absolute neighborhood retract—for details see McLennan (2014).

References

Berman, S. (1970): "Gaussian Processes with Stationary Increments: Local Times and Sample Function Properties," *Annals of Mathematical Statistics*, 41, 1173–1396.

Bertsekas, D., and S. Shreve (1996): *Stochastic Optimal Control: The Discrete Time Case*. Athena Scientific, Belmont, MA.

Federer, H. (1969): *Geometric Measure Theory*. Springer, New York.

Jarník, V. (1934): "Sur la Dérivabilité des Fonctions Continues," *Publications de la Faculty des Sciences de L'Université Charles*, 129, 9.

Kohlberg, E., and J.-F. Mertens (1986): "On the Strategic Stability of Equilibria," *Econometrica*, 54, 1003–1038.

Levy, Y. (2013): "Discounted Stochastic Games with No Stationary Nash Equilibrium: Two Examples," *Econometrica*, 81(5), 1973–2007.

McLennan, A. (2014): "Fixed Points of Parameterized Perturbations," forthcoming, *Journal of Mathematical Economics*.

Preiss, D., and L. . Zajïcek (2000): "On Dini and Approximate Dini Derivatives of Typical Continuous Functions," *Real Analysis Exchange*, 26, 401–412.

Saks, S. (1937): *The Theory of the Integral*, vol. 7 of *Monografie Matematyczne*. G.E. Stechert and Co., New York, 2nd edn.