Serial Dictatorship with Infinitely Many Agents*

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This Version: February 17, 2014

Abstract. This paper studies social choice correspondences assigning a set of choices to each pair consisting of a nonempty subset of the set of alternatives and a weak preference profile, which is called an *extended social choice correspondence* (ESCC). The ESCC satisfies unanimity if, when there is a weakly Pareto dominant alternative, the ESCC selects this alternative. Stability requires that the ESCC is immune to manipulation through withdrawal of some alternatives. Independence implies that the ESCC selects the same outcome from a subset of the set of alternatives for two preference profiles that are the same on this set. We characterize the ESCC satisfying the three axioms, when the set of alternatives is finite but includes more than three alternatives, and the set of voters can have any cardinality. Our main theorem establishes that the ESCC satisfying the three axioms is a serial dictatorship à la Eraslan and McLennan (2004). Our second theorem shows that a serial dictatorship includes "invisible serial dictators" à la Kirman and Sondermann (1972).

Key Words: Impossibility theorem, Social choice, Serial dictatorship, Ultrafilter.

JEL Classification Numbers: D71.

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*We would like to thank Andrew McLennan for continuous discussion and many suggestions to improve this paper. We are also grateful to Simon Grant, Chiaki Hara, Atsushi Kajii, John Quah, Satoru Takahashi, Rabee Tourky, and other participants in the Australasian Economics PhD Workshop, and the Econometric Society Australasian Meeting held in Sydney. We similarly acknowledge the useful suggestions of seminar participants at Kyoto University and the National University of Singapore. Takayama also gratefully acknowledges the hospitality of the Kyoto Institute of Economic Research and the University of British Columbia. All errors remaining are our own.

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1 Introduction

In this paper, we consider social choice correspondences assigning a set of choices to each pair consisting of a nonempty subset of the set of alternatives and a weak preference profile. We call such an object an *extended social choice correspondence* (ESCC) and impose three axioms, namely unanimity, stability, and independence. The ESCC satisfies unanimity if, when there is a weakly Pareto dominant alternative, the ESCC selects this alternative. Stability requires that the ESCC is immune to manipulation through withdrawal of some alternatives. Finally, independence implies that the ESCC selects the same outcome from a subset of the set of alternatives for two preference profiles that are the same on this set. We characterize the ESCC satisfying the three axioms, when the set of alternatives is finite but includes more than three alternatives, and the set of voters can have any cardinality. Our main theorem establishes that the ESCC satisfying the three axioms is a serial dictatorship à la Eraslan and McLennan (2004). Our second theorem shows that a serial dictatorship includes "invisible serial dictators" à la Kirman and Sondermann (1972).

Man and Takayama (2013) extend a social choice correspondence to an ESCC, characterize the ESCC satisfying the three axioms in the case of finitely many agents and derive many well-known impossibility theorems as corollaries to their main theorem, including Arrow's Impossibility Theorem (Arrow, 1959), the Gibbard–Satterthwaite Theorem (Gibbard, 1973; Satterthwaite, 1975), the theorem of serial dictators (Grether and Plott, 1982; Dutta, Jackson, and Le Breton, 2001; Eraslan and McLennan, 2004), and the characterization of game-theoretic solutions implementing only dictatorial social choices by Jackson and Srivastava (1996). In each case, the method of proof is to prove that the setting given by each theorem can be extended to, or used to derive, a "desirable" ESCC satisfying the three axioms.

With infinitely many voters, the method taken by Man and Takayama (2013) to identify each individual dictator does not work, because to identify an individual dictator, they use a Condorcet cycle and shrink a group that contains a dictator to one individual by changing preference profiles. Given a social welfare function satisfying Arrow's axioms of independence and unanimity, Fishburn (1970) shows that when the number of agents is infinite, it is impossible to identify a dictator whose preference dictates the outcome. In this context, Kirman and Sondermann (1972) show that, even in the case of infinitely many agents, a set of decisive coalitions is an ultrafilter and, in a sense, a dictatorship still persists. A filter on a ground set X is a collection of subsets of X that satisfies the three conditions: (1) the empty set is not included in the collection; (2) a superset of a set in the collection is also included; and (3) an intersection of finitely many sets in the collection is also included. An ultrafilter is a largest filter, which means that, if a set is not a member of the filter, then its complement is a member. An ultrafilter is called *principal* if it is generated by one individual, otherwise, it is called *non-principal*. As in Kirman and Sondermann

(1972), a non-principal ultrafilter can be thought of as an invisible dictator.

Our main theorem extends the result of invisible dictators by Kirman and Sondermann (1972) to a domain of weak preference profiles, and derives a serial dictatorship that generalizes the one in Eraslan and McLennan (2004) by using a *coherent hierarchy* of ultrafilters.

The existence of non-principal ultrafilters is a nontrivial consequence of the axiom of choice. In our analysis, the similar issue arises of how rich is the space of coherent hierarchies of ultrafilters. After the characterization of a serial dictatorship, the question remains if there exist invisible serial dictators that correspond to a coherent hierarchy of non-principal ultrafilters. By using Zorn's lemma (equivalent to the axiom of choice), we show that a coherent hierarchy of ultrafilters assigning non-principal ultrafilters to infinite sets exists. By our first main theorem, this result implies that there exists an ESCC that induces invisible serial dictators.

Since Schmeidler and Sonnenschein (1974), there have been studies on the relationship between Arrow's Impossibility Theorem and the Gibbard–Satterthwaite Theorem. Before the recent study by Man and Takayama (2013), Reny (2001) proves Arrow's Impossibility Theorem and the Gibbard–Satterthwaite Theorem in parallel arguments. Vohra (2011) proves Arrow's Impossibility Theorem, and then uses it to prove the Gibbard–Satterthwaite Theorem, and the impossibility theorem of strategic candidacy, by adopting integer programming techniques. In the case of finitely many agents, it is known that when preferences are strict, these two theorems share some common bases, whereas when preferences are weak, the Gibbard–Satterthwaite Theorem does not hold[1]. In this paper, we show that in the case of infinitely many agents, the Gibbard–Satterthwaite Theorem does not hold, even when preferences are strict. This implies that in the case of infinitely many agents, while a dictatorship persists when unanimity, independence, and stability hold, the Gibbard–Satterthwaite Theorem does not hold, even in strict preferences. Moreover, as our main theorem is not confined to the case of infinitely many agents, the direct consequence is the existence of serial dictatorship in the case of finitely many agents. In the literature, many preceding works (Geanakoplos, 2005; Yu, 2012; Barberà, 1980, 1983; Campbell and Kelly, 2002, for a comprehensive survey) provide another proof of Arrow's Impossibility Theorem in the case of finitely many agents.

The organization of the paper is as follows. Section 2 presents the model. Section 3 defines a generalized serial dictatorship. Section 4 provides the main theorem and proves it. Section 5 shows the existence of a hierarchy of non-principal ultrafilters. The last section concludes by providing an example to show that the Gibbard–Satterthwaite Theorem does not hold.

[1] An easy example is as follows. Let $X = \{a, b, c\}$ and suppose that there are three agents. Let a social choice function (SCF) be $f : \mathcal{R}^3 \to X$. Suppose that f selects outcomes so that it meets the following requirement: (1) if the set of most preferred alternatives for the first dictator 1 includes a, then agent 2 is the second dictator; (2) otherwise, agent 3 is the second dictator. We can verify that f satisfies the axioms for the theorem (which are stated in the last section of this paper), although this is not a serial dictatorship.

2 The Model

Let $\mathcal{X}$ be the set of potential alternatives. We assume that $3 \leq |\mathcal{X}| < \infty$. Let $\mathcal{N}$ be the set of agents. Let $\mathcal{R}$ be the space of weak preferences over $\mathcal{X}$. Let $\succsim_i \in \mathcal{R}$ be agent i's preference, and let $\succsim \in \mathcal{R}^{\mathcal{N}}$ be a preference profile of all the agents in $\mathcal{N}$. For each $x, y \in \mathcal{X}$, we say that $x \succ_i y$ if x is *strictly preferred* to y, i.e., $x \succsim_i y$ but not $y \succsim_i x$, and that $x \sim_i y$ if x is *indifferent* to y, i.e., $x \succsim_i y$ and $y \succsim_i x$.

For an arbitrary set Z, let $\mathscr{P}(Z) \equiv 2^Z \setminus \{\varnothing\}$. An *environment* is a pair $(X, \succsim) \in \mathscr{P}(\mathcal{X}) \times \mathcal{R}^{\mathcal{N}}$. An ESCC is a correspondence $\phi : \mathscr{P}(\mathcal{X}) \times \mathcal{R}^{\mathcal{N}} \to \mathcal{X}$ such that, for each $X \in \mathscr{P}(\mathcal{X})$,

$$\phi(X, \succsim) \in \mathscr{P}(X).$$

Next, we define three properties that we impose on ϕ. Alternative $x \in X$ is called *weakly Pareto dominant* in X if, for every $y \in X$, $x \succsim_i y$ for every $i \in \mathcal{N}$, and $x \succ_j y$ for at least one $j \in N$.

Definition (Unanimity). *An ESCC ϕ satisfies* unanimity *if $\phi(X, \succsim) = \{x\}$ holds for all $\succsim \in \mathcal{R}^{\mathcal{N}}$ such that $x \in X$ is weakly Pareto dominant in X.*

We say that $\succsim$ and $\succsim' \in \mathcal{R}^{\mathcal{N}}$ *agree* on X if $\succsim$ and $\succsim'$ are the same on X, which we denote by $\succsim =_X \succsim'$.

Definition (Independence). *An ESCC ϕ satisfies* independence *if, for each $X \in \mathscr{P}(\mathcal{X})$ and each $\succsim, \succsim' \in \mathcal{R}^{\mathcal{N}}$, $\succsim =_X \succsim'$ implies $\phi(X, \succsim) = \phi(X, \succsim')$.*

Definition (Stability). *An ESCC ϕ satisfies* stability *if, for each $X, X' \in \mathscr{P}(\mathcal{X})$ and each $\succsim \in \mathcal{R}^{\mathcal{N}}$, $X' \subset X$ and $\phi(X, \succsim) \cap X' \neq \varnothing$ imply $\phi(X', \succsim) = \phi(X, \succsim) \cap X'$.*

3 A Generalized Serial Dictatorship

In the literature of social choice theory, Eraslan and McLennan (2004) define a serial dictatorship in the case of finitely many agents. To extend their results in the case where the number of voters can be infinite, this section defines a "generalized serial dictatorship." To do this, we start by introducing our tool, an *ultrafilter*.

Definition (Ultrafilter). *A family $\mathscr{F} \subset 2^N$ is an* ultrafilter *of a set $N \subset \mathcal{N}$ if*

(1) $\varnothing \notin \mathscr{F}$.

(2) If $S \in \mathscr{F}$, and $S' \supset S$, then $S' \in \mathscr{F}$.

(3) If $S, S' \in \mathscr{F}$, then $S \cap S' \in \mathscr{F}$.

(4) If $S \subset N$, then either $S \in \mathscr{F}$ or $N \backslash S \in \mathscr{F}$.

An ultrafilter is called *principal* or *non-principal* when the intersection of all its members is nonempty or empty, respectively. If $\mathscr{F}$ is principal, then $\mathscr{F} = \{S \subset N : i \in S\}$ for some $i \in N$. If N is finite, all ultrafilters are principal.

For each $N \in \mathscr{P}(\mathcal{N})$ and $\succsim \in \mathcal{R}^{\mathcal{N}}$, let $\succsim|_N = (\succsim_i)_{i \in N} \in \mathcal{R}^N$. Let $\mathcal{U}_N$ be an ultrafilter over N. For each $R \in \mathcal{R}$, let $U_R \equiv \{i \in N \mid \succsim_i = R\}$, that is, U_R is the set of agents holding the preference R. As $\mathcal{X}$ is finite, there are finitely many possibilities for R, and so $\{U_R \subset N \mid R \in \mathcal{R}\}$ is a finite partition of N. Then, by properties (2) and (4) of ultrafilters, exactly one set in the partition belongs to $\mathcal{U}_N$. Thus, there exists a unique $R^* \in \mathcal{R}$ such that $U_{R^*} \in \mathcal{U}_N$. Then, we can use the preference of agents in U_{R^*} to represent ultrafilter $\mathcal{U}_N$'s preference, which we denote by $\succsim_{\mathcal{U}_N}$.

Definition. *A* coherent hierarchy of ultrafilters *$\mathscr{U}$ assigns an ultrafilter $\mathcal{U}_N$ over N to each $N \in \mathscr{P}(\mathcal{N})$ such that, for all $N, N' \in \mathscr{P}(\mathcal{N})$, if $N \in \mathcal{U}_{N'}$, then $M \in \mathcal{U}_{N'}$ if and only if $M \cap N \in \mathcal{U}_N$ for all $M \in \mathscr{P}(\mathcal{N})$.*

To explain the concept more clearly, we provide a simple example of violation of coherence. Let $\mathcal{N} = \{1, 2, 3, 4\}$. Suppose that we have identified a serial dictatorship and each number $i \in \mathcal{N}$ corresponds to their ranking within the dictatorship. For example, agent 1 is the first dictator in the serial dictatorship. In our definition, it implies $\mathcal{U}_{\mathcal{N}} = \{U : 1 \in U\}$. Suppose that there are two preference profiles $\succsim$ and $\succsim'$, each of which yields the following two cases:

- $U_1 = \{1, 4\}$ and $\mathcal{U}_{\{2,3\}} = \{U : 2 \in U\}$, and
- $U_1' = \{1\}$ and $\mathcal{U}_{\{2,3,4\}} = \{U : 3 \in U\}$.

Coherence is not satisfied by $\mathcal{U}_{\{2,3\}}$ and $\mathcal{U}_{\{2,3,4\}}$, because $\{3, 4\} \in \mathcal{U}_{\{2,3,4\}}$ but $\{3, 4\} \cap \{2, 3\} \notin \mathcal{U}_{\{2,3\}}$. The problem with the serial dictatorship is that even if agent 2 has not yet been chosen by the preceding ultrafilter $\mathcal{U}_{\mathcal{N}}$, the next ultrafilter $\mathcal{U}_{\{2,3,4\}}$ under $\succsim'$ exclusively chooses a set including agent 3. In this sense, agent 3 is chosen as a dictator before agent 2 in the second dictatorship that $\mathcal{U}_{\{2,3,4\}}$ defines.

To formally define a generalized serial dictatorship, we define a *top set* as follows: for each $R \in \mathcal{R}$ and nonempty $X \subset \mathcal{X}$, let

$$Top(X, R) \equiv \{x \in X \mid \text{for each } y \in X,\ x\, R\, y\}.$$

Finally, we provide a formal definition.

Definition. *An ESCC ϕ is a* generalized serial dictatorship *if there exists a coherent hierarchy of ultrafilters $\mathscr{U}$ and a tiebreaking rule $\rho \in \mathcal{R}$ such that, for each $(X, \succsim) \in \mathscr{P}(\mathcal{X}) \times \mathcal{R}^{\mathcal{N}}$, there exist $K(X, \succsim) \in \mathbb{N}_+$, and a series $\{T_k, C_k\}_{k=0}^{K(X,\succsim)}$ satisfying:*

1. *$T_0 = X$ and $C_0 = \mathcal{N}$,*

2. *for each $k \in \{0, \cdots, K(X, \succsim) - 1\}$, $C_k \neq \varnothing$,*

3. *for each $k \in \{1, \cdots, K(X, \succsim)\}$, $T_k = Top(T_{k-1}, \succsim_{\mathscr{U}(C_{k-1})})$,*

4. *for each $k \in \{1, \cdots, K(X, \succsim)\}$, $C_k = \{i \in \mathcal{N} \mid T_k \not\subset Top(T_{k-1}, \succsim_i)\}$,*

5. *$C_{K(X,\succsim)} = \varnothing$ and $\phi(X, \succsim) = Top(T_{K(X,\succsim)}, \rho)$.*

Having stated the definition of a generalized serial dictatorship in the environment where the number of agents can be infinite, we consider the connection of our definition with the definition of serial dictatorship for the case of finitely many agents found in Eraslan and McLennan (2004) by applying a concept of a non-principal or principal ultrafilter.

We say that there exists a hierarchy of dictators $\{i_1, \ldots, i_{|\mathcal{N}|}\}$, which is a permutation on $\mathcal{N}$ with $|\mathcal{N}| < \infty$, such that for all $(X, \succsim) \in \mathscr{P}(\mathcal{X}) \times \mathcal{R}^{\mathcal{N}}$, $\phi(X, \succsim) \subset Top(T_{n-1}, \succsim_{i_n})$ where $T_0 = X$ and $T_n = Top(T_{n-1}, \succsim_{i_n})$ for $n = 1, \ldots, |\mathcal{N}|$, and $\phi(X, \succsim) = Top(T_{i_{|\mathcal{N}|}}, \rho)$.

Next, let $\pi_1 = \bigcap\{C : C \in \mathscr{U}(C^0)\}$. When the ground set is finite, an ultrafilter is principal. Thus, when an ESCC $\phi_{\mathcal{N}}$ is a generalized serial dictatorship, π_1 is a singleton set and only contains i_1, who is the first dictator in the definition of Eraslan and McLennan (2004). Recursively, we can obtain a permutation of individuals.

4 The Main Theorem

Here, we state our main theorem.

Theorem 1. *An ESCC ϕ satisfies unanimity, independence, and stability if and only if ϕ is a generalized serial dictatorship.*

4.1 The Proof of the "If" Part

In this subsection, we assume that ϕ is a generalized serial dictatorship associated with a coherent hierarchy of ultrafilters $\mathscr{U}$. In each of the following three lemmas, we establish that ϕ satisfies one of the axioms.

Lemma 1. *If ϕ is a generalized serial dictatorship, then ϕ satisfies unanimity.*

Proof. Let $X \in \mathscr{P}(\mathcal{X})$ and $\succsim \in \mathcal{R}^{\mathcal{N}}$. Suppose that x is weakly Pareto dominant in X. Then, for each $k \in \{0, \cdots, K(X, \succsim)\}$, we have that $x \in T_k$. On the other hand, for each $y \neq x$, let $i_y \in \mathcal{N}$ be such that $x \succ_{i_y} y$. Then, there exists $k_y \in \{0, \cdots, K(X, \succsim) - 1\}$ such that $\succsim_{\mathcal{U}_{C_{k_y}}} = \succsim_{i_y}$. It implies that $y \notin T_{k_y+1}$. Therefore, $T_{K(X,\succsim)} = \{x\}$. Thus, we have that

$$\phi(X, \succsim) = Top(T_{K(X,\succsim)}, \rho) = \{x\}. \square$$

Lemma 2. *If ϕ is a generalized serial dictatorship, then ϕ satisfies independence.*

Proof. The result is direct, because $Top(X, \succsim)$ depends only on $\succsim|_X$. $\square$

Lemma 3. *If ϕ is a generalized serial dictatorship, then ϕ satisfies stability.*

Proof. Let $X \subset X' \subset \mathcal{X}$ and $\succsim \in \mathcal{R}^{\mathcal{N}}$. Let $\{T_m, C_m\}_{m=0}^{M}$ and $\{T'_l, C'_l\}_{l=0}^{L}$ be two sequences defined by a generalized serial dictatorship for $(X, \succsim)$ and $(X', \succsim)$, respectively. For each $m \in \{1, \cdots, M\}$ and $l \in \{1, \cdots, L\}$, let

$$U_m = \{i \in \mathcal{N} \mid T_m = Top(T_{m-1}, \succsim_i)\} \text{ and } U'_l = \{i \in \mathcal{N} \mid T'_l = Top(T'_{l-1}, \succsim_i)\}. \quad (1)$$

Suppose that $\phi(X', \succsim) \cap X \neq \varnothing$. We want to show that $T_M = T'_L \cap X$. Because $C_0 = C'_0$, $\succsim_{\mathcal{U}_{C_0}} = \succsim_{\mathcal{U}_{C'_0}}$ and $T_0 = T'_0 \cap X$. By induction, take $m \in \{1, \cdots, M\}$ and suppose that there exists an $l^* \in \{1, \cdots, L\}$ such that $T_{m-1} = T'_{l^*} \cap X$, $C_{m-1} \subset C'_{l^*}$, and $\succsim_{\mathcal{U}_{C_{m-1}}} = \succsim_{\mathcal{U}_{C'_{l^*}}}$.

Because $T_m = Top(T_{m-1}, \succsim_{\mathcal{U}_{C_{m-1}}})$ and $T'_{l^*+1} = Top(T'_{l^*}, \succsim_{\mathcal{U}_{C'_{l^*}}})$, we have that $T_m = T'_{l^*+1} \cap X$. By (1) and the definition of a generalized serial dictatorship,

$$U'_{l^*+1} = \{i \in \mathcal{N} \mid T'_{l^*+1} = Top(T'_{l^*}, \succsim_i)\} \in \mathcal{U}_{C'_{l^*}}. \quad (2)$$

For each $i \in U'_{l^*+1}$, as $T_m = T'_{l^*+1} \cap X$,

$$\begin{aligned} Top(T_{m-1}, \succsim_{\mathcal{U}_{C_{m-1}}}) &= Top(T'_{l^*}, \succsim_{\mathcal{U}_{C'_{l^*}}}) \cap X && (\succsim_{\mathcal{U}_{C_{m-1}}} = \succsim_{\mathcal{U}_{C'_{l^*}}}) \\ &= Top(T'_{l^*}, \succsim_i) \cap X && (i \in U'_{l^*+1} \text{ and } (2)) \\ &= Top(T_{m-1}, \succsim_i), && (T_{m-1} = T'_{l^*} \cap X), \end{aligned}$$

which implies that $U'_{l^*+1} \subset U_m$ by (1). As $C_m = C_{m-1} \backslash U_m$, $C'_{l^*+1} = C'_{l^*} \backslash U'_{l^*+1}$ and $C_{m-1} \subset C'_{l^*}$, $C_m \subset C'_{l^*+1}$. Then,

$$\{i \in \mathcal{N} \mid \succsim_i = \succsim_{\mathcal{U}_{C_m}}\} \in \mathcal{U}_{C_m} \text{ and } \{i \in \mathcal{N} \mid \succsim_i = \succsim_{\mathcal{U}_{C'_{l^*+1}}}\} \in \mathcal{U}_{C'_{l^*+1}}.$$

Now, we have to consider two cases: (1) $C'_{l^*+1} \backslash C_m \notin \mathcal{U}_{C'_{l^*+1}}$; and (2) $C'_{l^*+1} \backslash C_m \in \mathcal{U}_{C'_{l^*+1}}$.
(Case 1) Suppose that $C'_{l^*+1} \backslash C_m \notin \mathcal{U}_{C'_{l^*+1}}$. By coherence, $\{i \in \mathcal{N} \mid \succsim_i = \succsim_{\mathcal{U}_{C'_{l^*+1}}}\} \cap C_m \in \mathcal{U}_{C_m}$. It implies that $\{i \in \mathcal{N} \mid \succsim_i = \succsim_{\mathcal{U}_{C'_{l^*+1}}}\} \cap \{i \in \mathcal{N} \mid \succsim_i = \succsim_{\mathcal{U}_{C_m}}\} \neq \varnothing$. Hence, $\succsim_{\mathcal{U}_{C_m}} = \succsim_{\mathcal{U}_{C'_{l^*+1}}}$.

(Case 2) Suppose that $C'_{l^*+1} \backslash C_m \in \mathcal{U}_{C'_{l^*+1}}$. Then, there exists an $i^* \in C'_{l^*+1} \backslash C_m$ such that $\succsim_{i^*} = \succsim_{\mathcal{U}_{C'_{l_m}}}$. Therefore, $\succsim_{C_m} \neq \succsim_{\mathcal{U}_{C'_{l_m}}}$. Note that

$$T'_{l^*+2} = Top(T'_{l^*+1}, \succsim_{\mathcal{U}_{C'_{l^*+1}}}) = Top(T'_{l^*+1}, \succsim_{i^*}). \tag{3}$$

On the other hand, because $i^* \in U_m$, we have that

$$T_m = Top(T_{m-1}, \succsim_{i^*}). \tag{4}$$

Because $T_m \subset T'_{l^*+1}$ because $T_m = T'_{l^*+1} \cap X$, we have that

$$T_m = Top(T_{m-1}, \succsim_{i^*}) \subset Top(T'_{l^*+1}, \succsim_{i^*}) = T'_{l^*+2}. \tag{5}$$

Let $U'_{l^*+2} = \{i \in \mathcal{N} \mid T'_{l^*+2} = Top(T'_{l^*+1}, \succsim_i)\}$. Because $T_m \subset T'_{l^*+2} \subset T'_{l^*+1}$, we have that $U'_{l^*+2} \subset U_m$, because if $i \in U'_{l^*+2}$ and $a \in T_m$, then $a \in T'_{l^*+2}$, which indicates $i \in U_m$ by (5). By definition, $C_m \subset C'_{l^*+2} = C'_{l^*+1} \backslash U'_{l^*+2}$. Because there are finitely many different preferences among agents in $C'_{l^*+2} \backslash C_m$, we can repeat this elimination of C'_{l^*+j} for $j \in \{1, \cdots, J\}$ at most finite J times until $C'_{l^*+J} \backslash C_m \notin \mathcal{U}_{C'_{l^*+1}}$. Therefore, there exists an $l^{**} \in \{1, \cdots, L\}$ such that $T_m = T'_{l^{**}} \cap X$, $C_m \subset C'_{l^{**}}$, and $\succsim_{\mathcal{U}_{C_m}} = \succsim_{\mathcal{U}_{C'_{l^{**}}}}$. Hence, by induction, for each $m \in \{0, \cdots, M\}$, there exists an $l_m \in \{0, \cdots, L\}$ such that $T_m = T'_{l_m} \cap X$, $C_m \subset C'_{l_m}$, and $\succsim_{\mathcal{U}_{C_m}} = \succsim_{\mathcal{U}_{C'_{l_m}}}$.

Then, take a subsequence $\{T'_{l_m}, C'_{l_m}\}_{m=0}^M$ of the sequence $\{T'_l, C'_l\}_{l=0}^L$ such that, for each $m \in \{0, \cdots, M\}$, $T_m = T'_{l_m} \cap X$, $C_m \subset C'_{l_m}$, and $\succsim_{\mathcal{U}_{C_m}} = \succsim_{\mathcal{U}_{C'_{l_m}}}$. If $C'_{l_M} = \varnothing$, then $C'_{l_M} = C'_L$, which indicates $T_M = T'_L \cap X$ and $\phi(X, \succsim) = \phi(X', \succsim) \cap X$. If $C'_{l_M} \neq \varnothing$, by the same argument with (5),

$$T_M \subset T'_{l_M+1} = Top(T'_{l_M}, \succsim_{C'_{l_M}}).$$

Therefore, $T_M = T'_{l_M+1} \cap X$. Recursively, as $l_M \in \{0, \cdots, L\}$, we obtain $T_M = T'_L \cap X$ and $\phi(X, \succsim) = \phi(X', \succsim) \cap X$. □

4.2 The Proof of the "Only If" Part

Because Kirman and Sondermann (1972) only deal with the strict preferences and use a social welfare function, there are two issues that we need to take care of. First, we show that there exists a social preference whose top set on an arbitrary nonempty set X selects the same set chosen by the ESCC satisfying the three axioms. Then, we extend the proof in Kirman and Sondermann (1972) to the full domain of preferences.

An *environment for* N is a pair $(X, \succsim|_N) \in \mathscr{P}(\mathcal{X}) \times \mathcal{R}^N$. An ESCC for N is a correspondence $\phi_N : \mathscr{P}(\mathcal{X}) \times \mathcal{R}^N \to \mathcal{X}$ such that, for each nonempty $X \subset \mathcal{X}$, $\phi_N(X, \succsim|_N) \in \mathscr{P}(X)$. As before, we simply denote an ESCC for $\mathcal{N}$ by ϕ.

Lemma 4. *If ϕ_N satisfies unanimity, independence, and stability, then, for each $\succsim|_N \in \mathcal{R}^N$, there exists an $R \in \mathcal{R}$ such that, for each $X \subset \mathcal{X}$ with $|X| \geq 2$, $\phi_N(X, \succsim|_N) = Top(X, R)$.*

Proof. Let $\succsim|_N \in \mathcal{R}^N$. We define a binary relationship R on X by xRy if $x \in \phi_N(\{x,y\}, \succsim|_N)$. Then, because $\phi_N(\{x,y\}, \succsim|_N) \neq \varnothing$, R is complete.

Next, we show transitivity. Suppose that $x, y, z \in \mathcal{X}$ satisfy that xRy and yRz. Then

$$x \in \phi_N(\{x,y\}, \succsim|_N) \text{ and } y \in \phi_N(\{y,z\}, \succsim|_N).$$

We need to show $x\ R\ z$, namely $x \in \phi_N(\{x,z\}, \succsim|_N)$. Suppose that this is not true. Then, by stability,

$$x \notin \phi_N(\{x,y,z\}, \succsim|_N).$$

Hence, $y \notin \phi_N(\{x,y,z\}, \succsim|_N)$, because otherwise $\phi_N(\{x,y\}, \succsim|_N) = \{y\}$ would hold by stability. Hence, $\phi_N(\{x,y,z\}, \succsim|_N) = \{z\}$, which implies that $y \notin \phi_N(\{y,z\}, \succsim|_N)$ by stability. This is a contradiction. Thus, R is complete and transitive, that is, a weak preference.

Finally, we show that ϕ_N is the top set of R. Let $X \subset \mathcal{X}$ with $|X| \geq 2$. First, suppose that there exists $x \in \phi_N(X, \succsim|_N)$ but $x \notin Top(X, R)$. Then, there exists a $y \in X$ such that $y \in \phi_N(\{x,y\}, \succsim|_N)$ but $x \notin \phi_N(\{x,y\}, \succsim|_N)$ by the definition of R. However, because $\{x,y\} \subset X$, $x \in \phi_N(\{x,y\}, \succsim|_N)$ by stability. This is a contradiction. Thus, $\phi_N(X, \succsim|_N) \subset Top(X, R)$. Next, suppose that $x \in Top(X, R)$. Then, for each $y \in X$, xRy, which implies that, for each $y \in X$, $x \in \phi_N(\{x,y\}, \succsim|_N)$ and, thus, $x \in \phi_N(X, \succsim|_N)$ by stability. Hence, $Top(X, R) \subset \phi_N(X, \succsim|_N)$. □

By Lemma 4, we can define a social welfare function $\mathbf{R} : \mathcal{R}^N \to \mathcal{R}$ such that, for each $X \in \mathscr{P}(\mathcal{X})$ with $|X| \geq 2$,

$$\phi_N(X, \succsim|_N) = Top(X, \mathbf{R}(\succsim|_N)). \tag{6}$$

We write that $x\mathbf{P}_{\succsim|_N}y$ if x is strictly preferred to y, and that $x\mathbf{I}_{\succsim|_N}y$ if x is indifferent to y under $\mathbf{R}(\succsim|_N)$, respectively.

For each $U \subset N \in \mathscr{P}(\mathcal{N})$, $x, y \in \mathcal{X}$ and each $\succsim|_N \in \mathcal{R}^N$, we denote $x \succsim|_U y$ if, for each $i \in U$, $x \succsim|_{i \in N} y$. We consider three families of groups of agents.

- $\mathcal{F}_N$ is the set of subsets $U \subset N$ such that, for all $x, y \in \mathcal{X}$, and for all $\succsim|_N \in \mathcal{R}^N$, $x \succ|_U y$ and $y \succsim|_{N \setminus U} x$ imply $x\mathbf{P}_{\succsim|_N}y$;

- $\mathcal{F}'_N$ is the set of subsets $U \subset N$ such that there exist distinct $x, y \in \mathcal{X}$ where, for all $\succsim|_N \in \mathcal{R}^N$, $x \succ|_U y$ and $y \succsim|_{N \setminus U} x$ imply $x\mathbf{P}_{\succsim|_N}y$;

- $\mathcal{F}''_N$ is the set of subsets $U \subset N$ such that there exist distinct $x, y \in \mathcal{X}$ and an $\succsim|_N \in \mathcal{R}^N$ where the following hold: (1) $x \succ|_U y$, (2) $y \sim|_{\tilde{U}} x$, (3) $y \succ|_{N\setminus(U\cup\tilde{U})} x$, and (4) $x\mathbf{P}_{\succsim|_N} y$ for all $\tilde{U} \subset N\setminus U$.

Note that each member of $\mathcal{F}_N$ is a decisive coalition in the sense that as long as they strictly prefer some alternatives to others, those unfavored alternatives are never chosen.

We now show that $\mathcal{F}_N$ is an ultrafilter on N. We prove it by showing that (1) $\mathcal{F}_N = \mathcal{F}''_N$ and (2) $\mathcal{F}''_N$ is an ultrafilter. By definition, it is clear that

$$\mathcal{F}_N \subset \mathcal{F}'_N \subset \mathcal{F}''_N. \tag{7}$$

The following two lemmas show that the inverse inclusions also hold and these sets are not empty.

Lemma 5. $N \in \mathcal{F}''_N$.

Proof. In the definition of $\mathcal{F}''_N$, let $U = N$ and suppose that (1), (2), and (3) hold for some $x, y \in \mathcal{X}$ and $\succsim|_N \in \mathcal{R}^N$. Then, by unanimity, we obtain $\phi_N(X, \succsim|_N) = \{x\}$, which implies (4) $x\mathbf{P}_{\succsim|_N} y$ by (6). Thus, we obtain $N \in \mathcal{F}''_N$.

Lemma 6. $\mathcal{F}''_N \subset \mathcal{F}'_N$.

Proof. Let $U \in \mathcal{F}''_N$ and $U' \equiv N\setminus U$. Let $\succsim|_N \in \mathcal{R}^N$. Suppose that

$$x \succ|_U y \;\text{ and }\; y \succsim|_{U'} x \tag{8}$$

at $\succsim|_N$. Now, let $\tilde{U} \equiv \{i \in U' \mid x \sim_i y\}$, and let $U'' \equiv U'\setminus\tilde{U}$. Then, because $U \in \mathcal{F}''_N$, there exist $x, y \in \mathcal{X}$ and $\succsim^\alpha \in \mathcal{R}^N$ such that $x \succ^\alpha_U y$, $y \sim^\alpha_{\tilde{U}} x$, $y \succ^\alpha_{U''} x$, and $x\mathbf{P}_{\succsim^\alpha} y$, which implies $\succsim^\alpha =_{\{x,y\}} \succsim|_N$ by (8). Thus, by independence, $\phi_N(\{x, y\}, \succsim|_N) = \phi_N(\{x, y\}, \succsim^\alpha)$. Because $x\mathbf{P}_{\succsim^\alpha} y$, $\phi_N(\{x, y\}, \succsim|_N) = \phi_N(\{x, y\}, \succsim^\alpha) = \{x\}$. This implies that $x\mathbf{P}_{\succsim|_N} y$. Thus, $U \in \mathcal{F}'_N$. ☐

Lemma 7. $\mathcal{F}'_N \subset \mathcal{F}_N$.

Proof. Let $U \in \mathcal{F}'_N$, and let $U' \equiv N\setminus U$. Then, there exist distinct $x, y \in \mathcal{X}$ such that

$$\text{for all } \succsim|_N \in \mathcal{R}^N,\;\; x \succ|_U y \;\text{ and }\; y \succsim|_{U'} x \;\text{ imply }\; x\mathbf{P}_{\succsim|_N} y.$$

Let $z \in \mathcal{X}\setminus\{x, y\}$ and $\succsim|_N \in \mathcal{R}^N$. Suppose that $z \succ|_U y$ and $y \succsim|_{U'} z$. Now, we want to show that $z\ \mathbf{P}_{\succsim|_N}\ y$. Let $\succsim^\beta \in \mathcal{R}^N$ such that $z \succ^\beta_U x \succ^\beta_U y$, $y \succsim^\beta_{U'} z \succ^\beta_{U'} x$, and, for each $i \in U'$, $\succsim_i =_{\{y,z\}} \succsim^\beta_i$. Then, by unanimity, $\phi_N(\{x, z\}, \succsim^\beta) = \{z\}$. This implies that $z\mathbf{P}_{\succsim^\beta} x$. Now $x \succ^\beta_U y$ and $y \succ^\beta_{U'} x$. Because $U \in \mathcal{F}'_N$, we have $x\mathbf{P}_{\succsim^\beta} y$. Then, we have $z\mathbf{P}_{\succsim^\beta} y$ by transitivity.

Because $\succsim|_N =_{\{y,z\}} \succsim^\beta$, we have that $\phi_N(\{y,z\}, \succsim|_N) = \phi_N(\{y,z\}, \succsim^\beta)$. Therefore, $z\mathbf{P}_{\succsim|_N} y$, which shows that the preference of U dictates the social welfare between arbitrary z and y.

Finally, we show that it also dictates the social welfare between any arbitrary z and w. Let $w \in \mathcal{X}\backslash\{x,y,z\}$. Suppose that $z \succ|_U w$ and $w \succsim|_{U'} z$. Let $\succsim^{\beta'} \in \mathcal{R}^N$ such that $z \succ^{\beta'}_U y \succ^{\beta'}_U w$, $w \succsim^{\beta'}_{U'} z \succ^{\beta'}_{U'} y$, and, for each $i \in U'$, $\succsim_i =_{\{w,z\}} \succsim^{\beta'}_i$. By unanimity, $\phi_N(\{y,z\}, \succsim^{\beta'}) = \{z\}$. So, $z\mathbf{P}_{\succsim^{\beta'}} y$. Now, $y \succ^{\beta'}_U w$ and $w \succ^{\beta'}_{U'} y$. By the first part of this proof, we have $y\mathbf{P}_{\succsim^{\beta'}} w$, which implies $z\mathbf{P}_{\succsim^{\beta'}} w$ by transitivity. The rest follows in the same way as for the first part of this proof, by replacing y with w, and we obtain $z\mathbf{P}_{\succsim|_N} w$. Because $\succsim|_N$ is arbitrary, we obtain $U \in \mathcal{F}_N$. $\square$

Together with (7), Lemma 6 and Lemma 7 establish the following relationship.

Corollary 1. $\mathcal{F}_N = \mathcal{F}'_N = \mathcal{F}''_N$

Finally, we show that $\mathcal{F}_N$ is an ultrafilter.

Proposition 1. *$\mathcal{F}_N$ is an ultrafilter.*

Proof. By Corollary 1, it is enough to show that $\mathcal{F}''_N$ is an ultrafilter.
(Property 1) On the contrary, suppose $\varnothing \in \mathcal{F}''_N$. By taking $U = \varnothing$ and $\tilde{U} \neq N$, there exist $x, y \in \mathcal{X}$ and $\succsim|_N \in \mathcal{R}^N$ such that $y \succsim|_{\tilde{U}} x$, $y \succ|_{N\backslash\tilde{U}} x$, and $x\ \mathbf{P}_{\succsim|_N}\ y$. This contradicts unanimity.
(Property 3) Let $W_1, W_2 \in \mathcal{F}''_N$. We separate N into four disjoint subsets:

$$(1)\ V_1 \equiv W_1 \cap W_2;\ (2)\ V_2 \equiv W_1\backslash V_1;\ (3)\ V_3 \equiv W_2\backslash V_1;\ (4)\ V_4 \equiv N\backslash(W_1 \cup W_2).$$

Let $\{a,b,c\} \subset \mathcal{X}$ and $\tilde{V}_1 \subset N\backslash V_1$. Then, we can find $\succsim^\gamma \in \mathcal{R}^N$ such that (1) $c \succ^\gamma_{V_1} a \succ^\gamma_{V_1} b$; (2) $a \succ^\gamma_{V_2} b \succsim^\gamma_{V_2} c$; (3) $b \succsim^\gamma_{V_3} c \succ^\gamma_{V_3} a$; (4) $b \succsim^\gamma_{V_4} a \succsim^\gamma_{V_4} c$; (5) $b \sim^\gamma_{\tilde{V}_1} c$; (6) $b \succ^\gamma_{N\backslash(V\cup\tilde{V}_1)} c$.

Note that $a \succ^\gamma_{W_1} b$, and $b \succsim^\gamma_{N\backslash W_1} a$. By Corollary 1, we have $W_1 \in \mathcal{F}''_N = \mathcal{F}_N$. Therefore, $a\mathbf{P}_{\succsim^\gamma} b$. In the same way, $W_2 \in \mathcal{F}_N$, $c \succ^\gamma_{W_2} a$, and $a \succsim^\gamma_{N\backslash W_2} c$. This implies that $c\mathbf{P}_{\succsim^\gamma} a$. By transitivity, we have $c\mathbf{P}_{\succsim^\gamma} b$. Now, we also have that $c \succ^\gamma_{V_1} b$, $b \sim^\gamma_{\tilde{V}_1} c$, and $b \succ^\gamma_{N\backslash(V\cup\tilde{V}_1)} c$. Thus, by the definition of $\mathcal{F}''_N$, $V_1 = W_1 \cap W_2 \in \mathcal{F}''_N$.
(Property 4) Next, we show that for each $V \subset N$, $V \in \mathcal{F}''_N$ or $N\backslash V \in \mathcal{F}''_N$. If $V \in \mathcal{F}''_N$, then the result is immediate. Hence, we suppose that $V \notin \mathcal{F}''_N$. Then, as $\mathcal{F}''_N \neq \varnothing$ by Lemma 5 and $\mathcal{F}''_N = \mathcal{F}'_N$ by Corollary 1, for some $\{a,b,c\} \subset \mathcal{X}$, there exists a $\tilde{V} \subset N\backslash V$ with $\tilde{V} \in \mathcal{F}''_N$ such that for each $\succsim|_N \in \mathcal{R}^N$, if $b \succ|_V a$, $a \sim|_{\tilde{V}} b$, and $a \succ|_{N\backslash(V\cup\tilde{V})} b$, then $a\mathbf{P}_{\succsim|_N} b$. Now, let $\hat{V} \subset V$. Then, we can find $\succsim^\delta \in \mathcal{R}^N$ such that (1) $b \succ^\delta_{\hat{V}} c \sim^\delta_{\hat{V}} a$; (2) $b \succ^\delta_{V\backslash\hat{V}} c \succ^\delta_{V\backslash\hat{V}} a$; (3) $a \sim^\delta_{\tilde{V}} b \succ^\delta_{\tilde{V}} c$; (4) $a \succ^\delta_{N\backslash(V\cup\tilde{V})} b \succ^\delta_{N\backslash(V\cup\tilde{V})} c$.

By unanimity, $b\mathbf{P}_{\succsim^\delta} c$. Note that $b \succ^\delta_V a$, $a \sim^\delta_{\tilde{V}} b$, and $a \succ^\delta_{N\backslash(V\cup\tilde{V})} b$. By our initial assumption of $V \notin \mathcal{F}''_N$ and $\tilde{V} \in \mathcal{F}''_N$, we have that $a\mathbf{P}_{\succsim^\delta} b$. Thus, $a\mathbf{P}_{\succsim^\delta} c$. At the same time, we have that $a \succ^\delta_{N\backslash V} b$, $c \sim^\delta_{\hat{V}} a$, and $c \succ^\delta_{V\backslash\hat{V}} a$. Because $\hat{V}$ is an arbitrary subset of V, by the definition of $\mathcal{F}''_N$, we have that $N\backslash V \in \mathcal{F}''_N$.

(Property 2) On the contrary, suppose that there exist $W, U \subset N$ with $U \subset W$ such that $U \in \mathcal{F}''_N$ but $W \notin \mathcal{F}''_N$. Then, by Property 4, $N \backslash W \in \mathcal{F}''_N$. By Property 3, we have that $U \cap (N \backslash W) \in \mathcal{F}''_N$. However, $U \cap (N \backslash W) = \varnothing$, which contradicts Property 1. $\square$

The following lemma completes the proof of the "only if" part.

Lemma 8. *If ϕ satisfies unanimity, independence, and stability, then there exists a coherent hierarchy of ultrafilters $\mathscr{U}$ such that, for each $(X, \succsim) \in \mathscr{P}(\mathcal{X}) \times \mathcal{R}^{\mathcal{N}}$, there exist $K(X, \succsim) \in \mathbb{N}_+$ and $\{T_k, C_k\}_{k=0}^{K(X,\succsim)}$ satisfying 1-4 in the definition of the generalized serial dictatorship.*

Proof. Suppose that ϕ satisfies unanimity, independence, and stability. For each $N \in \mathscr{P}(\mathcal{N})$ and $\succsim|_N \in \mathcal{R}^N$, let $\succsim|_N = (\succsim|_{1\in N}, \cdots, \succsim|_{i\in N}, \cdots, \succsim|_{|N|\in N})$, and define $\succsim \in \mathcal{R}^{\mathcal{N}}$ by

$$\text{for each } i \in N, \ \succsim_i = \succsim|_{i\in N},$$
$$\text{for each } i \in \mathcal{N}\backslash N, \ \text{for each } x, y \in \mathcal{X}, \ x \sim_i y.$$

Let ψ_N be an ESCC such that, for each $(X, \succsim|_N) \in \mathscr{P}(\mathcal{X}) \times \mathcal{R}^N$, $\psi_N(X, \succsim|_N) \equiv \phi(X, \succsim)$. We show that ψ_N satisfies unanimity, independence, and stability. Let $x \in X$ and $\succsim|_N \in \mathcal{R}^N$.
(Unanimity) Suppose that for each $i \in N$ and $y \in X$, $x \succsim|_{i\in N} y$, and there exists $i^* \in N$ such that $x \succsim|_{i^*\in N} y$. Then, by construction, we have that for each $i \in \mathcal{N}$ and $y \in X$, $x \succsim_i y$. Because $i^* \in N$, we have that $\succsim|_{i^*\in N} = \succsim_{i^*}$. Therefore, $x \succ_{i^*} y$. By unanimity of ϕ, $\phi(X, \succsim) = \{x\}$. This implies that $\psi_N(X, \succsim|_N) = \{x\}$.
(Independence) Let $X \subset \mathcal{X}$, and $\succsim|_N, \tilde{\succsim}|_N \in \mathcal{R}^N$ be such that $\succsim|_N =_X \tilde{\succsim}|_N$. Then, for each $i \in N$, we have that $\succsim|_{i\in N} =_X \tilde{\succsim}|_{i\in N}$. For each $i \in \mathcal{N}\backslash N$, i is indifferent among $\mathcal{X}$ in both cases. Therefore, $\succsim|_N =_X \succsim^\delta$. Thus, we have that

$$\psi_N(X, \succsim|_N) = \phi(X, \succsim) = \phi_N(X, \succsim^\delta) = \psi_N(X, \tilde{\succsim}|_N).$$

The second equality is due to independence of ϕ.
(Stability) Let $Y \subset X \subset \mathcal{X}$ with $Y \neq \varnothing$. Let $\succsim|_N \in \mathcal{R}^N$. Suppose that $\psi_N(X, \succsim|_N) \cap Y \neq \varnothing$. Then, $\phi(X, \succsim) \cap Y \neq \varnothing$. Because ϕ satisfies stability,

$$\phi(Y, \succsim) = \phi(X, \succsim) \cap Y.$$

Therefore, $\psi_N(Y, \succsim|_N) = \psi_N(X, \succsim|_N) \cap Y$.

By Proposition 1, the set of decisive coalitions is an ultrafilter over N. We denote the ultrafilter as $\mathcal{U}_N$. Define a correspondence $\mathscr{U} : \mathscr{P}(\mathcal{N}) \to \mathscr{P}(\mathcal{N})$ such that, for every $N \in \mathscr{P}(\mathcal{N})$, $\mathscr{U}(N) = \mathcal{U}_N$, that is, $\mathscr{U}$ assigns a dictatorial ultrafilter $\mathcal{U}_N$, which is the set of decisive coalitions, to each nonempty N. We show that $\mathscr{U}$ is a coherent hierarchy of ultrafilters.

Let $N, N' \in \mathscr{P}(\mathcal{N})$ with $N \subset N'$. For each $\succsim|_N \in \mathcal{R}$, let $\succsim|_{N'}$ be such that (1) for each $i \in N$, $\succsim|_{i\in N} = \succsim|_{i\in N'}$, and (2) for each $i \in N'\backslash N$ and $x, y \in \mathcal{X}$, $x \sim|_{N'} y$. Then, we have that for each $(X, \succsim|_N) \in \mathscr{P}(\mathcal{X}) \times \mathcal{R}^N$, $\psi_N(X, \succsim|_N) = \psi_{N'}(X, \succsim|_{N'})$. Suppose that for $U \in \mathcal{U}_{N'}$, it holds that $U \subset N$. By construction, for each $\succsim \in \mathcal{R}^N$ and $x, y \in \mathcal{X}$, if $x \succ_U y$, $\psi_{N'}(\{x, y\}, \succsim|_{N'}) = \{x\}$. This implies that, for each $\succsim|_N \in \mathcal{R}^N$ and $x, y \in \mathcal{X}$, if $x \succ_U y$, $\psi_N(\{x, y\}, \succsim|_N) = \{x\}$. Thus, $U \in \mathcal{U}_N$.

Now, suppose that $N' \backslash N \notin U_{N'}$. Because $U_{N'}$ is an ultrafilter, $N \in U_{N'}$. If $M' \in \mathcal{U}_{N'}$, then $N \cap M' \in \mathcal{U}_{N'}$. The above argument implies that $N \cap M' \in \mathcal{U}_{N'}$. If $N \cap M' \in \mathcal{U}_N$, we want to show that $N \cap M' \in \mathcal{U}_{N'}$. Suppose this is not true. Because $N \in \mathcal{U}_{N'}$, we have that $N \backslash M' \in \mathcal{U}_{N'}$. Because $N \backslash M' \subset N$, we have that $N \backslash M' \in \mathcal{U}_N$. However, we assume that $N \cap M' \in \mathcal{U}_N$. This is a contradiction. Thus, $N \cap M' \in \mathcal{U}_{N'}$. Because $N \cap M' \subset M'$, we have that $M' \in \mathcal{U}_{N'}$.

It remains to show that ϕ is a generalized serial dictatorship. Let $(X, \succsim) \in \mathscr{P}(\mathcal{X}) \times \mathcal{R}^{\mathcal{N}}$. Let $T_0 \equiv X$ and $C_0 \equiv \mathcal{N}$. Because ultrafilter $\mathcal{U}_{C_0}$ picks out only one set from the equivalent classes with respect to their preferences, there exists a unique $\succsim^\circ \in \mathcal{R}$ such that $\{i \in \mathcal{N} \mid \succsim_i = \succsim^\circ\} \in \mathcal{U}_{C_0}$. We denote this $\succsim^\circ$ as $\succsim_{\mathcal{U}_{C_0}}$. Then, we have that $\phi(X, \succsim) \subset T_1 \equiv Top(T_0, \succsim_{\mathcal{U}_{C_0}})$.

Next, let $C_1 \equiv \{i \in \mathcal{N} \mid T_1 \not\subset Top(T_0, \succsim_i)\}$. Let $\succsim^{C_1} \in \mathcal{R}^{C_1}$ be such that, for each $i \in C_1$, $\succsim_i^{C_1} = \succsim_i$. Then, by construction, $\psi_{C_1}(T_1, \succsim^{C_1}) = \phi(T_1, \succsim)$. By stability, we have that $\psi_{C_1}(T_1, \succsim^{C_1}) = \phi(X, \succsim)$. In a similar fashion to $\succsim_{\mathcal{U}_{C_0}}$, we define $\succsim_{\mathcal{U}_{C_1}}$ so that we obtain $\psi_{C_1}(T_1, \succsim^{C_1}) \subset T_2 \equiv Top(T_1, \succsim_{\mathcal{U}_{C_1}})$. Because the equivalent classes with respect to the agents' preferences are only finite, this process ends at finite $K(\mathcal{X}, \succsim)$ steps. Thus, we construct our desired sequence, $\{T_k, C_k\}_{k=0}^{K(\mathcal{X}, \succsim)}$. $\square$

4.2.1 A Tiebreaking Rule

Finally, we show the existence of a tiebreaking rule.

Lemma 9. *There exists a $\rho \in \mathcal{R}$, such that, for each $X \in \mathscr{P}(\mathcal{X})$ and $\succsim \in \mathcal{R}^{\mathcal{N}}$, $\phi(X, \succsim) = Top(T_{K(X, \succsim)}, \rho)$.*

Proof. Let ρ be a binary relation on $\mathcal{X}$ such that, for each $x, y \in \mathcal{X}$, $x \rho y$ if and only if there exists a $\succsim \in \mathcal{R}^{\mathcal{N}}$ such that $\{x, y\} \subset T_{K(\mathcal{X}, \succsim)}$ and $x \in \phi(\mathcal{X}, \succsim)$. First, we show that ρ is a weak preference.

Let $x, y \in \mathcal{X}$. Then, we can find $\tilde{\succsim} \in \mathcal{R}^{\mathcal{N}}$ such that, for each $i \in \mathcal{N}$ and $z \in \mathcal{X} \backslash \{x, y\}$, $x \tilde{\sim}_i y$ and $x \tilde{\succ}_i z$. Then, $T_{K(\mathcal{X}, \tilde{\succsim})} = \{x, y\}$. Because $\phi(\mathcal{X}, \tilde{\succsim}) \subset T_{K(\mathcal{X}, \tilde{\succsim})}$ and $\phi(\mathcal{X}, \tilde{\succsim}) \neq \emptyset$, we have $x \in \phi(\mathcal{X}, \tilde{\succsim})$ or $y \in \phi(\mathcal{X}, \tilde{\succsim})$, that is, $x \rho y$ or $y \rho x$. Therefore, ρ is complete.

Next, we show transitivity. Let $x, y, z \in \mathcal{X}$ be such that $x \rho y$ and $y \rho z$. Then, there exists a $\succsim \in \mathcal{R}^{\mathcal{N}}$ such that $\{x, y\} \subset T_{K(\mathcal{X}, \succsim)}$ and $x \in \phi(\mathcal{X}, \succsim)$. In the same way, there also exists a $\succsim' \in \mathcal{R}^{\mathcal{N}}$ such that $\{y, z\} \subset T_{K(\mathcal{X}, \succsim')}$ and $y \in \phi(\mathcal{X}, \succsim')$. Then, by stability, $x \in \phi(\{x, y\}, \succsim)$ and $y \in \phi(\{y, z\}, \succsim')$. Now, let $\succsim^* \in \mathcal{R}^{\mathcal{N}}$ be such that for each $i \in \mathcal{N}$ and each $w \in \mathcal{X} \backslash \{x, y, z\}$, $x \sim^* y \sim_i^* z$ and $z \succ_i^* w$. Then, $T_{K(\mathcal{X}, \succsim^*)} = \{x, y, z\}$, which implies that

$\phi(\mathcal{X}, \succsim^*) \subset \{x, y, z\}$. By stability, we have that $\phi(\mathcal{X}, \succsim^*) = \phi(\{x, y, z\}, \succsim^*)$. Thus, it is enough to show that $x \in \phi(\{x, y, z\}, \succsim^*)$. Suppose that $x \notin \phi(\{x, y, z\}, \succsim^*)$. Because $\succsim^* =_{\{y,z\}} \succsim'$, we have that $y \in \phi(\{y, z\}, \succsim') = \phi(\{y, z\}, \succsim^*)$. Because $x \notin \phi(\{x, y, z\}, \succsim^*)$, stability implies that $y \in \phi(\{x, y, z\}, \succsim^*)$. Otherwise, $\phi(\{y, z\}, \succsim^*) = \{z\}$, which is a contradiction. By stability again, we have that $\phi(\{x, y\}, \succsim^*) = \phi(\{x, y, z\}, \succsim^*) \cap \{x, y\} = \{y\}$. However, note that $\succsim^* =_{\{x,y\}} \succsim$. Thus, we have that $\phi(\{x, y\}, \succsim) = \phi(\{x, y\}, \succsim^*)$. Because $x\rho y$, we have that $x \in \phi(\{x, y\}, \succsim) = \phi(\{x, y\}, \succsim^*)$, which is a contradiction.

It remains to show that, for each $X \in \mathscr{P}(\mathcal{X})$ and $\succsim \in \mathcal{R}^{\mathcal{N}}$, $\phi(X, \succsim) = Top(T_{K(X,\succsim)}, \rho)$. First, we show that, for each $X \in \mathscr{P}(\mathcal{X})$ and each $\succsim \in \mathcal{R}^{\mathcal{N}}$, $\phi(X, \succsim) \subset Top(T_{K(X,\succsim)}, \rho)$. Let $x \in \phi(X, \succsim)$ and $y \in Top(T_{K(X,\succsim)}, \rho)$. Then, $\{x, y\} \subset T_{K(X,\succsim)}$ and, by stability, $x \in \phi(\{x, y\}, \succsim)$. This implies that $x\rho y$. Because $y \in Top(T_{K(X,\succsim)}, \rho)$, we have $x \in Top(T_{K(X,\succsim)}, \rho)$.

Next, we show that, for each $X \in \mathscr{P}(\mathcal{X})$ and each $\succsim \in \mathcal{R}^{\mathcal{N}}$, $Top(T_{K(X,\succsim)}, \rho) \subset \phi(X, \succsim)$. Let $x \in Top(T_{K(X,\succsim)}, \rho)$ and $y \in \phi(X, \succsim)$. Because $x \ \rho \ y$, there exists $\succsim' \in \mathcal{R}^{\mathcal{N}}$ such that $\{x, y\} \subset T_{K(\mathcal{X},\succsim')}$ and $x \in \phi(\mathcal{X}, \succsim')$. Because $\{x, y\} \subset T_{K(X,\succsim)}$, we have that $\succsim =_{\{x,y\}} \succsim'$. Therefore, $x \in \phi(\{x, y\}, \succsim') = \phi(\{x, y\}, \succsim)$. By stability, we have that $x \in \phi(\mathcal{X}, \succsim)$. □

5 A Coherent Hierarchy of Non-principal Ultrafilters

In this section, we show that there exists a coherent hierarchy of ultrafilters that assigns non-principal ultrafilters to infinite sets. To do so, we define a consistent ultrafilter mapping that assigns a coherent ultrafilter to a collection of nonempty subsets of $\mathcal{N}$.

Definition. *Let $\mathcal{M} \subset \mathscr{P}(\mathcal{N})$. A correspondence $\mathscr{F} : \mathcal{M} \to \mathscr{P}(\mathcal{N})$ is a* consistent ultrafilter mapping *if (1) for each $N \in \mathcal{M}$, $\mathscr{F}(N)$ is an ultrafilter on N, and (2) for each $N, N' \in \mathcal{M}$, if $N \in \mathscr{F}(N')$, then*

$$\mathscr{F}(N') = \{Z \subset N' : Z \cap N \in \mathscr{F}(N)\}.$$

Suppose that $\mathscr{F}$ is a consistent ultrafilter mapping. Suppose $N \in \mathscr{F}(N')$ and $M \in \mathscr{F}(N')$. Then, $M \cap N \in \mathscr{F}(N)$. In a reverse way, if $M \subset N$ and $M \cap N \in \mathscr{F}(N)$, then $M \in \mathscr{F}(N')$. By replacing $\mathscr{F}(N)$ and $\mathscr{F}(N')$ with $\mathcal{U}_N$ and $\mathcal{U}_{N'}$ in the definition of coherence, we see that a consistent ultrafilter mapping, $\mathscr{F}$, can yield a family of coherent ultrafilters.

Proposition 2. *There exists a coherent hierarchy of ultrafilters that assigns a non-principal ultrafilter to each infinite set $N \in \mathscr{P}(\mathcal{N})$.*

Proof. Let $\mathfrak{F}$ be the class of pairs $(\mathcal{M}, \mathscr{F})$ satisfying: (i) $\mathcal{M} \subset 2^{\mathcal{N}}$ is a set of infinite sets and $\mathscr{F}$ is a consistent ultrafilter mapping whose domain is $\mathcal{M}$; (ii) if there exists a $Z \in \mathcal{M}$ and $Q \in \mathscr{F}(Z)$, then $Q \in \mathcal{M}$; (iii) if $Q \in \mathcal{M}$ and $Q \subset Z$, then $Z \in \mathcal{M}$; and (iv) $\mathscr{F}(Q)$ is a non-principal ultrafilter for every infinite set $Q \in \mathcal{M}$. Then, we can define a partial order $\leq$ on $\mathfrak{F}$

such that, for each $(\mathcal{M}, \mathscr{F})$ and $(\mathcal{M}', \mathscr{F}') \in \mathfrak{F}$, $(\mathcal{M}, \mathscr{F}) \leq (\mathcal{M}', \mathscr{F}')$ if $\mathcal{M} \subset \mathcal{M}'$ and, for each $M \in \mathcal{M}$, $\mathscr{F}(M) = \mathscr{F}'(M)$.

To begin with, we show that $\mathfrak{F}$ is not empty. Let $\mathscr{F}(\mathcal{N})$ be a non-principal ultrafilter on $\mathcal{N}$. Then, let $\mathcal{M} = \mathscr{F}(\mathcal{N})$ and for each $N \in \mathcal{M}$, $\mathscr{F}(N) = 2^N \cap \mathscr{F}(\mathcal{N})$. Then, $\mathscr{F}$ is consistent. In addition, condition (ii) is satisfied by construction of $\mathscr{F}$ for each $Q \in \mathcal{M}$. Moreover, by the second property of ultrafilters, condition (iii) is satisfied. Finally, (iv) is satisfied, because there exists a non-principal ultrafilter when its ground set is infinite.

We now show that there exists a maximal element in $\mathfrak{F}$, and consistency is also satisfied by the maximal element. Let $\mathscr{C}$ be a chain in $\mathfrak{F}$. Let $\bar{\mathcal{M}} \equiv \bigcup_{\mathcal{M} \in \mathscr{C}} \mathcal{M}$ and $\bar{\mathscr{F}} : \bar{\mathcal{M}} \to 2^{\mathcal{N}}$ be as follows:

$$\text{for each } M \in \bar{\mathcal{M}},\ \bar{\mathscr{F}}(M) \equiv \mathscr{F}(M),\ \text{ where } (\mathcal{M}, \mathscr{F}) \in \mathscr{C} \text{ and } M \in \mathcal{M}.$$

We show that $(\bar{\mathcal{M}}, \bar{\mathscr{F}}) \in \mathfrak{F}$.

Condition (i). Suppose that there exist $L, M \in \bar{\mathcal{M}}$ such that $L \subset M$ and $L \in \bar{\mathscr{F}}(M)$. Then, there exists $(\mathcal{M}, \mathscr{F}) \in \mathfrak{F}$ such that $L, M \in \mathcal{M}$. Note that $\mathscr{F}(L) = \bar{\mathscr{F}}(L)$ and $\mathscr{F}(M) = \bar{\mathscr{F}}(M)$. Thus, $\bar{\mathscr{F}}$ is a consistent ultrafilter mapping.

Condition (ii). Suppose that $Z \in \bar{\mathcal{M}}$ and $Q \in \bar{\mathscr{F}}(Z)$ for some Z. Then, there are $\mathcal{M}$ and $\mathscr{F}$ such that $Z \in \mathcal{M}$ and $Q \in \mathscr{F}(Z)$. Thus, $Q \in \bar{\mathcal{M}}$.

Condition (iii). Suppose that $Q \in \bar{\mathcal{M}}$ and $Q \subset Z$. Then, there is an $\mathcal{M}$ such that $Q \in \mathcal{M}$. Thus, $Z \in \mathcal{M}$. Hence, $Z \in \bar{\mathcal{M}}$.

Condition (iv). When each $\mathscr{F}(Q)$ is non-principal for each infinite set Q, clearly $\bar{\mathscr{F}}(Q)$ is a non-principal ultrafilter.

Finally, we conclude that $(\bar{\mathcal{M}}, \bar{\mathscr{F}}) \in \mathfrak{F}$ and it is an upper bound of $\mathscr{C}$. Therefore, by Zorn's lemma, there exists a maximal element in $\mathfrak{F}$. Let $(\mathcal{M}^*, \mathscr{F}^*) \in \mathfrak{F}$ be a maximal element. To show that it is a coherent hierarchy of ultrafilters, it is enough to show that $\mathcal{M}^* = 2^{\mathcal{N}}$. Suppose that $\mathcal{M}^* \neq 2^{\mathcal{N}}$. Then, there exists $Q \in 2^{\mathcal{N}} \backslash \mathcal{M}^*$. Because $\mathscr{F}$'s domain is assumed to be a set of infinite sets, we can assume that Q is infinite. Because condition (ii) of $\mathfrak{F}$ is satisfied by $\mathcal{M}^*$, there exists no $Z \in \mathcal{M}^*$ such that $Q \in \mathscr{F}^*(Z)$.

Let $\mathscr{U}_Q^*$ denote the set of non-principal ultrafilters on Q. Choose $U \in \mathscr{U}_Q^*$ arbitrarily. Let χ denote the set of supersets of elements in U that are not yet assigned any ultrafilters. Define $\hat{\mathscr{F}} : \mathcal{M}^* \cup \chi \to 2^{\mathcal{N}}$ as follows. Let $\hat{\mathscr{F}}(Q) = U$ and

$$\begin{aligned} &\text{for each } M \in \mathcal{M}^*, \quad \hat{\mathscr{F}}(M) = \mathscr{F}^*(M), \quad \text{and} \\ &\text{for each } M \in \chi, \quad \hat{\mathscr{F}}(M) = \{Z \subset M : Z \cap Q \in \hat{\mathscr{F}}(Q)\}. \end{aligned}$$

We show that $\hat{\mathscr{F}}$ is a consistent ultrafilter mapping. Take $L, M \in \mathcal{M}^* \cup \chi$ and $L \in \hat{\mathscr{F}}(M)$. Then, $L \subset M$. By condition (ii), if $M \in \mathcal{M}^*$, then $L \in \mathcal{M}^*$. By condition (iii), if $L \in \mathcal{M}^*$, then

$M \in \mathcal{M}^*$. Thus, there are two possible cases: (Case I) $L, M \in \mathcal{M}^*$; (Case II) $L, M \in \chi$. We need to show $\hat{\mathscr{F}}(M) = \{N \subset M : N \cap L \in \hat{\mathscr{F}}(L)\}$.

(Case I) $L, M \in \mathcal{M}^*$. First, let $S \in \hat{\mathscr{F}}(M)$ and $N \subset M$. Then, $S \in \mathcal{M}^*$ by condition (ii). Then, because $L \in \mathscr{F}^*(M)$ and $S \in \mathscr{F}^*(M)$, $L \cap S \in \mathscr{F}^*(M)$ by the second property of ultrafilters. Because $L \subset M$, and $\mathscr{F}^*$ is consistent, $S \cap L \in \mathscr{F}^*(L)$. Because $L \in \mathcal{M}^*$, $S \cap L \in \hat{\mathscr{F}}(L)$. Thus, we conclude

$$S \in \{Z \subseteq M : Z \cap L \in \hat{\mathscr{F}}(L)\}.$$

Second, suppose that $S \subset M$ and $S \cap L \in \hat{\mathscr{F}}(L)$. Then, because $L \in \mathcal{M}^*$, $S \cap L \in \mathscr{F}^*(L)$. Because $\mathscr{F}^*$ is consistent, we obtain $S \in \mathscr{F}^*(M)$, which implies $S \in \hat{\mathscr{F}}(M)$.

(Case II) $L, M \in \chi$. First, suppose that $S \in \hat{\mathscr{F}}(M)$. Then, $S \subset M$ and $S \cap Q \in \hat{\mathscr{F}}(Q)$. Because $L \in \hat{\mathscr{F}}(M)$, $L \subset M$ and $L \cap Q \in \hat{\mathscr{F}}(Q)$. Then, because the intersection of two sets in an ultrafilter is also in an ultrafilter, $(S \cap L) \cap Q \in \hat{\mathscr{F}}(Q)$. Because $S \cap L \subset L$, we obtain $S \cap L \in \hat{\mathscr{F}}(L)$. Thus, we conclude

$$S \in \{Z \subset M : Z \cap L \in \hat{\mathscr{F}}(L)\}.$$

Second, suppose that $S \subset M$ and $S \cap L \in \hat{\mathscr{F}}(L)$. Because $L \in \hat{\mathscr{F}}(M)$, $L \subset M$ and $L \cap Q \in \hat{\mathscr{F}}(Q)$. Also, because $N \cap L \in \hat{\mathscr{F}}(L)$, $(S \cap L) \cap Q \in \hat{\mathscr{F}}(Q)$. Because $L \in \chi$, $Q \subset L$ and so $(S \cap L) \cap Q = S \cap Q$. Hence, we conclude that $S \cap Q \in \hat{\mathscr{F}}(Q)$. Then, because $S \subset M$, $S \in \hat{\mathscr{F}}(M)$.

So, we conclude that $\hat{\mathscr{F}}$ is a consistent ultrafilter mapping and, thus, condition (i) is satisfied. Also, condition (iv) is satisfied by $\hat{\mathscr{F}}$. Thus, it remains to show that conditions (ii) and (iii) are satisfied. Suppose that $Z \in \mathcal{M}^* \cup \chi$ and $S \in \hat{\mathscr{F}}(Z)$. If $Z \in \mathcal{M}^*$, then obviously $Q \in \mathcal{M}^*$ by condition (ii) of $\mathscr{F}^*$. If $Z \in \chi$, then $S \cap Q \in \hat{\mathscr{F}}(Q)$. If S is assigned any ultrafilter, $S \in \mathcal{M}^*$. Otherwise, as χ includes all the supersets of elements in $U = \hat{\mathscr{F}}(Q)$ that are not yet assigned any ultrafilter, $S \in \chi$. Thus, condition (ii) is satisfied.

Next, suppose that $S \in \mathcal{M}^* \cup \chi$ and $S \subset Z$. If $S \in \mathcal{M}^*$, then there exists S' with $S \in \mathscr{F}^*(N')$. If $Z \subset S'$, then $Z \in \mathscr{F}^*(S')$ by the third property of ultrafilters and so $Z \in \mathcal{M}^*$. Otherwise, because $S \subset S' \cap Z$, $S' \cap Z \in \mathscr{F}^*(S')$ by the third property of ultrafilters. Then, condition (ii) implies that $S' \cap Z \in \mathcal{M}^*$ and because $S' \cap Z \subset Z$, condition (iii) indicates that $Z \in \mathcal{M}^*$. If $S \in \chi$, then obviously $Z \in \chi$. Hence, we obtain $Z \in \mathcal{M}^* \cup \chi$. Therefore, we conclude that the four conditions are satisfied by $(\mathcal{M}^* \cup \chi, \hat{\mathscr{F}})$ and $(\mathcal{M}^* \cup \chi, \hat{\mathscr{F}}) \in \mathfrak{F}$. By construction, we have $(\mathcal{M}^*, \mathscr{F}^*) < (\mathcal{M}^* \cup \chi, \hat{\mathscr{F}})$. However, this is a contradiction with the maximality of $(\mathcal{M}^*, \mathscr{F}^*)$. $\square$

Theorem 2. *There exists a hierarchy of ultrafilters such that (i) all the ultrafilters satisfy coherence and (ii) all the infinite sets are assigned non-principal ultrafilters.*

Proof. By Proposition 2, there exists a consistent ultrafilter mapping $\mathscr{F}$ that satisfies the second condition required in this theorem. Now, suppose that $N_0, N \subset \mathcal{N}$, $N_0 \subset N$ is a finite set and N is an infinite set. Then, $\mathscr{F}(N)$ only includes infinite sets and by the fourth property of ultrafilters, $N_0 \notin \mathscr{F}(N')$. Thus, we can see that coherence is not required between $\mathscr{F}(N_0)$ and $\mathscr{F}(N)$.

Now, by Zorn's lemma, there is a complete strict ordering for $\mathcal{N}$. There is a minimal element in this ordering for any finite set $N \subset \mathcal{N}$, which we denote by $\bar{m}_N$. For any finite set $N \subset \mathcal{N}$, define $\mathscr{F}(N)$ by

$$\mathscr{F}(N) = \{U \subset N : \bar{m}_N \in U\}.$$

We show that $\mathscr{F}$ satisfies consistency. Take two finite sets N_0 and N_1 with $N_0 \subset N_1 \subset \mathcal{N}$. Suppose that $N_0 \in \mathscr{F}(N_1)$. Then, $\bar{m}_{N_1} \in N_0$. Because $N_0 \subset N_1$, $\bar{m}_{N_1} = \bar{m}_{N_0}$. If $M \in \mathscr{F}(N_1)$, then $\bar{m}_{N_0} \in M$ and, thus, $\bar{m}_{N_0} \in M \cap N_0$, which implies $M \cap N_0 \in \mathscr{F}(N_0)$. On the other hand, take $M \subset N_1$ and let $M \cap N_0 \in \mathscr{F}(N_0)$. Then, $\bar{m}_{N_0} \in M \cap N_0$. Thus, because $\bar{m}_{N_1} = \bar{m}_{N_0}$, $\bar{m}_{N_1} \in M$. Thus, $M \in \mathscr{F}(N_1)$.

Therefore, we conclude that there exists a hierarchy of ultrafilters for which condition (i) is satisfied for any finite sets. This completes the proof. $\square$

6 Concluding Remarks

To conclude, we discuss the Gibbard–Satterthwaite Theorem with infinitely many agents. In this example, we restrict our attention to strict preference profiles[2]. Let $\mathcal{P}$ be the entire space of strict preferences over $\mathcal{X}$. Let $\succ_i \in \mathcal{P}$ be agent i's strict preference. An SCF is a mapping from $\mathcal{P}^{\mathcal{N}}$ to $\mathcal{X}$. By an abuse of notation, let $(\succ_i', \succ_{-i})$ denote a preference profile such that only $\succ_i$ is replaced by $\succ_i'$ while holding everything else the same as in $\succ$. Now, we introduce two properties about SCFs.

Definition (Truthfully Implementable in Dominant Strategies)**.** *An SCF $f : \mathcal{P}^{\mathcal{N}} \to \mathcal{X}$ is* truthfully implementable in dominant strategies *if, for each $i \in \mathcal{N}$, each $\succ_i, \succ_i' \in \mathcal{P}$, and each $\succ_{-i} \in \mathcal{P}^{\mathcal{N}-1}$, $f(\succ_i, \succ_{-i}) \succsim_i f(\succ_i', \succ_{-i})$.*

Note that $f(\succ_i, \succ_{-i}) \sim_i f(\succ_i', \succ_{-i})$ if and only if $f(\succ_i, \succ_{-i}) = f(\succ_i', \succ_{-i})$. This indicates that it is always best for the agents to report their true preferences to the social planner.

Definition (Dictatorial)**.** *An SCF $f : \mathcal{P}^{\mathcal{N}} \to \mathcal{X}$ is dictatorial if there exists $i^* \in \mathcal{N}$ such that, for each $\succ \in \mathcal{P}^{\mathcal{N}}$ and each $x \in \mathcal{X}$, $f(\succ) \succsim_{i^*} x$.*

[2]In the case of weak preference profiles, the Gibbard–Satterthwaite Theorem does not hold even in the case of finitely many agents, as described in the Introduction. Man and Takayama (2013) also provide an example of a SCF that does not satisfy a dictatorship when it does not satisfy the set of axioms for the Gibbard–Satterthwaite Theorem.

The Gibbard–Satterthwaite Theorem states the following.

Theorem (Gibbard–Satterthwaite)**.** *Let $f : \mathcal{P}^{\mathcal{N}} \to \mathcal{X}$. Suppose that (i) $\mathcal{N}$ is finite; (ii) for every $x \in \mathcal{X}$, there is a $\succ \in \mathcal{P}^{\mathcal{N}}$ with $f(\succ) = x$; and (iii) f is truthfully implementable in dominant strategies. Then f is dictatorial.*

We say that $\succ$ is a *modification* of $\succ'$ if $\succ_i \neq \succ'_i$ for only finitely many $i \in N$. We say that $\succ$ and $\succ'$ are *close* if there exists a finite sequence $\{\succ_m\}_{m=0}^{M}$ such that $\succ_0 = \succ$, $\succ_M = \succ'$ and for each $m \in \{0, \cdots, M-1\}$, $\succ_{m+1}$ is a modification of $\succ_m$.

We denote a set of preference profiles that are close to $\succ$ by $A(\succ)$. Then, $A(\succ)$ is an equivalence class of $\succ$. The axiom of choice implies that there is an SCF f with the following properties:

(I) f satisfies unanimity;

(II) for every $\succ \in \mathcal{P}^{\mathcal{N}}$ and $\succ' \in A(\succ)$, $f(\succ) = f(\succ')$.

Then, f is truthfully implementable in dominant strategies, because $(\succ'_i, \succ_{-i}) \in A(\succ)$ and

$$f(\succ) \sim_i f(\succ'_i, \succ_{-i})$$

for all $\succ \in \mathcal{P}^{\mathcal{N}}$, all $i \in \mathcal{N}$ and all $\succ'_i \in \mathcal{P}$.

However, because of (II) in the definition of f, a dictatorship does not hold with f. Therefore, we can say that the Gibbard–Satterthwaite Theorem does not hold when there are infinitely many agents. Note that, when there are only finitely many agents, there is only one set of preference profiles that is close to $\succ$. Thus, unanimity makes (II) impossible. Therefore, a dictatorship can be proved as in Geanakoplos (2005).

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