

A NOTE ON THE EQUILIBRIUM EXISTENCE PROBLEM IN DISCONTINUOUS GAMES*

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Abstract

In this note we prove an equilibrium existence theorem for games with discontinuous payoffs and convex and compact strategy spaces. It generalizes the classical result of Reny (1999) [*Econometrica* **67**, p. 1029-1056], as well as the recent paper of McLennan, Monteiro, and Tourky (2011) [*Econometrica* **79**, p. 1643-1664]. Our condition is simple and easy to verify. Importantly, an example of a spatial location model shows that our conditions allow for economically meaningful payoff discontinuities, that are not covered by other conditions in the literature.

KEYWORDS: Nash equilibrium, discontinuous payoffs, better reply security, location games

1 Introduction

Our purpose is to study the question of existence of Nash equilibrium in games in which players' strategy spaces are convex and compact, and payoffs are quasiconcave, but discontinuous. A classical reference for the theory of equilibrium existence in games with discontinuous payoffs is Reny (1999). Because Reny's hypotheses are sufficiently general and at the same time so easy to verify, his results have been applied to a variety of significant economic problems. Indeed, Reny (1999) shows that very general multi-unit pay-your-bid auctions has pure-strategy Nash equilibria. Jackson and Swinkels (2005) use better-reply security to establish the existence of equilibrium in a large class of private value auctions, including double auctions. Monteiro and Page Jr (2008) show that the mixed extension of a game in which each seller competes for a buyer of unknown type by offering a catalog of products and pricing is better reply secure, and thus has a Nash equilibrium. Duggan (2007) derives a condition for a class of zero-sum games that include spatial models of elections in which the main theorem in Reny (1999) can be applied to show existence of equilibrium in mixed strategies.

Reny's result have been extended in several directions. Bich (2009) went in the direction of relaxing the convexity assumption on the payoffs. Bagh and Jofre (2006), Carmona (2009), Monteiro and Page Jr (2007), and Prokopovych (2011) relax conditions studied by Reny (1999) that imply better-reply security. For a survey of the theory of existence of equilibrium in discontinuous games, see Carmona (2011).

In a recent generalization of Reny's results, McLennan, Monteiro, and Tourky (2011) weaken the condition of better-reply security by allowing agents to use several securing strategies. Using this,

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they are able to prove existence of pure equilibria in a class of finite games, in which agents' choice sets are nonempty subsets of a given set.

Our result is more general than Reny (1999) and McLennan, Monteiro, and Tourky (2011) and allows for economically meaningful payoff discontinuities not covered by either work. We relax Reny's requirement for a single securing strategy and McLennan, Monteiro, and Tourky (2011)'s multiple securing strategies by allowing for the securing strategies to vary upper hemicontinuously in response to changes in other players strategies.

As for the technique of proof, our approach is based on the essential intuition behind the concept of better reply security that securing (or dominant) strategies need to be robust to the other players' small deviations. Informally, we use these securing strategies to construct selections of the strict upper contour set of the players' preferences, and shows that, if these selections are sufficiently well behaved, then one can expect the game to have an equilibrium in pure strategies. The simplicity of the approach allows us to take some steps further. First, and in line with the better reply security logic, we can allow different players to be activated locally. That is, we can weaken the requirement that the securing strategy is contained in the strict contour set of all players to only the player being activated locally. Second, because we will eventually use a fixed point theorem, the selections don't need to be continuous, but they can be any mapping that has the fixed point property. In particular, it can be a sufficiently well behaved correspondence, which then allows the securing strategies to vary in semi-continuous manners.

The note is divided in 4 parts. Section 2 is the heart of this note. We give sufficient conditions on the payoffs for existence of equilibrium in games with convex and compact strategy spaces. We also provide generalizations of the local conditions known in the literature. In Section 3, a few simple examples in the unit square illustrate what kind of discontinuities our results allow. Section 4 presents a location model that shows the practical use of our approach in applied work. It should be emphasized that this example shows how the construction of the correspondence implicit in our main condition, multi-player security, is easy and informative of the nature of the strategic interaction in a game. Section 5 provides proofs of the results.

2 Existence of Nash equilibria

Let N be the finite set of players. Each player $i \in N$ has a pure strategy set X_i , which is a nonempty, convex and compact subset of a Hausdorff locally convex topological vector space, and a payoff function $u_i : X \rightarrow \mathbb{R}$. Product sets are endowed with the product topology and we use X to denote $\times_{i \in N} X_i$, and X_{-i} to denote $\times_{j \neq i} X_j$, with typical element x_{-i} .

Denote by $G = (X_i, u_i)_{i \in N}$ the *normal form game*. A **pure strategy Nash equilibrium** of G is a profile $x^* \in X$ such that $u_i(x^*) \geq u_i(x_i, x_{-i}^*)$ for all $x_i \in X_i$, and all $i \in N$.

A correspondence $\varphi : Y \rightarrow Z$ between two topological spaces Y and Z is said to be *upper hemicontinuous* at the point x if for any open neighborhood V of $\varphi(x)$ there exists a neighborhood U of x such that $\varphi(x)$ is a subset of V for all x in U . For any set $K \subseteq X$, the *convex hull* of K is denoted $\text{co } K$.

The following condition, continuous security, generalizes the *multiple security* in McLennan, Monteiro, and Tourky (2011).¹

Definition 2.1. A game $G = (X_i, u_i)_{i \in N}$ is **continuously secure** at $x \in X$ if there is $\alpha \in \mathbb{R}^N$, an open neighborhood V of x and an upper hemicontinuous correspondence $\varphi : V \rightarrow X$ such that

- (a) $\varphi_i(y) \subseteq B_i(\alpha_i, y)$ for every $i \in N$ and every $y \in V$
- (b) for each $y \in V$ there exists i with $y_i \notin \text{co } B_i(\alpha_i, y)$,

¹McLennan, Monteiro, and Tourky (2011) have an additional term in their definition, the restriction operator $\mathcal{R} : X \rightarrow X_i$. The extension of our definitions and results to include such operator is obvious.

where $B_i(\alpha_i, x) = \{y_i \in X_i : u_i(y_i, x_{-i}) \geq \alpha_i\}$.

A game is **continuously secure** if it is continuously secure at each x that is not an equilibrium.

We can now state our main existence theorem. The proof is in Section 5.

Theorem 2.2. *A continuously secure game $G = (X_i, u_i)_{i \in N}$ has a pure-strategy Nash equilibrium.*

The difference between continuous security and multiple security in McLennan, Monteiro, and Tourky (2011) is simple. Multiple security implies that there exist a finite number of (constant) robust profitable deviations at a neighborhood of a point that is not an equilibrium. Continuous security is more permissive, as it allows for these robust profitable deviations to vary continuously in that neighborhood, allowing thus for an infinite number of profitable deviations as long as they satisfy the continuity condition.

Likewise, it is possible to generalize the condition *better-reply security* in Reny (1999). That is exactly what the following definition does: whenever better-reply security requires constant profitable deviations in a neighborhood of any point that is not an equilibrium, we allow for general mappings, as long as these profitable deviations are sufficiently well-behaved in small neighborhoods.

If φ is a correspondence, let $Gr(\varphi)$ denote its graph. Let $\Gamma = Gr(u)$ be the graph of the game's vector payoff function, and let $\bar{\Gamma}$ be its closure.

Definition 2.3. A game $G = (X_i, u_i)_{i \in N}$ is called **generalized better reply secure** if whenever $(x, u) \in \bar{\Gamma}$ and x is not an equilibrium, there exists a player i and a triple $\{\varphi_i, V_x, \alpha_i\}$ where V_x is an open neighborhood of x , $\varphi_i : V_x \rightarrow X_i$ is an upper hemicontinuous correspondence and $\alpha_i > u_i$, such that $u_i(z_i, y_{-i}) \geq \alpha_i$ for every $(y, z_i) \in Gr(\varphi_i)$.

The combination of *better reply security* and own-strategy quasiconcavity, used in Reny (1999), implies continuous security, as the following proposition shows.

Proposition 2.4. *A generalized better reply secure and own-strategy quasiconcave game $G = (X_i, u_i)_{i \in N}$ is continuously secure. In particular, such a game has an equilibrium.*

Proof. Suppose the game G is generalized better reply secure and own-strategy quasiconcave, and take any x that is not an equilibrium. Let J be the set of players that can secure some payoff limit u_i at x , that is $(x, u) \in \bar{\Gamma}$. There exist $\{\phi_i, V_x, \alpha_i\}$ for each $i \in J$. Because J is finite the neighborhood V_x can be made small enough such that for each $y \in V_x$, there is some player $i \in J$ for which $u_i(y) < \alpha_i$.

Define the correspondence $\varphi_x : V_x \rightarrow X$ by the product $\varphi_x = \times_i \varphi_{x,i}$ where

$$\varphi_{x,i}(y) = \begin{cases} \phi_i(y) & \text{if } i \in J, \\ X_i & \text{otherwise.} \end{cases}$$

Moreover, let $\alpha_x = \times_i \alpha_{x,i}$ where

$$\alpha_{x,i} = \begin{cases} \alpha_i & \text{if } i \in J, \\ m_i & \text{otherwise,} \end{cases}$$

with m_i being the lower bound on the payoff of player i .

By construction, the pair $\{\varphi_x, \alpha_x\}$ satisfies part (a) of continuous security at x for the neighborhood V_x . Also, since for each $y \in V_x$, there is some player $i \in J$ for which $u_i(y) < \alpha_i$, quasiconcavity of the payoffs implies that part (b) of continuous security is satisfied. $\square$

In the same fashion, it is possible to strengthen the following results: Baye, Tian, and Zhou (1993), Bagh and Jofre (2006), Carmona (2009), and Tian and Zhou (1995).

3 Two-player examples

We now present two simple examples in which players choose real numbers between zero and one that illustrate the novel classes of discontinuities our results allow.

Example 3.1. There are two players with strategy sets $X_i = [0, 1]$, $i = 1, 2$. For some integer $m \geq 3$, let $\varepsilon \in (0, \frac{1}{m})$. The payoff functions are given by

$$u_1(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 = x_2, \\ 0 & \text{otherwise} \end{cases}$$

and

$$u_2(x_1, x_2) = \begin{cases} 1 & \text{if } x_2 = 1, \\ k & \text{if } x_1 + \varepsilon < \frac{k}{m} = x_2 \text{ for } k = 1, 2, \dots, m-1, \\ 0 & \text{otherwise.} \end{cases}$$

We can think of it as a game of product quality choices of two firms: firm 1 wants to match the choice of firm 2; firm 2 has some quality levels (e.g. $\frac{2}{4}$ and $\frac{3}{4}$ when $m = 4$) where it makes a good profit in case it beats firm 1; and also has a somewhat safe quality level $x_2 = 1$. The profile $x_1 = x_2 = 1$ is the Nash equilibrium.

Player 2's strict upper contour set is not lower hemicontinuous, and we have $x_2 \in \text{co}\{y_2 \in X_2 : u_2(x_1, y_2) > u_2(x)\}$ for some profiles x (for instance, $x = (0, \frac{1}{m-1})$). Therefore, this game is not own-strategy quasiconcave and the result of Reny (1999) cannot be applied. However, this game is continuously secure, as we show in what follows.

Let $f: X \rightarrow X$ be given by $f(x) = (x_2, 1)$, and put $f_x = f|_{V_x}$ for any $x \neq (1, 1)$ and V_x an open neighborhood not including $(1, 1)$. Then we have continuous security: for every profile outside of the diagonal, pick a neighborhood that does not meet the diagonal; for that neighborhood, player 1 is the player for which part (b) of continuous security is satisfied. For any profile x in the diagonal, pick an open ball with radius $\frac{\varepsilon}{2}$ centered at x as V_x . Even though we have profiles with $x_2 \in \text{co}\{y_2 \in X_2 : u_2(x_1, y_2) > u_2(x)\}$, it is simple to verify that part (b) of the condition is satisfied for player 2. $\square$

Example 3.2. There are two players with strategy sets $X_i = [0, 1]$, $i = 1, 2$. Let $k_i: X \rightarrow \mathbb{R}$, $l_i: X \rightarrow \mathbb{R}$ and $m_i: X \rightarrow \mathbb{R}$ be functions satisfying: (i) $k_i(s) > l_i(s) > m_i(s)$ for every $s \in [0, 1]$, and (ii) k_i is non-decreasing and lower semicontinuous, $i = 1, 2$. Player i 's payoffs are given by

$$u_i(x_i, x_j) = \begin{cases} k_i(x_i) & \text{if } x_i < x_j, \\ l_i(x_i) & \text{if } x_i = x_j, \\ m_i(x_j) & \text{if } x_i > x_j. \end{cases}$$

Assume that all functions involved are bounded. For $i, j = 1, 2$, $i \neq j$, because l_i is lower semicontinuous, there exists a continuous function $\delta_i: X_j \rightarrow \mathbb{R}_+$ with $x_j - \delta_i(x_j) > 0$ for all $x_j \in (0, 1]$, $\delta_i(x_j) = 0$ only if $x_j = 0$, and $l_i(x_j - \delta_i(x_j)) \geq \phi_i(x_j)$. Let $\varphi(x) = (x_2 - \delta_1(x_2), x_1 - \delta_2(x_1))$.² For each non equilibrium x , let V_x be an open neighborhood of x not containing an equilibrium, and set $\varphi_x = \varphi|_{V_x}$. Then continuous security is verified: player 1 satisfies part (b) of continuous security below the diagonal, player 2 satisfies part (b) of continuous security above the diagonal, and either player satisfies part (b) of continuous security for profiles at the diagonal.

This example illustrates the idea described in the introduction that continuous security allows for different players to be locally activated in a given open neighborhood. In this example, the neighborhoods V_x are being broken up into two halves: above and below the diagonal. $\square$

²Notice that all functions involved are allowed to be discontinuous, contrary to the case considered by Reny (1999). Also note that the case considered by Bagh and Jofre (2006), with $k_i(s) = 10$, $m_i(s) = -10$ and $l_i(s) \in \{0, 1\}$ for all $s \in [0, 1]$, is a special case.

4 Finite location model with a continuous choice variable

The model in this section is an extension of the finite location model in McLennan, Monteiro, and Tourky (2011). In this extension, the players' decision problem involves two simultaneous choices. The first choice is analogous to the problem the players face in the location example in McLennan, Monteiro, and Tourky (2011), that is, each agent chooses a location to visit from a predefined set of finite locations, and agents' payoffs are affected by the number of desirable versus undesirable players choosing to visit the same location. The second choice is a continuous variable (a real number between zero and one), that can be understood as price, quantity or effort. The specific example that we have in mind is an oligopoly setting in which firms choose from a finite number of technologies and prices. As in McLennan, Monteiro, and Tourky (2011) each player groups the other players into desirable and undesirable players, and avoid locations with too many undesirable players.

There is a finite set $N = \{1, 2, \dots, n\}$ of players, who choose two strategic variables. The first dimension of the players' problem is the spatial location. There is a common finite set S of strategic (or geographical) centers that, for simplicity, are affinely independent points distributed on $\mathbb{R}^{|S|-1}$. Player i chooses a point x_i in $X_i = \text{co } S$, and given this choice, they have the right to access (or visit) the center (or centers) close to x_i . For each player i , let $D_i \subset N \setminus \{i\}$ and $U_i \subset N \setminus \{i\}$ be the sets of players that i considers desirable and undesirable, respectively. In the oligopoly example, D_i could be a set of firms producing goods that are complementary to i , and U_i could be a set of competitors of i . To make the problem interesting, assume that D_i and U_i are non-empty.

The idea is that players prefer locations visited by desirable players and want to avoid locations with too many undesirable players. If player i chooses $x_i \in X$, the locations he has access to are given by the mapping $\phi: X_i \rightarrow 2^S$, where

$$\phi(x_i) = \{s_k \in S: x_i = \sum_k c_k s_k \text{ for some } c_k \in (0, 1]\}.$$

Figure 1 shows an example of the available choices of an agent in the simple case when $S = \{s_1, s_2, s_3\}$.

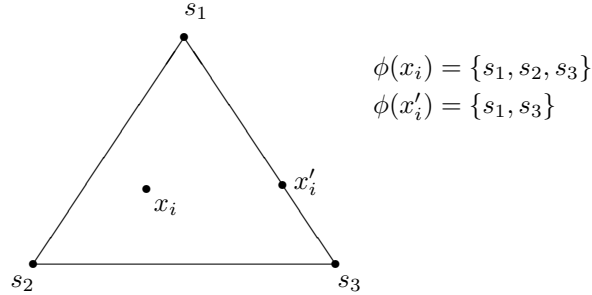


Figure 1: Two choices of agent i : x_i and x'_i

For a given profile of locations x_{-i} chosen by i 's opponents, and each $s \in S$, set

$$d_i(s, x_{-i}) = |\{j \in D_i: \phi(x_j) = \{s\}\}|, \text{ and}$$

$$u_i(s, x_{-i}) = |\{j \in U_i: s \in \phi(x_j)\}|.$$

As in McLennan, Monteiro, and Tourky (2011), if player i access center s , given the other players choice of x_{-i} , he gets a bonus (or a penalty) according to the function $y_i: S \times X_{-i} \rightarrow \mathbb{R}$, given by

$$y_i(s, x_{-i}) = \begin{cases} 0 & \text{if } u_i(s, x_{-i}) > d_i(s, x_{-i}), \\ 1 & \text{otherwise.} \end{cases}$$

Define $g_i(x) = \min_{\{s: s \in \phi(x_i)\}} y_i(s, x_{-i})$.

The second dimension of interest to each firm is a variable chosen from the interval $[0, P]$. Consistent with the oligopoly example, assume that p_i is the price chosen by firm i , in which case, given the price vector p , firm i 's profits are given by the function $f_i: [0, P]^N \rightarrow \mathbb{R}$

$$f_i(p) = \begin{cases} \frac{M}{k} p_i & \text{if } p_i \in \arg \min\{p_1, \dots, p_n\} \\ 0 & \text{otherwise,} \end{cases}$$

where k is the number of firms tied with the lowest price and M is the size of the market.

Then, the payoff of player i is given by

$$v_i(x, p) = g_i(x) f_i(p).$$

The location game is given by $(S, X, P, (v_i)_{i=1}^N)$. To see that it satisfies multi-player security, let

$$C_i = \{x_{-i} : \max_{s \in S} y_i(s, x_{-i}) = 1\}.$$

The idea is that the set C_i identifies a subset of the space of other players' spatial strategies where player i can potentially have a profitable deviation. To see that it is a closed set, take a sequence $x_{-i}^n \rightarrow x_{-i}$ with $x_{-i}^n \in C_i$ for every n . By finiteness of S , there is an s such that $d_i(s, x_{-i}^n) \geq u_i(s, x_{-i}^n)$ for all n in a subsequence. Furthermore, for n large enough, $\phi(x_j) \subset \phi(x_j^n)$ for all $j \neq i$, thus $d_i(s, x_{-i}^n) \leq d_i(s, x_{-i})$ and $u_i(s, x_{-i}^n) \geq u_i(s, x_{-i})$. Therefore, $y_i(s, x_{-i}) = 1$ and $x_{-i} \in C_i$.

Define a correspondence $\varphi_i: C_i \rightarrow X$ as

$$\varphi_i(x_{-i}) = \{x_i : g_i(x_i, x_{-i}) = 1\}$$

and note that it is non-empty, convex and closed-valued, and has a closed graph. In fact, take $x_{-i}^n \rightarrow x_{-i}$ in C_i and $x_i^n \in \varphi_i(x_{-i}^n)$ with $x_i^n \rightarrow x_i$. Then for $s \in \phi(x_i)$, we must have $s \in \phi(x_i^n)$ as well for n large enough, hence $y_i(s, x_{-i}^n) = 1$. Again, by the same argument, $y_i(s, x_{-i}) = 1$.

Using Theorem 2.4 in Tan and Wu (2002), there is an upper hemicontinuous extension of φ_i to $\times_{j \neq i} \Delta(S_j)$. From this point on, φ_i will refer to such extension.

Additionally, define the function $\psi_i: [0, P]^N \rightarrow [0, P]$ by

$$\psi_i(p) = \frac{3}{4} \min_j \{p_j\}.$$

For any profile (x, p) that is not an equilibrium, there exists a neighborhood V of (x, p) small enough such that

$$\varphi_i(x_{-i}) \times \psi_i(p) \subset \{(x'_i, p'_i) : v_i(x'_i, x_{-i}, p'_i, p_{-i}) > \tilde{v}_i\}$$

for some i , where $\tilde{v}_i = \sup_{(\tilde{x}, \tilde{p}) \in V} v_i(\tilde{x}, \tilde{p})$. Therefore, part (a) of continuous security is verified, whereas part (b) follows from the quasiconcavity of the payoffs.

Notice that the example in McLennan, Monteiro, and Tourky (2011) is a special case of this model, for f_i constant and equal to one. Thus, as in McLennan, Monteiro, and Tourky (2011), this extension does not satisfy the other conditions in the literature, including better reply security and Tian and Zhou (1995)'s condition (diagonal transfer continuity). Moreover, it is easy to see that our conditions would allow for a richer family of payoffs. For example, if the mappings f_i corresponded to payoffs of a coordination game, one can show that continuous security is still satisfied, whereas the multiple securing strategies needed in McLennan, Monteiro, and Tourky (2011) fail to exist.

5 Proof of Theorem 2.2

The following proof is similar to the argument of the main theorem of McLennan, Monteiro, and Tourky (2011). Notice, however, that the Hausdorff and local convexity assumptions are only used at the end of the proof, where the fixed point argument requires them due to the possibility of having an infinite number of profitable deviations in arbitrarily small neighborhoods. The reason why McLennan, Monteiro, and Tourky are able to avoid these assumptions is because their conditions imply the existence of only a finite number of deviations (or a single deviation in the case of Reny (1999)) in small neighborhoods of the strategy space.

Proof of Theorem 2.2. Suppose that the game G is continuously secure and has no equilibrium. For each $x \in X$, there exists $\{\alpha_x, V_x, \phi_x\}$ satisfying parts (a) and (b) of the continuous security condition. Because X is regular, each x has a closed neighborhood $\tilde{V}_x \subset V_x$. Moreover, the cover $\{\tilde{V}_x\}_{x \in X}$ has a finite subcover $\{\tilde{V}_k\}_{k=1, \dots, K}$.

Define the function $\beta: X \rightarrow \mathbb{R}$ by the product $\beta = \times_{i \in N} \beta_i$ where

$$\beta_i(x) = \max_{x \in \tilde{V}_k} \alpha_{k,i},$$

and notice that β is upper semicontinuous and finite-valued. Therefore, for each $x \in X$, there is a neighborhood U_x such that $\beta(y) \leq \beta(x)$ for all $y \in U_x$ and $U_x \subset \cap_{\{k: x \in \tilde{V}_k\}} \tilde{V}_k$. Define the correspondence $\varphi_x: U_x \rightarrow X$ by the product $\varphi_x = \times_{i \in N} \varphi_{x,i}$ where

$$\varphi_{x,i}(y) = \phi_{k,i}(y),$$

for some k such that $\alpha_{k,i} = \max_{\{k': x \in \tilde{V}_{k'}\}} \alpha_{k',i}$. Notice that $\varphi_{x,i}$ is upper hemicontinuous.

Again, for each x there is a closed neighborhood $\tilde{U}_x \subset U_x$, and the cover $\{\tilde{U}_x\}_{x \in X}$ has a finite subcover $\{\tilde{U}_k\}_{k=1, \dots, L}$. Let $\Phi: X \rightarrow X$ be the correspondence given by

$$\Phi(x) = \text{co} \cup_{\{k \in \{1, \dots, L\}: x \in \tilde{U}_k\}} \varphi_k(x).$$

Φ is non empty, convex and compact valued by construction. Also, because it is the convex hull of a finite union of upper hemicontinuous correspondences defined on closed sets, it has closed graph. Thus, Φ has a fixed point (by Glicksberg (1952)). To see that it is a contradiction, take any x and let $J = \{k \in \{1, \dots, L\}: x \in \tilde{U}_k\}$ and $J' = \{k \in \{1, \dots, K\}: x \in \tilde{V}_k\}$. Notice that part (a) of continuous security implies that

$$\varphi_k(x) \subset B(x, \alpha_{k'})$$

for all $k \in J$ and all $k' \in J'$. Thus

$$\Phi(x) \subset \text{co} B(x, \max_{k' \in J'} \alpha_{k'}).$$

However, part (b) of continuous security implies that for each $k' \in J'$, we can find $y_i \notin \text{co} B_i(y, \alpha_{k',i})$ for some i . □

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