A Model of Two-stage Electoral Competition with Strategic Voters*

Shino Takayama†
University of Queensland

Abstract. This paper proposes a two-party spatial model of policy and valence issues for office-seeking candidates who face a two-stage electoral process with strategic voters. We study how the difference in valences among candidates affects the equilibrium outcomes when voters are strategic and candidates consider their winning chances in general and primary elections. Our results indicate that compared with the case of just maximizing her party median voters expected payoff, a forward-looking challenger chooses a more moderate policy so that she can appeal to the general population, and that the winning probability in the general election for the winning candidate in the primary election increases because of the more moderate policy promise that she chooses. The model is analytically tractable, and provides a vehicle for answering normative questions about holding primary elections. Finally, we provide empirical predictions on primaries and the roles of valences in elections.

Key Words: primary election, median voter, uncertainty, valence.
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†Takayama; School of Economics, University of Queensland, Level 6 - Colin Clark Building (39), St Lucia, QLD 4072, Australia; e-mail: s.takayama@economics.uq.edu.au; tel: +61-7-3346-7379; fax: +61-7-3365-7299.

1 Introduction

In this paper, we develop a theoretical framework of two-stage electoral competitions starting at primaries and study whether or not holding a primary election actually enhances a party's chances in the general election, and how it affects policy promises. Despite many real examples around the world, theoretical works on party primaries are quite scarce in the literature. We propose a two-party spatial model to analyze policy and valence issues for office-seeking candidates who face a two-stage electoral process where voters vote strategically in the sense that they also consider each candidate's winning probability in the general election at the time of the primary election. We provide an equilibrium analysis that is analytically tractable and deals with the multidimensional issues of policy promises and valence, which a forward-looking politician faces between the two stages of elections.

In our model, voters judge candidates on two aspects: policy promises and valence, and primary election candidates choose policy promises. In the first stage, the challenging party holds a primary election. The challenging party has two candidates and one candidate has higher valence than the other. Voters in the primary choose strategically between the two candidates. In the second stage, the winning candidate in the primary election competes with the incumbent, where there is uncertainty about the realization of the median voter's bliss point in the general election. In the model, we make the *no etch-a-sketch* assumption that politicians are committed to one policy for both elections. We define a political equilibrium by using a notion of Nash equilibrium. Our result demonstrates that in equilibrium, the challenger with higher valence chooses the maximizer of the winning probability in the general election subject to just beating the other challenger in the primary election. Then, we compare the political equilibrium outcome with the outcome from a different scenario such that a challenger tries to maximize the expected payoff of the median voter in their party. We find that under a political equilibrium, a more moderate policy, which is further away from the bliss point of their own party's median voter and more appealing to the general population, is selected, and the winning probability of a candidate with higher valence increases.

Finally, we obtain the following predictions from our model:

- as the incumbent's valence increases, the challenging candidate chooses a policy promise that is less favorable to their own party's median voter in order to appeal to the general population;
- as the ideological difference between the incumbent party's policy and the challenging party's median policy decreases, valence becomes more important to the electoral outcome.

In practice, primary elections are commonly used to select a party representative who competes with other parties' nominees in the subsequent general election. Galderisi and Ezra (2001) state that in general, the direct primary was thought to be a method of controlling the corrupting influences of the urban political machine. Ware (2002) studies the political history of the US, and claims that the direct primary was the result of an attempt by politicians to subject their previously informal procedures to formal rules, which started in the late 1880s. Chhibber and Kollman (2009) study the origins of political parties in Canada, Great Britain, India, and the US. They state that the earliest party caucuses by Hamilton and Jefferson in the 1790s were created to organize the House of Representatives into policy-making blocs. Recently, primaries have been adopted in many countries other than the US, including some European and South American countries (for empirical evidence, see Morton, 2006). In 2009, the UK Conservative Party experimented with the use of open primaries to select two of its parliamentary candidates. Political parties throughout Latin America are relying increasingly on primary elections to select presidential candidates (see Carey and Polga-Hecimovich, 2006).

There is a large body of empirical work on valences and policy promises in the literature. Many studies have analyzed how party elites' valences affect voters' electoral support for parties (McCurley and Mondak, 1995, Mondak, 1995, Stone and Simas, 2010), although the relationship between valences, policy decisions, and winning chances remains uncertain. Scholarly evidence from studies of the US, and from recent cross-country research of Western Europe, suggests that changes to parties' or candidates' valences carry important electoral consequences and shows that there is some relationship between valence, voters' choices, and the ideological distance between the candidates (see Abney *et al.*, 2013, Clark and Leiter, 2013, Green and Hobolt, 2008, Adams *et al.*, 2012, Buttice and Stone, 2012).

Recently, Hirano and Snyder (2014) have argued that the literature underestimates the value of primaries, and primary elections are most needed in safe constituencies, where the advantaged party's candidate can usually win the general election. On the one hand, an increasing body of empirical research has considered the effects of primaries on electoral prospects using more recent or international data sets (for example, see Carey and Polga-Hecimovich, 2006, Stone *et al.*, 1992). However, until recently, in the theoretical literature, most models of electoral competition assume that candidates representing their political parties compete in one-stage elections. Recently, within the framework proposed by Coleman (1971), Owen and Grofman (2006) develop a model of two-stage electoral competition between two parties in one dimension and study candidates' policy positions. Furthermore, Serra (2011) proposes a model of party elites such that party elites who have a different bliss point with the party's median voter decide whether or not to hold a primary election.

In the literature (e.g., see Adams and Merrill, 2008), it is known that there are two opposing forces: how do office-seeking candidates balance the *centrifugal* incentive to appeal to their primary voters against the *centripetal* motivation to appeal to voters in the general election if they have to commit to the same policy promises at both stages of the electoral competition? A difficulty is to formulate what is an optimal policy choice for each candidate who faces the two opposing incentives, especially when voters in primaries are strategic. The literature so far has assumed that primary voters vote sincerely, or that there is a party bliss point such as a party's median voter, where each candidate in the primary tries to maximize the expected payoff (see McGann, 2002, Adams and Merrill, 2008, Owen and Grofman, 2006), although empirical studies suggest that some primary voters are strategic and consider the candidates' prospects in the general election (Abramowitz, 1989).

This paper aims to provide a micro-founded tractable model of a two-stage electoral process with policy and valence issues in the presence of primaries. The model is analytically tractable, and can consider both valence and policy promises. It provides a vehicle for answering normative questions about holding a primary election. Our findings are testable and generally consistent with the empirical research on primaries and the roles of valences in elections. Our analysis shows how the difference in the valences of politicians affects policy promises to achieve a balance between the centrifugal and centripetal motivations in equilibrium.

The closest analysis to ours is Adams and Merrill (2008), who propose a model where there are two candidates in one party who try to maximize the joint winning probability in both the general election and the primary election, and voters vote sincerely. Then, they prove the existence of a Nash equilibrium when the voter's utility is concave and peaks at the voter's ideal point. Furthermore, they show that when the candidates have the same valence, they choose the same policy position in equilibrium, and as the other party's candidates' valence increases, candidates have incentives to shift unilaterally toward the median voter in the general election. The key difference in our analysis is that voters also care about the prospect of each candidate in the general election, while in their analysis voters are assumed to vote sincerely, and thus the median voter theorem in the primary election is assumed to hold. We show that even when voters vote strategically in the sense that primary voters consider the candidates' prospective appeal in the general election, and the expected induced payoffs of primary voters may not be single-peaked, the median voter theorem in the primary election still holds under a mild technical assumption. Furthermore, Adams and Merrill (2008) numerically demonstrate that primaries benefit weaker parties in that the primary candidates locate their policy closer to the center to become more competitive in the general election. In their analysis, when neither party has a primary election, each candidate locates at the position of the overall median voter and each has an equal chance of winning the general election. Intuitively, a candidate with lower valence needs to choose a more moderate policy in order to appeal to the general electorate. When the valence becomes more important compared with policy in voters' preferences, this tendency becomes stronger, and so a party with weaker candidates benefits from holding primaries more than a stronger party because of the more moderate policy choice.

In this paper, we further provide a closed-form analysis on the relationship between the primary advantage and the salience of valences in voters' preferences, and also a set of testable hypotheses on the relationship between valences of candidates and the policy promises that they would choose. In contrast with the numerical analysis in Adams and Merrill (2008), our results show that a candidate with higher valence may be able to choose a more moderate policy in order to appeal to the general electorate while still winning the primary election.

The remainder of the paper is organized as follows. Section 2 details the model. Section 3 undertakes the equilibrium analysis of policy positions by candidates. Section 4 provides some numerical examples and discusses our findings and testable hypotheses. The final section discusses possible extensions of the model.

2 The Model

In the model, we assume that there are two parties: an incumbent party and a challenging party. The incumbent party has one politician, who we call the *incumbent*. The challenging party has two candidates, who are called the *challengers*. We assume that there are two stages of the election: a primary and a general election. The two challengers first compete in the primary election, then the winner of the primary competes with the incumbent in the general election, and the winner in the general election obtains office.

The timing of the game is as follows.

Stage 1 At the beginning of the game, the challenging candidates simultaneously choose their policy promises, p_L and p_H, respectively. We assume that a politician in the challenging party chooses one policy promise that is common in the primary election and the general election if she wins the candidacy in the primary election. We call this the *no etch-a-sketch* assumption.[1]

Stage 2 The primary election is held for the challenging party.

Stage 3 The general election is held.

[1] Asked whether Mr. Romney had moved too far to the right for the general election in 2012, Mr. Fehrnstrom said that they would hit a reset button for the fall campaign, "almost like an Etch A Sketch." Then, two other candidates from the Republican Party held up an Etch A Sketch toy as a visual aid to their audiences and criticized the remark (`http://en.wikipedia.org/wiki/Mitt_Romney_presidential_campaign,_2012`).

Each politician is characterized by two aspects: policy promise and valence. The valence variable is fixed exogenously, and benefits all voters independently of their political opinions. The incumbent's policy promise is fixed. The policy space is $[0, 1]$. Let $p \in [0, 1]$ denote a policy promise and $v \in I\!R$ the valence measured in utility terms. We identify each voter with her bliss point. The utility of voter i when the winning candidate has valence v and policy p is: [2]

$$u_i(v, p) = v - |p - i|. \tag{1}$$

Assume that the incumbent's policy promise is $\bar{p}$. We assume that, perhaps because of a commitment to the previous policy, $\bar{p}$ is fixed and publicly known. There are three politicians in the model. The incumbent's valence is denoted by $\bar{v} \in I\!R$. The two challengers, called challenger L and challenger H, have valences v_L and v_H, respectively, with $v_L < v_H$. We assume that $\bar{v}$, v_L, and v_H are public information. Let $d_k = \bar{v} - v_k + \bar{p}$ for each $k = H, L$. Then, $u_i(v_k, d_k) = u_i(\bar{v}, \bar{p})$ for all $i \geq \max\{d_k, \bar{p}\}$.

There is a measure μ of the right-wing party voters. Let the right-wing party median voter's bliss point be $R > \bar{p}$. We assume that v_L ensures that the support of the measure μ is contained in $\delta = (\frac{d_L+R}{2}, 1]$. Then, for almost all $r \in \delta$,

$$u_r(v_L, R) > u_r(\bar{v}, \bar{p}). \tag{2}$$

This will have the consequence that, in equilibrium, there is no voter in the primary election who is motivated to vote for a candidate whom the incumbent can beat more easily.

We denote the bliss point of the overall median voter by M. This is assumed to be randomly distributed in $[\bar{p}, R]$, with a cumulative distribution function, denoted by F. We assume that F is concave and twice differentiable, and the hazard rate $\frac{F'(x)}{1-F(x)}$ is increasing in the region. We assume that F is common knowledge.

We assume that for each $k = H, L$,

$$|\bar{v} - v_k| < (R - \bar{p}). \tag{3}$$

This assumption rules out two cases.

- $\bar{v} - \bar{p} + \bar{p} < v_H - R + \bar{p}$ – when this holds, the incumbent cannot win the general election. Even if challenger H chooses R, she can win the general election with a probability of one. Thus, challenger H chooses some policy promise that will beat challenger L in the primary election, and then wins the general election with a probability of one.

[2]One may wonder if we can extend our analysis to a more general case where $u_i(v, p) = h(v) - g(|p - i|)$ with some more general forms of g. We discuss this point in Proposition 1. We will analyze the relationship between salience of valences for the electorate and the winning probability by using a function of v rather than v itself and discuss our findings in relation to the empirical literature in Section 4.

- $v_L + R < \bar{v} + \bar{p}$ – when this holds, challenger L is too uncompetitive and becomes irrelevant in the game. If challenger H has higher valence than the incumbent, challenger H can set any policy with which she can win the general election with a probability of one, otherwise challenger H sets a policy promise that appeals to the median voter in the general election with the highest probability.

In addition, note that (3) is consistent with (2), because almost every voter $r \in \delta$ has a bliss point $r > \bar{p}$, and thus the cutoff point $\frac{d_L+R}{2}$ must be greater than $\bar{p}$. When $\frac{d_L+R}{2}$, it requires $R - \bar{p} > \bar{v} - v_L$.

One challenger is denoted by $k = H, L$ and the other challenger is denoted by $-k$ (i.e., when $k = L$, $-k = H$). Define the interval I_k for each $k = H, L$ by

$$I_k = [\max\{2\bar{p} - d_k, d_k\}, 2R - d_k],$$

and let I_k° denote the interior of I_k. As we will explain later, for each challenger k there is an equilibrium policy promise that belongs to this interval. In particular, when the equilibrium is unique, the equilibrium policy promises belong to this interval. Any policy outside of I_k gives a constant winning probability of 0 or 1.

Let $\pi(v_k, p_k)$ denote the probability that candidate k beats the incumbent in the general election. Then, by the median voter theorem, for each $k = H, L$, $\pi(v_k, p_k)$ is indeed the probability that M votes for challenger k:

$$\pi(v_k, p_k) = \Pr\left[u_M(v_k, p_k) \geq u_M(\bar{v}, \bar{p})\right]. \tag{4}$$

Then, for each $p_k \in I_k$:

$$\pi\left(v_k, p_k\right) = 1 - F\left(\frac{d_k + p_k}{2}\right). \tag{5}$$

For each $k = H, L$, when challenger k wins the primary election, voter r's expected utility is given by:

$$\mathrm{E}u_r(v_k, p_k) = (1 - \pi(v_k, p_k))u_r(\bar{v}, \bar{p}) + \pi(v_k, p_k)u_r(v_k, p_k). \tag{6}$$

The following definition expresses the notion of Nash equilibrium for our model.

Definition 1. *A* political equilibrium in a game of two-stage electoral competition *is a pair of policy promises (p_L^*, p_H^*) and a pair of winning probabilities in the primary election (w_L^*, w_H^*) such that:*

(1) $w_L^* + w_H^* = 1$;

(2) if $\mathrm{E}u_R(v_k, p_k^*) > \mathrm{E}u_R(v_{-k}, p_{-k}^*)$, then $w_k^* = 1$;

and for each $k = H, L$, there is no p'_k such that:

(3) $\mathrm{E}u_R(v_k, p'_k) > \mathrm{E}u_R(v_{-k}, p^*_{-k})$*; and*

(4) $\pi(v_k, p'_k) > w^*_k \cdot \pi(v_k, p^*_k)$.

In our model, candidates try to maximize the probability of eventually winning the general election subject to the constraints for the primary election.

3 Equilibrium Analysis

3.1 An Equilibrium with the Primary

In this section, we solve for the equilibrium in our game. Our main theorem demonstrates the following equilibrium. Challenger L chooses a policy promise that maximizes median voter R's expected payoff. Because challenger H can always select a policy promise that will beat challenger L in the primary election, challenger H always wins the primary. As a result, challenger H chooses the maximizer of the winning probability in the general election subject to just beating challenger L in the primary election.

Note that the following three cases are possible.

1. when $\bar{v} > v_H$ holds. In this case, there is a unique equilibrium and the equilibrium policy promise belongs to each I_k.

2. when $v_H \geq \bar{v} \geq v_L$ holds. In this case, there is an interval of policy promises for which challenger H can win the general election with a probability of one, and there exist multiple equilibria where challenger H wins the general election with a probability of one.

3. when $v_L > \bar{v}$ holds. In this case, there is an interval of policy promises for which each can win the general election with a probability of one. Unlike the second case, challenger H's decision is affected by challenger L's decision, although the minimal point of I_H still dominates any policy promise chosen by challenger L in terms of median voter R's expected payoff, and still there exist multiple equilibria.

In what follows, we first focus on the intervals I_L and I_H. Under any of the three cases, there is an equilibrium policy for each challenger that belongs to the intervals. It is easy to verify that any policy p_k, which is smaller than d_k or greater than $2R - d_k$, gives $\pi(v_k, p_k) = 0$. In case 1, if a challenger chooses a policy promise in the interval outside of I_k, the opponent can easily beat her in the primary election by choosing a policy promise in I_k and still obtain a positive winning probability in the general election. Therefore, in case 1, policy promises outside of I_k cannot be supported in equilibrium. Moreover, in both case 2 and case 3, because the minimal point of I_H gives the winning probability of one for challenger H, this choice by challenger H can beat challenger L even if challenger L chooses the maximizer of median voter R's expected payoff, which is also located in I_L.

Hence, first we show that the equilibrium in the intervals exists uniquely, and then demonstrate other equilibria outside of the interval when multiple equilibria exist. To prove the theorem, we start by showing that in the primary election, the median voter theorem holds in that median voter R's most favorable policy promise can beat any other policy promise in the set of each challenger's possible choices. In our model, the primary voters do not necessarily hold single-peaked induced preferences. In particular, the voters whose bliss points are more leftish do not have single-peaked induced preferences, while in the interval of more rightish points, their induced preferences are U-shaped. The payoff when the incumbent wins becomes larger than the payoff when a challenger choosing these extreme policy promises wins. However, by (2), they still prefer a challenger if she chooses median voter R's optimal policy. As such, median voter R's most favorable policy beats any other policy promises and obtains the majority.

Second, we will show that median voter R's maximal expected payoff from challenger H is higher than that from challenger L, and that her equilibrium payoff from challenger H is also higher than that from challenger L. Then, by using these two results, we prove the theorem.

We say that for each $k = H, L$, p_k *weakly beats* p_{-k} if given p_{-k}, the following holds:

$$\mathrm{E}u_R(v_k, p_k) \geq \mathrm{E}u_R(v_{-k}, p_{-k}). \tag{7}$$

For each $k = H, L$, define the set of challenger k's promises that weakly beat p_{-k} as follows:

$$\mathcal{U}_k(p_{-k}) = \{p_k \in [0,1] \ : \ \mathrm{E}\Pi_R(v_k, p_k) \geq \mathrm{E}\Pi_R(v_{-k}, p_{-k})\}.$$

Let

- $p_L^* = \hat{p}_L$;
- $p_H^* = \min\{p \in I_H : p \in \mathcal{U}_H(p_L^*)\}$.

Theorem 1. *There exists a unique equilibrium* (p_L^*, p_H^*) *in* I_L *and* I_H *with* $w_L^* = 0$ *and* $w_H^* = 1$.

A policy p_k^c is called a *Condorcet winner for challenger* k if for any $p_k \in I_k$,

$$\mu\left(\{r \in \delta : \mathrm{E}u_r(v_k, p_k^c) \geq \mathrm{E}u_r(v_k, p_k)\}\right) \geq \frac{\mu(\delta)}{2}.$$

Let

$$\hat{p}_k := \arg\max_{p \in I_k} \mathrm{E}u_R(v_k, p).$$

The next proposition shows that such a $\hat{p}_k$ exists uniquely in I_k.

Proposition 1. *For each candidate $k = H, L$, in the interval I_k, (i) median voter R's expected payoff is single-peaked at $\hat{p}_k$; (ii) $\hat{p}_k$ is a Condorcet winner for challenger k; (iii) $\hat{p}_k \leq R$ and if $\hat{p}_k \neq R$, $\frac{\partial}{\partial p_k}\left(\mathrm{E}u_R(v_k, p_k)\right) = 0$ is satisfied.*[3]

We present the proof of Proposition 1 together with the necessary lemmas in the Appendix. The second result states that challenger H can yield a higher maximal expected payoff to median voter R than challenger L, and thus in equilibrium she can choose a policy promise that yields a higher expected payoff to median voter R than challenger L.

Lemma 1. *The following holds:*

- $\mathrm{E}u_R(v_H, \hat{p}_H) > \mathrm{E}u_R(v_L, \hat{p}_L)$.
- $\mathrm{E}u_R(v_H, p_H^*) \geq \mathrm{E}u_R(v_L, p_L)$ *for any* $p_L \in I_L$.

Proof. The first statement is direct from $\mathrm{E}u_R(v_H, \hat{p}_L) > \mathrm{E}u_R(v_L, \hat{p}_L)$. To prove the second statement, on the contrary suppose $\mathrm{E}u_R(v_H, p_H^*) < \mathrm{E}u_R(v_L, p_L)$ for some $p_L \in I_L$. Then, the equilibrium condition (2) requires that (p_H^*, p_L), which yields $w_H^* = 0$ and $w_L = 1$ in equilibrium. However, by the first statement, $\hat{p}_H$ yields $\hat{w}_H = 1$ and contradicts the equilibrium conditions (3) and (4). $\square$

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. First, we show that (p_H^*, p_L^*), and $w_H^* = 1$ constitute an equilibrium. To begin with, we show that $p_L^* = \hat{p}_L$ in equilibrium. Suppose not. Then, by Proposition 1, $\mathrm{E}\Pi_R(v_L, p_L^*) < \mathrm{E}\Pi_R(v_L, \hat{p}_L)$. Then, challenger H selects p_H^* as

$$\mathrm{E}\Pi_R(v_L, p_L^*) + \epsilon = \mathrm{E}\Pi_R(v_L, p_H^*),$$

[3]When the utility from policy difference is of a general form, we can change the definition of d_k accordingly and still prove the rest of the results. However, one technical problem arises for this proposition. Specifically, Y_k defined by

$$\bar{v} - v_k = g(Y_k(p_k) - \bar{p}) - g(p_k - Y_k(p_k))$$

is part of the derivation of the winning probabilities (in our current setting, $Y_k(p_k) = \frac{d_k + p_k}{2}$). Therefore, we cannot guarantee that $(\log \pi(v_k, p_k))'$ is decreasing, which is needed to ensure that this theorem holds. In other words, as long as the maximizers of voters r and r''s expected payoffs, p_k^r and $p_k^{r'}$, satisfy $p_k^r < p_k^{r'}$ for $r < r'$ and $(\log \pi(v_k, p_k))'$ is decreasing, we can obtain this proposition and our remaining results still work for a general case of g. Studying the conditions for g, Y_k, and F is analytically complicated and goes beyond our scope. Therefore, we keep this simple setting in this paper. The proofs for a general g are available upon request.

for some ϵ, so that $w_H^* = 1$. However, because F is strictly increasing in (5), in I_H, $\pi(v_H, p_H)$ is decreasing, and so in order to maximize the winning probability in the general election, challenger H further selects $p_H^* - \delta$ with $\mathrm{E}\Pi_R(v_L, p_L^*) < \mathrm{E}\Pi_R(v_L, p_H^* - \delta)$. Thus, p_H^* is not an equilibrium strategy for challenger H, and for the same reason, $p_H^* - \delta$ is not an equilibrium strategy. Thus, in equilibrium, it must be the case that $p_L^* = \hat{p}_L$.

Then, by the equilibrium conditions (3) and (4), challenger H chooses p_H^*, which yields a higher expected payoff to median voter R in the primary election than the expected payoff from challenger L, and at the same time maximizes the winning probability in the general election. By (2), we obtain $w_H^* = 1$.

Second, we show that the equilibrium is unique. By Proposition 1, $\hat{p}_L$ is unique. Thus, p_L^* is unique. Furthermore, because in I_H $\pi(v_H, p_H)$ is a strictly decreasing function of p_H, $\pi(v_H, p_H)$ is uniquely maximized at p_H^* in I_H, and thus p_H^* is also unique. □

Theorem 1 formally proves that challenger H selects the equilibrium policy that maximizes the winning probability of the general election while just beating challenger L in the primary election. In equilibrium, challenger H wins the primary election and the equilibrium condition (4) requires that challenger L maximizes median voter R's expected payoff. By Proposition 1, this maximizer exists uniquely and thus the equilibrium policy promise p_L^* is equal to $\hat{p}_L$ for challenger L.

Let

$$\Psi_H = [d_H, \max\{d_H, 2\bar{p} - d_H\}].$$

Then, Ψ_H is an interval of policy promises for challenger H that yields winning probability 1 in the general election. Note that $\Psi_H \setminus \{d_H\} \neq \emptyset$ if and only if $v_H > \bar{v}$.

In case 1, there is no policy promise for which the winning probability for challenger H is one. Therefore, challenger H chooses a policy promise that is just enough to beat challenger L, and at the same time tries to maximize the winning probability in a general election.

On the other hand, in case 2 and case 3 where $v_H \geq \bar{v}$, challenger H is strong enough to obtain the winning probability of one in the general election, but at the same time also needs to consider beating challenger L in the primary election. By Lemma 1, $\mathcal{U}_H(p_L^*) \neq \emptyset$. In case 2, obviously $\Psi_H \cap \mathcal{U}_H(p_L^*) \neq \emptyset$ and any policy in $\Psi_H \cap \mathcal{U}_H(p_L^*)$ can beat challenger L in the primary election and yield the winning probability in the general election for challenger H. Thus, there exist multiple equilibrium policy promises for challenger H in these cases.

Case 3 is a little more complicated. In particular, when challenger L's valence is much higher than the incumbent's, $\Psi_H \cap \mathcal{U}_H(p_L^*)$ can be empty. This arises when in case 3, the challengers' valences are quite close to each other. Because $\mathrm{E}u_R(v_H, 2\bar{p} - d_H) = 2v_H - \bar{v} - R + \bar{p}$, and comparing it with $\mathrm{E}u_R(v_L, 2\bar{p} - d_H + \delta)$, it may be possible for challenger L to beat $2\bar{p} - d_H$ in the primary election by choosing $2\bar{p} - d_H + \delta$ with δ to satisfy

$$\frac{(2 - \pi(v_L, 2\bar{p} - d_H + \delta))}{\pi(v_L, 2\bar{p} - d_H + \delta)}(v_H - \bar{v}) + (\bar{v} - v_L) < \delta. \tag{8}$$

It is easy to see that when v_H is higher or v_L is lower, it is more difficult for δ to satisfy (8).

Finally, we obtain the following proposition, which summarizes the above discussion and describes the equilibrium policy promises in each case. The above discussion replaces the formal proof.

Proposition 2. *In every case, p_L^* is the only equilibrium policy promise for challenger L. In addition,*

1. *In case 1, p_L^* is the unique equilibrium policy promise;*
2. *In case 2, any policy promise belonging to $\Psi_H \cap \mathcal{U}_H(p_L^*)$ is challenger H's equilibrium policy promise;*
3. *In case 3, if $\Psi_H \cap \mathcal{U}_H(p_L^*) \neq \emptyset$, then any policy promise belonging to $\Psi_H \cap \mathcal{U}_H(p_L^*)$ is challenger H's equilibrium policy promise, otherwise p_H^* is the unique equilibrium policy promise.*

3.2 Comparative Analysis

In this section, we compare the policy promises resulting from the following two scenarios.

(A) A challenger tries to maximize her winning probability in a general election subject to the primary constraint, which is given by our equilibrium definition;

(B) A challenger tries to maximize the expected payoff of the median voter in their party by choosing the p_k that solves

$$\max_p \mathrm{E}u_R(v_k, p). \tag{9}$$

The comparison highlights two results. First, we show that weaker parties can benefit more from a primary election in the sense that the winning probability under (A) is strictly higher than the one under (B). Second, we show that the policy promise chosen under (A) is more moderate than the one chosen under (B), in the sense that it is closer to the incumbent.

To present these results, we use the relationship between $\hat{p}_H$ and p_H^*. The main message here is that in the policy space, p_H^* is smaller, which implies that its value is closer to $\bar{p}$ compared with $\hat{p}_H$ or $\hat{p}_L$. Note that even when $\Psi_H \cap \mathcal{U}_H(p_L^*)$ contains a strictly positive interval of policy promises and there are multiple equilibria, p_H^* is the rightmost in the set $\Psi_H \cap \mathcal{U}_H(p_L^*)$.

Although p_H^* is defined as the minimal point in the set of $I_H \cap \mathcal{U}_H(p_L^*)$, in this case it yields the winning probability by noting $\Psi_H \cup I_H = \{p_H^*\}$, and the most rightish policy among the equilibrium policy promises in Proposition 2. Therefore, the result that p_H^* is smaller than $\hat{p}_H$ implies that even when there are multiple equilibria, any other equilibrium policy promises are also smaller than $\hat{p}_H$.

In Roemer (1997) and Owen and Grofman (2006), they consider the problem given by (9). In our model, challengers need to consider the winning probability in the general election. This is the main point on which our model differs from Owen and Grofman (2006) and Roemer (1997). We consider how our model makes a difference to the solution from Problem (9) under (B), and how the equilibrium changes if the objective of challengers is to maximize median voter R's expected utility rather than the winning probability of one in the general election.[4]

As a corollary to Theorem 1, we obtain the following result.

Corollary 1. *The condition* $\pi(v_H, p_H^*) \geq \pi(v_H, \hat{p}_H)$ *holds with equality only if* $\pi(v_H, p_H^*) = \pi(v_H, \hat{p}_H) = 1$.

Proof. By Theorem 1, because $\hat{p}_H \in \mathcal{U}_H(p_L^*)$, it is clear that $p_H^* \leq \hat{p}_H$. For every $p_H \in I_H$, $\pi(v_H, p_H)$ is strictly decreasing and for every $p_H \in I_H$, $\pi(v_H, p_H) = 1$. Thus, we obtain the "only if" part. ☐

An implication of this corollary is that unless the challenging party has a very strong candidate, for which $\pi(v_H, p_H^*) = \pi(v_H, \hat{p}_H) = 1$ holds, the challenging party obtains a higher chance of winning in the general election by using a primary election. This result is consistent with the empirical finding in Carey and Polga-Hecimovich (2006) that primary elections can select candidates who are stronger than those selected by other procedures. Moreover, the condition $\pi(v_H, p_H^*) = \pi(v_H, \hat{p}_H) = 1$ indicates that primaries tend to benefit weaker parties, which is also shown numerically by Adams and Merrill (2008). They label this effect the *weaker party's competitive primary advantage*. Corollary 1 also implies that through primaries, weaker parties' candidates may select policies that are more competitive in the general election. The following theorem states that challenger H can select a more moderate policy promise through the primary election in equilibrium.

Theorem 2. *Let* p_H^* *be challenger* H*'s equilibrium strategy. Then,*

[4] In the literature on voting theory, Myerson (1993) compares, under different electoral systems, the incentives for candidates to create inequalities among otherwise homogeneous voters by making a promise that favors small groups, rather than appealing equally to all voters. Myerson and Weber (1993) also show that the set of equilibria can vary substantially with the choice of voting system. In a different approach, Myerson (1994) broadly argues that the foundations of any social institution may be understood in terms of more fundamental games that have multiple equilibria by using the concept of focal points proposed in Schelling (1960).

(a) $p_H^* < \hat{p}_H$;

(b) $p_H^* < \hat{p}_L$.

Proof. First, by Theorem 1, $p_H^*, \hat{p}_H \in \mathcal{U}_H(p_L^*)$. By Lemma 1, $\mathrm{E}u_R(v_H, \hat{p}_H) > \mathrm{E}u_R(v_L, \hat{p}_L)$. Furthermore, $\pi(v_H, p_H)$ is a strictly decreasing function of p_H. Hence, we obtain (a).

To prove (b), note that $\hat{p}_L \in \mathcal{U}_H(\hat{p}_L)$. By Theorem 1, $p_H^* \leq \hat{p}_L$. Then, again by $\mathrm{E}u_R(v_H, \hat{p}_L) > \mathrm{E}u_R(v_L, \hat{p}_L)$ and because $\pi(v_H, p_H)$ is a strictly decreasing function of p_H in I_H, we conclude that $p_H^* < \hat{p}_L$. $\square$

Theorem 2 implies that under (A), challenger H chooses a more moderate policy promise that is more appealing to the electorate in the general election. It stems from the condition that challenger H tries to maximize the winning probability in the general election while just barely beating challenger L in the primary election.

4 Comparative Statics

4.1 Some Examples

Up to now, we have not assumed any particular distribution for the overall median voter position, M. In this section, we present three sets of examples by assuming a particular distribution function. The first example assumes a linear distribution function. The second and third examples assume a triangular distribution function, which allows us to study the situation where the likelihood of the median voter's position increases up to some point and then decreases.

Now, we present the first example where F is assumed to be linear in the interval $[\bar{p}, R]$. The implication of this assumption is that the median voter position is uniformly distributed between $\bar{p}$ and R.

Example 1. *Suppose that* $F(x) = \frac{x-\bar{p}}{R-\bar{p}}$ *and that* $\bar{v} > v_H > v_L$.[5] *Then, for each* $k = H, L$, $p_L^* = \hat{p}_k = R$. *Moreover,* p_H^* *is the minimal* p_H *that satisfies*

$$v_H - R + p_H = \frac{R - d_L}{2R - d_H - p_H} v_L + \frac{R - p_H + d_L - d_H}{2R - d_H - p_H} \cdot (\bar{v} - R + \bar{p}). \tag{10}$$

We present the proof in the Appendix. Note that when $v_L = v_H = 0$, $p_H^* = R$ by (10). This is a natural result, because in this situation there is no difference between challenger H and challenger L. Now, because $\mathrm{E}u_R(v_H, R) > \mathrm{E}u_R(v_L, R)$, we must have $p_H^* \neq R$. In a sense, the primary election causes a more moderate policy to be selected and increases the probability

[5] This function F does not satisfy twice-differentiability. Our analysis is still valid.

that a candidate of higher valence wins. Moreover, in example 1, (3) is critical. Due to (3), it holds that

$$\mathrm{E}u_R(v_L, R) > \min_{p_H \in I_H} \mathrm{E}u_R(v_H, p_H) = u_R(\bar{v}, \bar{p}),$$

because $\pi(v_H, 2R - d_H) = 0$. However, if challenger H is too strong, challenger H can simply choose R and still win the general election with a probability of one.

Owen and Grofman (2006) assume that the electorate may shift ideologically from one side to the other, and that this shift can be expressed as a random change in M, which is normally distributed. The next examples have similar features in the sense that there is a single peak in the likelihood of the median voter's bliss point. Moreover, unlike Example 1, challenger L's equilibrium position may be to the left of R.

We start with the case where challenger H's valence is higher than the incumbent's. This example corresponds to case 2. Figure 1 demonstrates the winning probabilities in the general election for each challenger. In this case, challenger H has the best valence and challenger L has the worst valence among the three politicians. Because $v_H > \bar{v}$, any policy in the interval $[0.2, 0.5]$ yields the winning probability of one. Therefore, there are multiple equilibria, and $p_H^* = 0.5$, which can be chosen when we apply the criterion for median voter R by Proposition 1.

As shown in Figure 1, in the entire line $[0, 1]$, challenger H's winning probability is greater than 0. Challenger H can choose any point in $[0, 1]$ because the minimized median voter R's expected payoff from challenger H exceeds the maximized median voter R's expected payoff from challenger L. A result such as Example 1 does not hold in this case.

Example 2. *Suppose that* $\bar{p} = 0.2, R = 0.8$*, and* $v_L = 0.2, \bar{v} = 0.5, v_H = 0.8$*. Suppose that*

$$F(x) = \begin{cases} 2 \times \frac{(x-\bar{p})^2}{(R-\bar{p})^2} & \textit{for } x \in (\bar{p}, \frac{\bar{p}+R}{2}]; \\ 1 - 2 \times \frac{(R-x)^2}{(R-\bar{p})^2} & \textit{for } x \in (\frac{\bar{p}+R}{2}, R] \end{cases}.$$

Then, $\hat{p}_L = p_L^* = 0.7$*,* $\hat{p}_H = 0.8$*, and* $p_H^* = 0.5$ *with* $\pi(v_H, p_H) = 1$ *for all* $p_H \in [\bar{p}, 0.5]$*.*

Proof. Note that $d_H = -0.1$ and $d_L = 0.5$. First, we consider the first derivative F' as follows:

$$F'(x) = \begin{cases} 4 \times \frac{(x-\bar{p})}{(R-\bar{p})^2} & \text{for } x \in (\bar{p}, \frac{\bar{p}+R}{2}]; \\ 4 \times \frac{(R-x)}{(R-\bar{p})^2} & \text{for } x \in (\frac{\bar{p}+R}{2}, R] \end{cases}.$$

By calculation, we can show that $\frac{F'(x)}{1-F(x)}$ is increasing for $x \in [\bar{p}, R]$. Without loss of generality, fix $k = H$ or L. We denote the first derivative of the winning probability $\pi(v_k, p_k)$ with respect to p_k by $\pi'(v_k, p_k)$. For some $p_k \in I_k^\circ$, if $\frac{d_k+p_k}{2} \leq \frac{\bar{p}+R}{2}$,

$$\begin{aligned} \pi'(v_k, p_k) &= -\tfrac{1}{2}F'(\tfrac{d_k+p_k}{2}) = -2 \times \frac{\frac{d_k+p_k}{2} - \bar{p}}{(R-\bar{p})^2} \\ \pi(v_k, p_k) &= 1 - F(\tfrac{d_k+p_k}{2}) = 1 - 2 \times \frac{(\frac{d_k+p_k}{2} - \bar{p})^2}{(R-\bar{p})^2}, \end{aligned}$$

and if $\frac{d_k+p_k}{2} > \frac{\bar{p}+R}{2}$,

$$\begin{aligned}\pi'(v_k,p_k) &= -\tfrac{1}{2}F'(\tfrac{d_k+p_k}{2}) = -2\times\frac{R-\frac{d_k+p_k}{2}}{(R-\bar{p})^2}\\ \pi(v_k,p_k) &= 1-F(\tfrac{d_k+p_k}{2}) = 2\times\frac{(R-\frac{d_k+p_k}{2})^2}{(R-\bar{p})^2}.\end{aligned}$$

First, suppose that $\frac{d_k+p_k}{2} \leq \frac{\bar{p}+R}{2}$. Note that

$$\mathrm{E}\Pi'_R(v_k,p_k) = 1-\frac{d_k+p_k-2\bar{p}}{(R-\bar{p})^2}\left(\frac{3p_k-d_k-2\bar{p}}{2}\right). \tag{11}$$

Thus, $\mathrm{E}\Pi'_R(v_k,p_k) > 0$. Then, $\mathrm{E}u_R(v_k,p_k)$ takes a maximum when $p_k = \min\{\bar{p}+R-d_k, R\}$ for each $k = H, L$. Therefore, in the first case, we obtain our candidate solution $\hat{p}_H = 0.8$ and $\hat{p}_L = 0.5$ by Lemma 2.

Second, suppose that $\frac{d_k+p_k}{2} > \frac{\bar{p}+R}{2}$. Then, the first-order condition yields

$$\begin{aligned}\mathrm{E}\Pi'_R(v_k,p_k) &= 2\times\frac{(R-\frac{d_k+p_k}{2})^2}{(R-\bar{p})^2} + 2\times\frac{R-\frac{d_k+p_k}{2}}{(R-\bar{p})^2}\times(d_k-p_k)\\ &= 2\times\frac{R-\frac{d_k+p_k}{2}}{(R-\bar{p})^2}\times\left(R-\frac{d_k+p_k}{2}+d_k-p_k\right).\end{aligned}$$

When $\mathrm{E}\Pi'_R(v_k,p_k) = 0$, because $(R-\frac{d_k+p_k}{2}) > 0$,

$$2R-d_k-p_k+2d_k-2p_k = 0.$$

Then, we obtain $\hat{p}_k = \frac{2R+d_k}{3}$, which implies $\hat{p}_H = 0.5$ and $\hat{p}_L = 0.7$. However, $\hat{p}_H = 0.5$ is impossible because $\frac{\hat{p}_H+d_H}{2} < \frac{\bar{p}+R}{2}$. Thus, we conclude $\hat{p}_H = 0.8$. By Lemma 2 and by the first case, we conclude $\hat{p}_L = 0.7$.

Finally, we are interested in $p_H \in I_H$, which maximizes $\pi(v_H,p_H)$ by satisfying

$$\mathrm{E}u_R(v_H,p_H) \geq \mathrm{E}u_R(v_L,0.7).$$

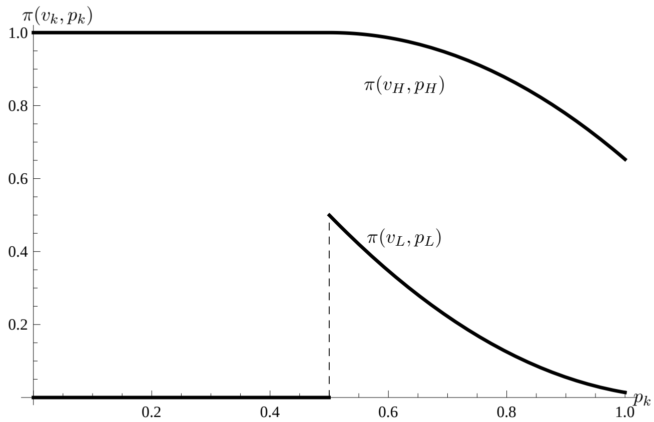
Note that $\pi(v_H,0.5) = 1$, $\pi(v_H,p_H) = 1$ for $p_H \in [0.2,0.5]$. By calculation, we obtain $\mathrm{E}u_R(v_H,p_H) \geq \mathrm{E}u_R(v_L,0.7)$ for $p_H \in [0.2,0.5]$. Furthermore, we obtain $p^*_H = 0.5$. □

The following examples show how the relative changes of valences between challengers affect equilibrium outcomes.

Example 3. *Suppose that F is defined as in Example 2, and that $\bar{p} = 0.2 and R = 0.8$.*

(a) Let $\bar{v} = 0.8, v_L = 0.3, v_H = 0.4$. Then, $\hat{p}_H = 0.73$ and $\hat{p}_L = p^*_L = 0.77$, and there exists a unique $p^*_H = 0.61$ with $\pi(v_H,p^*_H) = 0.21$.

Figure 1: Winning Probabilities for Example 2



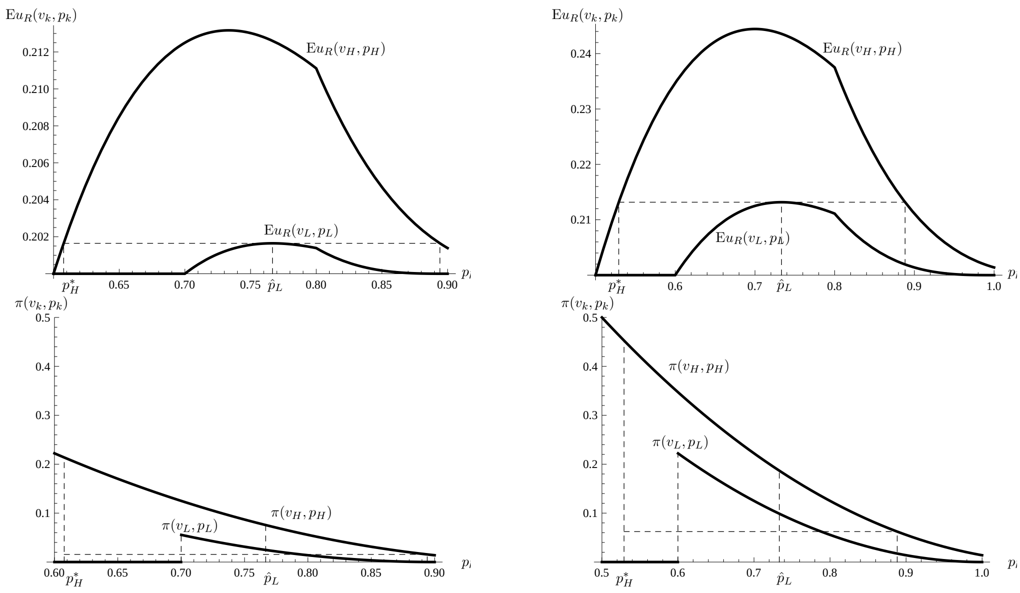
(b) Let $\bar{v} = 0.8, v_L = 0.4, v_H = 0.5$. *Then,* $\hat{p}_H = 0.7$ *and* $\hat{p}_L = p_L^* = 0.73$*, and there exists a unique* $p_H^* = 0.53$ *with* $\pi(v_H, p_H^*) = 0.45$.

(c) Let $\bar{v} = 0.3, v_L = 0.5, v_H = 0.8$. *Then,* $\hat{p}_H = 0.8$ *and* $\hat{p}_L = p_L^* = 0.77$*, and there exists a unique* $p_H^* = 0.7$ *with* $\pi(v_H, p_H) = 1$ *for all* $p_H \in [0, 0.7]$.

In Example 3, (a) and (b) are the cases where the incumbent is better than the challengers in terms of valences, and (c) is the opposite case such that the incumbent has the worst valence among the three politicians. The first two examples correspond to case 1 and the last example corresponds to case 3 in Section 3.1. First, compare parts (a) and (b) of Figure 2. As Theorem 2 implies, p_H^* is more moderate than $\hat{p}_L$ and $\hat{p}_H$, and the winning probability of p_H^* is higher than that for $\hat{p}_H$. As the incumbent's valence is higher than challenger H's, the winning probability of p_H^* is not equal to one and there is no other equilibrium policy. Therefore, we can uniquely determine challenger H's equilibrium policy in these two cases.

On the other hand, in Figure 3, the incumbent is much weaker compared with the challengers. Thus, even for challenger L, there is an interval of policy promises with which she can win the general election with a probability of one. Similarly with Example 2, there are multiple equilibria because there is an interval of policy promises such that challenger H can win the general election with a probability of one.

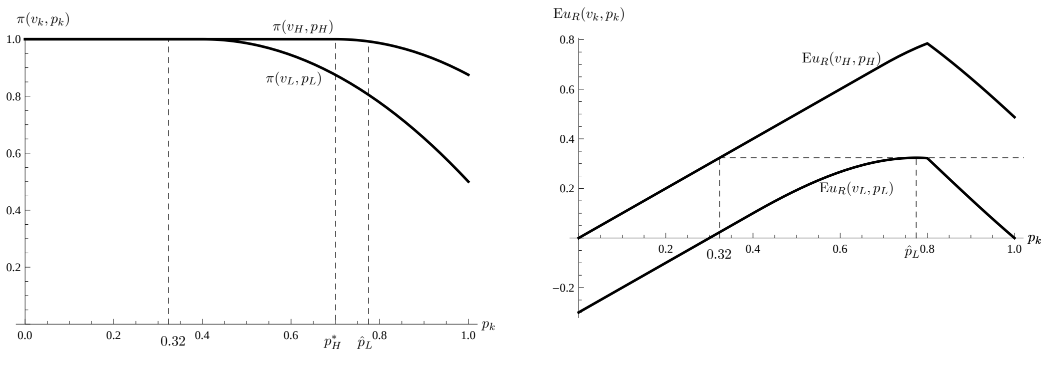
Figure 2: Winning Probabilities and Expected Payoffs for (a) and (b) of Example 3



(a) $\bar{v} = 0.8, v_L = 0.3, v_H = 0.4$

(b) $\bar{v} = 0.8, v_L = 0.4, v_H = 0.5$

Figure 3: Winning Probabilities and Expected Payoffs for (c) of Example 3



3.1 - Winning Probability

3.2 - Expected Payoff

4.2 Primary Advantages

In this section, we study how the change in the salience of valence changes our equilibrium outcome. For this purpose, we redefine voters' utility when a winning politician has valence v and implements policy p by

$$u_i(v, p) = \lambda v - |p - i| \tag{12}$$

for some constant $\lambda > 0$. This λ captures the salience of valence in voters' preferences.[6] To convey a simple intuition, we make the assumption of a uniform distribution F under Example 1 so that we can focus on the effects of valences on winning chances. In the new setting, assumption (3) is replaced with the following: for each $k = H, L$,

$$|\lambda(\bar{v} - v_k)| < (R - \bar{p}). \tag{13}$$

To measure the magnitude of the advantage of holding primaries, we define a *primary advantage* by a function $S : I\!R \times [0, 1] \times [0, 1] \to I\!R$ such that $S(v_H, p^*_H, \hat{p}_H) = \pi(v_H, p^*_H) - \pi(v_H, \hat{p}_H)$. Then, the party's winning chance increases by $S(v_H, p^*_H, \hat{p}_H)$ because of the primary election.[7] By Corollary 1, $S(v_H, p^*_H, \hat{p}_H)$ is always positive.

We start with the case where the challenging party is weaker than the incumbent party. The following proposition shows that a weaker party usually benefits from a primary election, as the salience of valence λ increases, although in the extreme case, as we will explain later, the opposite result may hold.

Proposition 3. *Suppose that $F(x) = \frac{x-\bar{p}}{R-\bar{p}}$ and that $\bar{v} > v_H > v_L$. Then, the change in the primary advantage when λ changes is given by*

$$\frac{\partial S(v_H, p^*_H, \hat{p}_H)}{\partial \lambda} = \frac{(R - \bar{p} - \lambda\bar{v})(v_H - v_L) + \lambda(v_H - v_L)(v_H + v_L - \bar{v}) - (R - p^*_H)\bar{v}}{2(R - \bar{p})\left(R - p^*_H + \lambda\bar{v}\right)}.$$

Proof. Note that the primary advantage is given by

$$S(v_H, p^*_H, \hat{p}_H) = \frac{R - p^*_H}{2(R - \bar{p})}. \tag{14}$$

[6]This is the simplest way to incorporate the salience of valence in voters' preferences, and obviously there are many other possibilities for modifying the functional form. Clark and Leiter (2013)'s empirical study of nine Western European countries over the period 1976 to 2003 indicates that when voters face more ideologically distinct policy positions they weigh valences more heavily, and concludes that voters may view valence and policy positions as complementary. It may be interesting to see how different forms of the salience of valence relative to policy positions would result in different equilibrium outcomes.

[7]Serra (2011) defines the "primary skill bonus" as the increase of the expected campaigning skill of their nominee. As in Serra (2011), we obtain a similar result for the primary skill bonus such that challenger H always wins. Theorem 1 confirms this result.

By taking the derivative of (10) in Proposition 1 with respect to λ,[8] we obtain

$$\frac{\partial(R-p_H^*)}{\partial\lambda}=\frac{(R-\bar{p}-\lambda\bar{v})(v_H-v_L)+\lambda(v_H-v_L)(v_H+v_L-\bar{v})-(R-p_H^*)\bar{v}}{R-p_H^*+\lambda\bar{v}}.$$

By substituting $\frac{\partial(R-p_H^*)}{\partial\lambda}$ into (14), we obtain the desired result. $\square$

When $\bar{v}$ is sufficiently high, the advantage could decrease, because the incumbent wins the general election with a much higher probability and it is almost impossible for challenger H to win the general election. Then, the maximum of R's induced expected preference from challenger L, $\mathrm{E}u_R(v_L, R)$, increases as λ increases, because the incumbent having very high valence wins the general election with an even higher probability. Thus, challenger H may need to shift her policy promise toward R in order to meet $\mathrm{E}u_R(v_L, R)$ and win the primary election.

As we have seen, the benefit of the primary election rests on the differences between politicians' valences. Next, we consider the case of $v_H > \bar{v} > v_L$, which is the simplest case in terms of calculation. For notational convenience, we denote $\Delta_k = v_k - \bar{v}$ for each $k = H, L$. Then, $\Delta_H > 0 > \Delta_L$ and $\pi(v_H, p_H^*) = 1$ hold. We compare the primary advantages under the two situations: $\Delta_H \in \{\Delta_H^1, \Delta_H^2\}$. We denote the primary advantages at each Δ_H^1 and Δ_H^2 by $S_1(v_H^1, p_1^*, \hat{p}_1)$ and $S_2(v_H^2, p_2^*, \hat{p}_2)$, respectively.

Proposition 4. *Suppose that $F(x) = \frac{x-\bar{p}}{R-\bar{p}}$ and that $v_H > \bar{v} > v_L$ so that $\pi(v_H, p_H^*) = 1$ holds. Then, the difference between the two primary advantages, $S_2(v_H^2, p_2^*, \hat{p}_2) - S_1(v_H^1, p_1^*, \hat{p}_1)$, is given by $\frac{\lambda(\Delta_H^2-\Delta_H^1)}{2(R-\bar{p})}$, and the primary advantage at Δ_H^2 is higher than the one at Δ_H^1 if and only if $\Delta_H^1 < \Delta_H^2$ holds.*

Proof. The detailed calculation for $S_2(v_H^2, p_2^*, \hat{p}_2) - S_1(v_H^1, p_1^*, \hat{p}_1)$ is found in the Appendix. Then, the rest follows. $\square$

When $\Delta_H^1 < \Delta_H^2$ holds, as the salience λ increases, the challenging party benefits more from holding the primary election. This is similar to the finding in Adams and Merrill (2008). However, in the opposite case where voters care more about valence being reasonable, as λ increases, the advantage decreases.

Finally, we investigate the relationship between valences, policy promises, and winning chances. Let $P : I\!R \to (\bar{p}, R)$ satisfy $\mathrm{E}u_R(h(v_L), R) = \mathrm{E}u_R(h(v_H), P(v_H))$. Then, by Example 1, there exists a $p'_H = P(v_H)$ to satisfy (10), and by the implicit function theorem, P is continuously differentiable. We characterize how a policy bias $R - P(v_H)$ or the winning probability $\pi(v_H, P(v_H))$ changes when v_H changes in the next proposition.

[8] A detailed calculation is available in the Appendix.

Proposition 5. *Suppose that* $F(x) = \frac{x-\bar{p}}{R-\bar{p}}$ *and fix three politicians' valences as* $\bar{v} > \bar{v}_H > \bar{v}_L$*. Suppose that* (10) *holds in the* ϵ *neighborhood of* $\bar{v}_H$ *for sufficiently small* $\epsilon > 0$*. Then, for any* v_H *in the* ϵ *neighborhood of* $\bar{v}_H$*,*

(A) $\frac{\mathrm{d}(R-P(v_H))}{\mathrm{d}(R-\bar{p})} = \frac{\lambda(v_H-v_L)}{R-P(v_H)+\lambda\bar{v}}$ *holds and as* $R-\bar{p}$ *increases,* $R-P(v_H)$ *increases;*

(B) $\frac{\mathrm{d}\left(\frac{R-P(v_H)}{R-\bar{p}}\right)}{\mathrm{d}v_H} = \frac{\lambda}{R-P(v_H)}\left(\frac{\lambda(v_H-\bar{v})}{R-\bar{p}}+1\right)$*;*

(C) $\frac{\mathrm{d}\ln\pi(v_H,P(v_H))}{\mathrm{d}v_H} = \frac{\lambda}{R-P(v_H)}$*.*

As $\bar{v} > \bar{v}_H$ and by (13), $\lambda(v_H - \bar{v}) + R - \bar{p} > 0$ holds for any v_H in the ϵ neighborhood of $\bar{v}_H$. Therefore, in (B), $\frac{\mathrm{d}\left(\frac{R-P(v_H)}{R-\bar{p}}\right)}{\mathrm{d}v_H} > 0$. This implies that a candidate with higher valence can choose a larger $\frac{R-P(v_H)}{R-\bar{p}}$. Intuitively, if a candidate is stronger, she can try to appeal to the general electorate while still winning the primary election. This is in contrast with the finding in Adams and Merrill (2009), which numerically compares the two cases where the two challengers with the same valence have the same or higher valence relative to the incumbent's valence, and shows that in the case where the three politicians have the same valence, the challengers need to choose a more moderate policy to appeal to the general electorate. In Proposition 5, when v_H increases, because of this increment in her valence, challenger H can choose a more moderate policy, which increases her winning probability in the general election.

As the difference between both parties' median bliss points decreases, Proposition 5 implies that by (B), the effect of valence on policy position relative to the ideological distance $R-\bar{p}$ decreases, and by (C), the effect of valence on winning chances increases. Buttice and Stone (2012) study US House elections and show that the effect of candidate quality increases with reduced differences between candidates on ideology. One interpretation of Proposition 5's (A) and (C) is that as $R-\bar{p}$ decreases, the policy deviation from R, $R-P(v_H)$ decreases and the effect of valence on winning chances increases, which is consistent with the finding in Buttice and Stone (2012), even with the presence of primary elections.[9]

4.3 Testable Hypotheses

This section summarizes the testable hypotheses from our model for empirical research. Theorem 1 indicates that any force expanding the set $\mathcal{U}_H(p_L^*)$ may possibly shift challenger H's

[9]Some recent empirical research has found that as parties have converged ideologically, valence considerations have become more important in voters' choices (see Green and Hobolt, 2008).

policy promise toward the overall median voter. In particular, if challenger L's valence decreases or challenger H's valence increases, challenger H can shift her policy promise toward $\bar{p}$ to appeal more to the general electorate. The following proposition formally states this result.

Proposition 6. *Holding all else constant, p_H^* decreases*

- *as v_L decreases; or*
- *as v_H increases.*

Proof. First, when v_L decreases, $\mathrm{E}u_R(v_L, p_L)$ decreases for any given $p_L \in I_L$, because $u_R(v_L, p_L)$ ($> u_R(\bar{v}, \bar{p})$ by (3)) decreases and $\pi(v_L, p_L)$ also decreases. Because the maximum of $\mathrm{E}u_R(v_L, p_L)$ decreases, the set of possible p_H^* to beat the maximum of $\mathrm{E}u_R(v_L, p_L)$ expands. On the other hand, because others are fixed, I_H does not change. Therefore, we can say that p_H^* decreases.

Second, to study the case where v_H increases, if $v_H \geq \bar{v}$, because $p_H^* = \min\{p \in I_H : p \in \mathcal{U}_H(p_L^*)\}$ and $\min I_H = \bar{p} + v_H - \bar{v} \in \mathcal{U}_H(p_L^*)$ imply that $p_H^* = \bar{p} + v_H - \bar{v}$, the result is direct. Therefore, suppose that $v_H < \bar{v}$. Note that when v_H increases, for a similar reason as $\mathrm{E}u_R(v_L, p_L^*)$ decreasing in the first case, $\mathrm{E}u_R(v_H, p_H)$ increases for any $p_H \in I_H$. Because $\mathrm{E}u_R(v_L, \hat{p}_L)$ is fixed and $\mathrm{E}u_R(v_H, p_H)$ is increasing in p_H by Lemma 2 in the Appendix, p_H^* decreases to satisfy $\mathrm{E}u_R(v_L, \hat{p}_L) = \mathrm{E}u_R(v_H, p_H^*)$. $\square$

Now we consider the case when $\bar{v}$ changes. When $v_H \geq \bar{v}$, the result is direct, because $p_H^* = \bar{p} + v_H - \bar{v}$, as shown in the second case of Proposition 6.

Proposition 7. *Holding everything else constant, when $v_H \geq \bar{v}$, p_H^* decreases as $\bar{v}$ increases.*

The other case of $v_H \geq \bar{v}$ is more complicated compared with the other two cases in Proposition 6 because $\bar{v}$ reduces the winning probability for a challenger; however, at the same time, it increases $u_R(\bar{v}, \bar{p})$. Thus, the result depends on the shape of u_R and the distribution F. This can also be seen from (10), although under the assumption of Example 1 the result still holds.

Proposition 8. *Suppose that $F(x) = \frac{x - \bar{p}}{R - \bar{p}}$ and $v_H < \bar{v}$. Holding everything else constant, p_H^* decreases, as $\bar{v}$ increases.*

5 Concluding Remarks

In this paper, we have proposed a simple model of a two-stage election with a primary and a general election. By allowing voters to care both about policy promises and valence, we showed that the existence of primary elections brings out a more moderate policy promise and, at the same time, increases the probability of a candidate with valence winning. Our model is simple and tractable. At the same time, it is stable enough to lend itself to various extensions.

Recently, Hirano and Snyder (2012) studied the impact of scandals on primary and general election competition using data for US House incumbents running for reelection over the period 1978 to 2008. They find that incumbents involved in relatively serious scandals are very likely to be challenged in the primary and lose. In that study, they point out that primary elections can improve accountability by removing incumbents who face reelection in districts dominated by one political party.

In our current setting, only the challenging party holds a primary election; however, we can interpret our result in the context of a primary election held by the incumbent party. Supposing that the expected value of a policy promise from the challenging party is fixed at $\bar{p}$, our analysis shows that a low-valence incumbent loses the primary election to a high-valence challenger. There are a number of directions in which our model can be extended. First, an important assumption in this paper is that there is no primary election for the incumbent party. Modelling primary elections in both parties and analyzing the equilibrium is an obvious extension, although technically it would complicate the analysis substantially. Specifically, each politician on each side has to predict who would be their opponent in the general election at the time of the primaries. Adams and Merrill (2008) show that when both parties hold primaries and two candidates in each party have the same level of valence, then both candidates within the party choose the same policy promise. Furthermore, they show that when one candidate's valence increases, this may shift the other party's candidates' positions toward the median voter in the general electorate. Studying whether this tendency still holds in our framework when voters vote strategically and candidates' valences are not necessarily the same is a promising future research direction.

Next, one may try to relax the no etch-a-sketch assumption by allowing politicians to revise a policy promise within a certain constraint in the general election. As there is valence in the model and valence also captures the honesty of candidates, it may be interesting to see how the equilibrium policy choices vary, if candidates can choose a different policy in the general election within a certain distance, which is a function of valence, from their policy promise in the primary.

Furthermore, studying how different institutional arrangements affect the equilibrium outcomes in our model would be interesting. One example is crossover voting. Chen and Yang (2002) and Cho and Kang (2014) study an open primary election with two competing candidates and the possibility of crossover voting. In our current model, crossover voting is eliminated by assumption and our setting is more akin to closed primaries.

One interesting question is whether the open primary system leads to more moderate outcomes than our current model. Gerber and Morton (1998) empirically examine how different primary election laws affect the types of candidates elected in nonpresidential American elections. It would be useful to study whether the theoretical prediction matches the empirical observations in Gerber and Morton (1998).

Another direction for future research is introducing a certain type of error to median voter R's position. In this paper, we have assumed that it is publicly known and exogenously given. The extension would require careful consideration of the modelling of the interaction between the two distributions for overall median voter and this error. Although this might be technically challenging, it may provide more realistic predictions about the equilibrium outcome in the primaries.

Moreover, in this paper, we have assumed that valence and a policy promise are independent and voters' utility is a function of the two variables. In the original model of Roemer (1997), it is assumed that the policy promise relates to the tax rate to fund public goods, and expenditure on public goods is also included in the analysis. Although our settings are quite different, Anh and Oliveros (2012) studies elections that simultaneously decide multiple issues, where voters have independent private values over bundles of issues, while Lizzeri and Persico (2001) studies public goods provision under various electoral incentives. It would be interesting to examine the equilibrium outcome and who would benefit from the existence of primaries by analyzing our model using the original setting in Roemer (1997).

Furthermore, Adams and Merrill (2009) have developed a model of policy-seeking parties in a parliamentary democracy, and in their model, party elites are uncertain about voters' evaluations of the parties' valence. Adams *et al.* (2013) extend that model to situations where voters hold coalitions of parties collectively responsible for their valence-related performances. In our paper, it is assumed that the valence values are exogenously given. It would be interesting to add uncertainty about valence and consider the aspect of collective responsibility for it, as in Adams *et al.* (2013).

Appendix: Lemmas and Proofs

Lemma 2. *Take some* $k = H, L$ *and* $r \in [\max\{\bar{p}, d_k\}, 1]$. *Then, voter* r*'s expected payoffs are single-peaked at* $\min\{r, \bar{p}_k\}$ *in the interval* $[\max\{\bar{p}, d_k\}, r]$ *for each* $k = H, L$ *where* $1 = -\frac{\pi'(v_k,\bar{p}_k)(u_r(v_k,\bar{p}_k)-u_r(\bar{v},\bar{p}))}{\pi(v_k,\bar{p}_k)}$.

Proof. As we restrict our attention to the interval $[\max\{\bar{p}, d_k\}, r]$ in this lemma, we assume $p_k \leq r$. Then,

$$\mathrm{E}u_r(v_k, p_k) = (1 - \pi(v_k, p_k))(\bar{v} - (r - \bar{p})) + \pi(v_k, p_k)(v_k - (r - p_k)).$$

Taking the first and second derivatives, then

$$\begin{aligned} \frac{\partial}{\partial p_k}\left(\mathrm{E}u_R(v_k, p_k)\right) &= \pi'(v_k, p_k)(u_r(v_k, p_k) - u_r(\bar{v}, \bar{p})) + \pi(v_k, p_k) \\ \frac{\partial^2}{\partial p_k^2}\left(\mathrm{E}u_R(v_k, p_k)\right) &= \pi''(v_k, p_k)(u_r(v_k, p_k) - u_r(\bar{v}, \bar{p})) + 2\pi'(v_k, p_k). \end{aligned}$$

Let p_k^r maximize r's expected payoff in this case. If $\pi(v_k, p_k^r) = 0$, $\mathrm{E}u_r(v_k, p_k^r) = u_r(\bar{v}, \bar{p})$ would hold, and by (2) this is a contradiction. Therefore, we can assume that $\pi(v_k, p_k^r) \neq 0$. Then, note that

$$-\frac{\pi'(v_k, p_k^r)}{\pi(v_k, p_k^r)} = \frac{\frac{1}{2}F'(\frac{d_k+p_k^r}{2})}{1 - F(\frac{d_k+p_k^r}{2})}. \tag{15}$$

As (15) is strictly positive, by the definition of p_k^r, we must have $u_r(v_k, p_k^r) > u_r(\bar{v}, \bar{p})$. Because $\frac{F'(\frac{d_k+p_k^r}{2})}{1-F(\frac{d_k+p_k^r}{2})}$ is increasing, $-\frac{\pi'(v_k,p_k^r)}{\pi(v_k,p_k^r)}$ is increasing. Therefore,

$$\left(\frac{\pi'(v_k, p_k^r)}{\pi(v_k, p_k^r)}\right)' = \frac{\pi''(v_k, p_k^r)}{\pi(v_k, p_k^r)} - \frac{(\pi'(v_k, p_k^r))^2}{(\pi(v_k, p_k^r))^2} < 0. \tag{16}$$

As $\pi(v_k, p_k^r) > 0$, we obtain

$$\pi''(v_k, p_k^r) < \frac{(\pi'(v_k, p_k^r))^2}{(\pi(v_k, p_k^r))}.$$

As $\pi'(v_k, p_k^r)) < 0$, we obtain

$$\frac{\pi''(v_k, p_k^r)}{\pi'(v_k, p_k^r)} > \frac{\pi'(v_k, p_k^r)}{(\pi(v_k, p_k^r))}.$$

Then, we obtain

$$\frac{\pi''(v_k, p_k^r)}{\pi'(v_k, p_k^r)}(u_r(v_k, p_k^r) - u_r(\bar{v}, \bar{p})) > \frac{\pi'(v_k, p_k^r)}{\pi(v_k, p_k^r)}(u_r(v_k, p_k^r) - u_r(\bar{v}, \bar{p})).$$

Thus, because $\pi'(v_k, p_k^r) < 0$ and $\frac{\pi'(v_k,p_k^r)}{\pi(v_k,p_k^r)}(u_r(v_k, p_k^r) - u_r(\bar{v}, \bar{p})) + 1 = 0$,

$$\begin{aligned} \frac{\partial^2}{\partial p_k^2}\left(\mathrm{E}u_R(v_k, p_k)\right) &= \pi'(v_k, p_k^r)\left(\frac{\pi''(v_k,p_k^r)}{\pi'(v_k,p_k^r)}(u_r(v_k, p_k^r) - u_r(\bar{v}, \bar{p})) + 2\right) \\ &< \pi'(v_k, p_k^r)\left(\frac{\pi'(v_k,p_k^r)}{\pi(v_k,p_k^r)}(u_r(v_k, p_k^r) - u_r(\bar{v}, \bar{p})) + 2\right) < 0. \end{aligned} \tag{17}$$

Thus, if $p_k < p_k^r$, then $\mathrm{E}u_r'(v_k, p_k) > 0$ and if $p_k > p_k^r$, then $\frac{\partial}{\partial p_k}(\mathrm{E}u_R(v_k, p_k)) < 0$. Now we show $p_k^r = \bar{p}_k$ for any r. Note that as $u_r(v, p) = v - (r - p)$, $u_r(v_k, \bar{p}_k) - u_r(\bar{v}, \bar{p}) = v_k - h(\bar{v}) - (\bar{p} - p_k)$, which does not depend on r. Therefore, we obtain a maximizer $\bar{p}_k$ that does not depend on r in this case. □

Lemma 3. *Take some* $k = H, L$ *and* $r \in [\max\{\bar{p}, d_k\}, 1]$.

(i) In the interval $[r, 2R - d_k]$*, voter* r*'s expected payoff is* U*-shaped with a minimum at* $\min\{\tilde{p}_k^r, 2R - d_k\}$ *if there is* $\tilde{p}_k^r$ *to satisfy* $1 = \frac{\pi'(v_k, \tilde{p}_k^r)(u_r(v_k, \tilde{p}_k^r) - u_r(\bar{v}, \bar{p}))}{\pi(v_k, \tilde{p}_k^r)}$*, or decreasing if such a* $\tilde{p}_k^r$ *does not exist.*

(ii) It holds that $\mathrm{E}u_r(v_k, R) > \mathrm{E}u_r(v_k, 2R - d_k)$.

(iii) If $r < R$*, then* $\mathrm{E}u_r(v_k, r) > \mathrm{E}u_r(v_k, R)$.

Proof. We restrict our attention to the interval $[r, 2R - d_k]$ in this lemma, and so we can assume $p_k \in [r, 2R - d_k]$, therefore

$$\mathrm{E}u_r(v_k, p_k) = (1 - \pi(v_k, p_k))(\bar{v} - (r - \bar{p})) + \pi(v_k, p_k)(v_k - (p_k - r)).$$

Note that

$$\begin{aligned}
\mathrm{E}u_r(v_k, 2R - d_k) &= \bar{v} - (r - \bar{p}) \\
\mathrm{E}u_r(v_k, r) &= (1 - \pi(v_k, r))(\bar{v} - (r - \bar{p})) + \pi(v_k, r) \cdot v_k \\
\mathrm{E}u_r(v_k, R) &= (1 - \pi(v_k, R))(\bar{v} - (r - \bar{p})) + \pi(v_k, R)(v_k - (R - r)).
\end{aligned}$$

It is clear that $\mathrm{E}u_r(v_k, R) > \mathrm{E}u_r(v_k, 2R - d_k)$ by (2). Thus, we obtain (ii). Moreover, if $r < R$, then $r \in I_k$ and $\pi(v_k, r)$ is decreasing. By (2), $u_r(v_k, r) > u_r(v_k, R) > u_r(\bar{v}, \bar{p})$. Thus, we obtain (iii). Taking the first and second derivatives of the expected payoff, then

$$\begin{aligned}
\frac{\partial}{\partial p_k}(\mathrm{E}u_R(v_k, p_k)) &= \pi'(v_k, p_k)(u_r(v_k, p_k) - u_r(\bar{v}, \bar{p})) - \pi(v_k, p_k); \\
\frac{\partial^2}{\partial p_k^2}(\mathrm{E}u_R(v_k, p_k)) &= \pi''(v_k, p_k)(u_r(v_k, p_k) - u_r(\bar{v}, \bar{p})) - 2\pi'(v_k, p_k).
\end{aligned}$$

By (2), $u_r(v_k, R) > u_r(\bar{v}, \bar{p})$. Thus, for p_k sufficiently close to R with $p_k \geq R$, $u_r(v_k, p_k) > u_r(\bar{v}, \bar{p})$. As $\pi'(v_k, p_k) < 0$, $\mathrm{E}u_r'(v_k, p_k) < 0$. Now, suppose that there is a $\tilde{p}_k^r \in [R, 2R - d_k]$ to satisfy $1 = \frac{\pi'(v_k, \tilde{p}_k^r)(u_r(v_k, \tilde{p}_k^r) - u_r(\bar{v}, \bar{p}))}{\pi(v_k, \tilde{p}_k^r)}$. Because F is concave, $\pi''(v_k, p_k) > 0$, which indicates that $\mathrm{E}u_r''(v_k, p_k) > 0$. Thus, we conclude that if $p_k < \tilde{p}_k^r$, $\mathrm{E}u_r'(v_k, p_k) < 0$ and if $p_k > \tilde{p}_k^r$, $\mathrm{E}u_r'(v_k, p_k) > 0$, and we obtain the desired result (i). □

Proof of Proposition 1. To start with, note that $\delta = (\frac{\bar{p}+d_L}{2}, 1]$ contains the support of the primary voters' bliss points, and their bliss points are greater than $\bar{p}$. Because $[\max\{\bar{p}, d_k\}, 1]$ is contained in δ, Lemmas 2 and 3 hold for the primary voters.

Fix $k = H, L$. To prove part (i), we first claim that there is no $\tilde{p}_k^R$ to satisfy $1 = \frac{\pi'(v_k,\tilde{p}_k^R)(u_R(v_k,\tilde{p}_k^R)-u_R(\bar{v},\bar{p}))}{\pi(v_k,\tilde{p}_k^R)}$. Note that

$$u_R(v_k, 2R - d_k) = v_k - (2R - d_k - R) = -R + \bar{p} + \bar{v} = u_R(\bar{v}, \bar{p}).$$

As $u_R(v_k, p_k)$ is decreasing in p_k at the interval $[R, 2R - d_k)$, we conclude that for any $p_k \in [R, 2R-d_k)$, $u_R(v_k, p_k) > u_R(\bar{v}, \bar{p})$ holds. Thus, because $\pi'(v_k, p_k)$ is negative in this interval, we conclude that there is no $\tilde{p}_k^R \in [R, 2R - d_k)$ that satisfies $1 = \frac{\pi'(v_k,\tilde{p}_k^R)(u_R(v_k,\tilde{p}_k^R)-u_R(\bar{v},\bar{p}))}{\pi(v_k,\tilde{p}_k^R)}$. Thus, Lemma 3 implies that $\mathrm{E}u_R(v_k, p_k)$ is decreasing in the interval $[R, 2R - d_k]$. On the other hand, Lemma 2 completes our claim (i) for the interval $[d_k, R]$.

Moreover, by Lemma 2, it is clear that median voter R's expected payoff is single-peaked at $\min\{R, \bar{p}_k\}$. It is also clear that if $\hat{p}_k \neq R$, then $\hat{p}_k = \bar{p}_k$, which satisfies the first-order condition. Thus, we obtain (iii). Finally, we will complete our proof by proving part (ii).

(Case I) $\hat{p}_k < R$. Suppose that $p_k < \hat{p}_k$. Consider voter r with $r \geq R$. As shown in Lemma 2, voter r's expected payoff is increasing in $[p_k, \hat{p}_k]$. Because r is arbitrary in the set of $r \geq R$,

$$\mu\left(r \in \delta : \mathrm{E}u_r(v_k, \hat{p}_k) \geq \mathrm{E}u_r(v_k, p_k)\right) \geq \frac{\mu(\delta)}{2}.$$

Similarly, we can prove that $\hat{p}_k$ beats any policy $p_k \geq \hat{p}_k$. Consider voter r with $r \leq R$. By Lemma 3, voter r's most preferred policy promise is R in the interval $[R, 2R - d_k]$. By Lemma 2, voter r's expected payoff is decreasing in $[\hat{p}_k, R]$. Therefore, voters who have bliss points $r \leq R$ strictly prefer $\hat{p}_k$.

(Case II) $\hat{p}_k = R$. Note that in this case, $\bar{p}_k \geq R$. Suppose that $p_k \leq R$. Consider voter r with $r \geq R$. By Lemma 2, voter r strictly prefers R, and so R beats p_k. Similarly with Case I, this establishes our claim in this case. Now, suppose that $p_k > R$. Consider voter r with $r \leq R$. By Lemma 3, voter r strictly prefers R. Therefore, R strictly beats p_k. $\square$

Proof of Example 1. Fix $k = H, L$. For any $p_k \in I_k$,

$$\begin{aligned} \pi'(v_k, p_k) &= -\tfrac{1}{2}F'(\tfrac{d_k+p_k}{2}) = -\tfrac{1}{2(R-\bar{p})} \\ \pi(v_k, p_k) &= 1 - F(\tfrac{d_k+p_k}{2}) = 1 - \tfrac{d_k+p_k-2\bar{p}}{2(R-\bar{p})}. \end{aligned}$$

By the first-order condition $\frac{\partial}{\partial p_k}\left(\mathrm{E}u_R(v_k, p_k)\right) = \pi'(v_k, p_k)(p_k - d_k) + \pi(v_k, p_k)$,

$$\frac{\partial}{\partial p_k}\left(\mathrm{E}u_R(v_k, p_k)\right) = \frac{d_k - p_k}{2(R - \bar{p})} + 1 - \frac{d_k + p_k - 2\bar{p}}{2(R - \bar{p})} = 1 - \frac{p_k - \bar{p}}{R - \bar{p}}.$$

Thus, the above is greater than zero if and only if $p_k \leq R$ and equal to zero only when $p_k = R$. By Proposition 2, we obtain $\hat{p}_k = R$. Next, observe that

$$\begin{aligned}
\mathrm{E}u_R(v_L, R) &= F(\tfrac{d_L+R}{2})(\bar{v} - R + \bar{p}) + (1 - F(\tfrac{d_L+R}{2})) \cdot v_L \\
\mathrm{E}u_R(v_H, p_H) &= F(\tfrac{d_H+p_H}{2})(\bar{v} - R + \bar{p}) + (1 - F(\tfrac{d_H+p_H}{2}))(v_H - R + p_H).
\end{aligned}$$

Because $\mathrm{E}u_R(v_H, p_H)$ is continuous in $p_H \in I_H$, by the intermediate value theorem, if

$$\mathrm{E}u_R(v_L, R) > \min_{p_H \in I_H} \mathrm{E}u_R(v_H, p_H),$$

that is,

$$F(\frac{d_L+R}{2})(\bar{v} - R + \bar{p}) + (1 - F(\frac{d_L+R}{2})) \cdot v_L > \min_{p_H \in I_H} \mathrm{E}u_R(v_H, p_H), \tag{18}$$

together with Lemma 1, there exists a p'_H that satisfies

$$\mathrm{E}u_R(v_H, p'_H) = \frac{d_L + R - 2\bar{p}}{2(R-\bar{p})}(\bar{v} - R + \bar{p}) + \frac{R - d_L}{2(R-\bar{p})} \cdot v_L. \tag{19}$$

Note that by Lemma 3,

$$\min_{p_H \in I_H} \mathrm{E}u_R(v_H, p_H) = \mathrm{E}u_r(v_H, 2R - d_H) = \bar{v} - (R - \bar{p}). \tag{20}$$

By (3), (18) holds. Now, (19) can be rewritten as the RHS of (10). Finally, note that because $\bar{v} > v_H > v_L$, and by Lemma 2, for any $p_H < p'_H$, $\mathrm{E}u_R(v_H, p_H) < \mathrm{E}u_R(v_H, p'_H)$. Because $p^*_H \in \mathcal{U}_H(p^*_L)$, we obtain the desired result. $\square$

The detailed calculation for Proposition 3. Modifying (10) in Example 1 for λv, instead of v, we obtain

$$\lambda v_H - R + p_H = \frac{R + \lambda\Delta_L - \bar{p}}{2R + \lambda\Delta_H - \bar{p} - p_H}\lambda v_L + \frac{R - p_H + \lambda(\Delta_H - \Delta_L)}{2R + \lambda\Delta_H - \bar{p} - p_H} \cdot (\lambda\bar{v} - R + \bar{p}). \tag{21}$$

Let $R - p_H = D$ and $R - \bar{p} = \bar{D}$. By multiplying both sides of (21) by $2R + \lambda\Delta_H - \bar{p} - p_H = D + \bar{D} + \lambda\Delta_H$, the LHS becomes

$$\begin{aligned}
(\lambda v_H - D)\left(D + \bar{D} + \lambda\Delta_H\right) &= -(D - \lambda v_H)\left(D + (\bar{D} + \lambda\Delta_H)\right) \\
&= -\left(D^2 + D\left(\bar{D} + \lambda\bar{v}\right) - \lambda v_H\left(\bar{D} + \lambda\Delta_H\right)\right),
\end{aligned}$$

and the RHS is

$$(\bar{D} + \lambda\Delta_L)\lambda v_L + (D + \lambda(\Delta_H - \Delta_L))(\lambda\bar{v} - \bar{D}).$$

Thus, (21) can be rewritten as

$$\begin{aligned}
-D^2 - 2D\lambda\bar{v} &= -\lambda v_H\left(\bar{D} + \lambda\Delta_H\right) + (\bar{D} + \lambda\Delta_L)\lambda v_L + \lambda(\Delta_H - \Delta_L)(\lambda\bar{v} - \bar{D}) \\
&= -2\bar{D}\lambda(\Delta_H - \Delta_L) + \lambda^2 v_L\Delta_L - \lambda^2 v_H\Delta_H + \lambda^2\bar{v}(\Delta_H - \Delta_L).
\end{aligned} \tag{22}$$

Thus, (21) becomes

$$D^2 + 2D\lambda\bar{v} \;=\; 2\bar{D}\lambda(\Delta_H - \Delta_L) + \lambda^2 v_H \Delta_H - \lambda^2 v_L \Delta_L - \lambda^2 \bar{v}(\Delta_H - \Delta_L). \tag{23}$$

By taking the derivative, we obtain

$$D' = \frac{(\bar{D} - \lambda\bar{v})(\Delta_H - \Delta_L) + \lambda(v_H\Delta_H - v_L\Delta_L) - D\bar{v}}{D + \lambda\bar{v}}.$$

Therefore, we obtain the desired result. □

The detailed calculation for Proposition 4. Note that

$$F(\frac{\bar{p} + R - \lambda\Delta_H}{2}) = \frac{1}{2} - \frac{\lambda\Delta_H}{2(R - \bar{p})}. \tag{24}$$

By substituting F from (24), the primary advantage is given by

$$S(v_H, p_H^*, \hat{p}_H) = \pi(v_H, p_H^*) - \pi(v_H, R) = \frac{1}{2} + \frac{\lambda\Delta_H}{2(R - \bar{p})}. \tag{25}$$

By (25), the difference in the two primary advantages is

$$S_2(v_H, p_2^*, \hat{p}_2) - S_1(v_H, p_1^*, \hat{p}_1) = \frac{-\lambda\Delta_H^1}{2(R - \bar{p})} + \frac{\lambda\Delta_H^2}{2(R - \bar{p})}. \tag{26}$$

□

Proof of Proposition 5. By taking a derivative of (23), we obtain

$$\frac{\mathrm{d}(R - P(v_H))}{\mathrm{d}(R - \bar{p})} = \frac{\lambda(\Delta_H - \Delta_L)}{R - P(v_H) + \lambda\bar{v}}. \tag{27}$$

By Example 1, $\hat{p}_L = R$ and by (b) of Theorem 2, $R > P(v_H)$. Therefore, we obtain $\frac{\mathrm{d}(R - P(v_H))}{\mathrm{d}(R - \bar{p})} > 0$.

Next, by substituting (24) into $\mathrm{E}u_R(v_L, R) = \mathrm{E}u_R(v_H, P(v_H))$, we obtain

$$\begin{aligned}
&\left(\tfrac{1}{2} - \tfrac{\lambda\Delta_L}{2(R-\bar{p})}\right)(\lambda\bar{v} - R + \bar{p}) + \left(\tfrac{1}{2} + \tfrac{\lambda\Delta_L}{2(R-\bar{p})}\right)\lambda v_L \\
&\quad = \left(\tfrac{P(v_H)-\bar{p}}{2(R-\bar{p})} - \tfrac{\lambda\Delta_H}{2(R-\bar{p})}\right)(\lambda\bar{v} - R + \bar{p}) \\
&\qquad + \left(1 - \tfrac{P(v_H)-\bar{p}}{2(R-\bar{p})} + \tfrac{\lambda\Delta_H}{2(R-\bar{p})}\right)(\lambda v_H - R + P(v_H)).
\end{aligned} \tag{28}$$

Taking the first derivative of (28) with respect to v_H, we obtain

$$\begin{aligned}
0 = &\left(\frac{\frac{\mathrm{d}P(v_H)}{\mathrm{d}v_H}}{2(R-\bar{p})} - \frac{\lambda}{2(R-\bar{p})}\right)(\lambda\bar{v} - R + \bar{p}) - \left(\frac{\frac{\mathrm{d}P(v_H)}{\mathrm{d}v_H}}{2(R-\bar{p})} - \frac{\lambda}{2(R-\bar{p})}\right)(\lambda v_H - R + P(v_H)) \\
&+ \left(1 - \tfrac{P(v_H)-\bar{p}}{2(R-\bar{p})} + \tfrac{\lambda\Delta_H}{2(R-\bar{p})}\right)\left(\lambda + \tfrac{\mathrm{d}P(v_H)}{\mathrm{d}v_H}\right).
\end{aligned} \tag{29}$$

Arranging orders in (29), we obtain

$$\frac{\mathrm{d}P(v_H)}{\mathrm{d}v_H} = \frac{\lambda((R-\bar{p})+\lambda\Delta_H)}{P(v_H)-R}. \tag{30}$$

Organizing terms in (30), we obtain (B). Next, because $F\left(\frac{d_H+P(v_H)}{2}\right) = \frac{\frac{d_H+P(v_H)}{2}-\bar{p}}{R-\bar{p}}$, we obtain

$$\frac{\partial}{\partial v_H}F\left(\frac{d_H+P(v_H)}{2}\right) = \frac{1}{2(R-\bar{p})}\cdot\left(\frac{\mathrm{d}P(v_H)}{\mathrm{d}v_H}-\lambda\right). \tag{31}$$

Substituting (30) to (31), we obtain

$$\frac{\partial}{\partial v_H}F\left(\frac{d_H+P(v_H)}{2}\right) = \frac{\lambda}{2(R-\bar{p})}\cdot\left(\frac{\lambda\Delta_H+R-\bar{p}}{P(v_H)-R}-1\right). \tag{32}$$

Thus, by (32),

$$\begin{aligned} \frac{\mathrm{d}\pi(v_H,P(v_H))}{\mathrm{d}v_H} &= -\frac{\partial}{\partial v_H}F\left(\frac{d_H+P(v_H)}{2}\right) \\ &= \frac{\lambda}{2(R-\bar{p})}\left(1-\frac{\lambda\Delta_H+R-\bar{p}}{P(v_H)-R}\right). \end{aligned} \tag{33}$$

Because $\pi(v_H,P(v_H)) = 1-F\left(\frac{-\lambda\Delta_H+\bar{p}+P(v_H)}{2}\right)$, dividing the above by $\pi(v_H,P(v_H))$ yields our desired result. □

Proof of Proposition 8. By taking the derivative of (23) with respect to $\bar{v}$, we obtain

$$\frac{\partial(R-p_H^*)}{\partial\bar{v}} = \frac{\lambda v_L-\lambda v_H-2D}{2(D+\lambda\bar{v})}\cdot\lambda.$$

Because $v_L < v_H$, we obtain the desired result. □

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