

TRUTHFUL IMPLEMENTATION AND PREFERENCE AGGREGATION IN RESTRICTED DOMAINS

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ABSTRACT. In a setting where agents have quasi-linear utilities over social alternatives and a transferable commodity, we consider three properties that a social choice function may possess: truthful implementation (in dominant strategies); monotonicity in differences; lexicographic affine maximization. We introduce the notion of a flexible domain of preferences that allows elevation of pairs and study which of these conditions implies which others when the domain is flexible. We provide a generalization of the theorem of Roberts (1979) in restricted valuation domains. Flexibility holds (and the theorem is not vacuous) if the domain of valuation profiles is restricted to the space of continuous functions defined on a topological space, or the space of piecewise linear functions defined on an affine space, or the space of smooth functions defined on a compact differentiable manifold. Our results can be applied in both public and private goods allocation settings, with finite or infinite alternative sets.

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1. INTRODUCTION

The relationship between the implementability of a social choice function and other properties that this aggregation device may possess is a central theme in the mechanism design and social choice literatures. In reaction to the fundamental result of Gibbard (1973) and Satterthwaite (1975), a natural direction for investigation is to restrict the domain of profiles, aiming either at similar results with weaker hypotheses or positive results¹. The most obvious restrictions on the domain of the social choice function are obtained by restricting each agent to have a preference

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¹The dictatorial conclusion of the Gibbard-Satterthwaite theorem holds even if the domain of preferences is restricted to the space of all continuous functions defined on a metric space; cf. Barberà and Peleg (1990). In rich domains consisting of strict preferences, Dasgupta, Hammond, and Maskin (1979) show that a social choice function is truthfully implementable in dominant strategies if and only if it is monotonic, in the sense that for any two utility profiles, if alternative a is selected by the choice function under the first utility profile but a is not selected under the second profile, then there exist an alternative b and an agent in the society that weakly prefers a to b at the first situation but strictly prefers b to a at the second one. See also Maskin (1985).

drawn from some subset of the space of strict preference orderings², and our work considers only domain restrictions of this sort.

A particularly important possibility is that side payments may be feasible. Such environments occur naturally in connection with, for example, public good provision models with projects of variable size (Green and Laffont (1977), Green and Laffont (1979), Laffont and Maskin (1980)), cost sharing agreements (Moulin (1988), Moulin (2010)), allocation models of pollution permits and other common divisible resources (Dasgupta, Hammond, and Maskin (1980), Duggan and Roberts (2002), Montero (2008)), models of private good allocation with externalities (Jehiel, Moldovanu, and Stacchetti (1999), Jehiel and Moldovanu (2006) and references therein), one-sided matching models with monetary transfers (Miyagawa (2001), Mishra and Roy (2011), also Babaioff, Lavi, and Pavlov (2005)) and so forth.

Roberts (1979) considers an environment in which an outcome of the social choice process consists of an element of a finite set A of social alternatives and a vector of side payments. Roberts restricts agents to have preferences that are quasilinear: an agent's utility is the sum of a utility given by the societal alternative and the monetary transfer she receives. In the literature related to Roberts' work a *social choice function* (SCF) is a function from the domain of preference profiles to A . Such an SCF is said to be *truthfully implementable (in dominant strategies)* if it can be combined with a payment function, mapping preference profiles to vectors of side payments, to create a social choice function, in the more complete sense of Gibbard and Satterthwaite, for which truth telling is weakly dominant.

An SCF is an *affine maximizer* if it maximizes a social welfare function that is a weighted sum of the agents' utilities plus a function that may be thought of as representing societal values, such as externalities or the welfare of future generations, that are not captured by individual preferences. In this definition the weights on the individual utilities are required to be nonnegative, with at least one individual having a positive weight. Roberts (1979) asserts that if $3 \leq |A| < \infty$ and the preference domain is unrestricted—that is, the agents can attach any values to the elements of A —then an SCF is truthfully implementable in dominant strategies if and only if it is an affine maximizer, and that in turn these conditions hold if and only if a third condition called *positive association of differences* (PAD) holds. However, this is not quite correct because an affine maximizer can fail to be truthfully implementable³. The complication arises when some of the agents' preferences have zero weight in the affine combination. At each profile of types for the agents with positive weight there is an induced SCF whose domain is the set of type profiles for the agents with zero weight and whose range is the set of alternatives maximizing the affine functional; without an additional restriction these induced SCF's may not be truthfully implementable. Example 3.2 gives a concrete instance.

If the given SCF is truthfully implementable, then so is each derived SCF, so it must be an affine maximizer when its image has three or more elements. Developing

This monotonicity condition is sometimes referred to as strong positive association; cf. Muller and Satterthwaite (1977).

²Single peakedness (Black (1958)) is an example of a well studied domain restriction that does not have this form.

³A referee informed us that this problem, and the resolution we describe, was independently observed before by Rozenztrum (1999).

this condition recursively leads to the notion of a *lexicographic affine maximizer*, and the corrected result asserts that if $3 \leq |A| < \infty$ and the preference domain is unrestricted, then truthfully implementability, PAD, and lexicographic affine maximization are equivalent.

The assumption that A is finite is unnatural in many applications, e.g., typically the quantity of a public good is modeled as a real variable. Previous results characterizing implementable choice functions as affine maximizers⁴ are not appropriate to handle such situations. When A is infinite the equivalence asserted by Roberts' theorem continues to hold in a formal sense, but one can easily construct profiles for which affine maximization is undefined, so the result is vacuous in this case. In order to have a meaningful generalization of Roberts' result in this direction, one must introduce restrictions on the domain. One of our major objectives is to extend Roberts' theorem to the case in which the set of alternatives is a compact metric space and the admissible preferences are continuous (or differentiable) functions on A . Our strategy is to impose conditions on the spaces of individual preferences that ensure that they are "rich enough" to support arguments leading to the conclusions of Roberts' theorem but are at the same time satisfied in a wide variety of applications.

The rest of this section describes the structure of the paper. Section 2 introduces the social choice setting and characterizes truthful implementation when the range of the SCF has one or two elements. Section 3 defines what it means for an SCF to be an affine maximizer, and a lexicographic affine maximizer. A lexicographic affine maximizer is always truthfully implementable. In Section 3 we also introduce a condition called *monotonicity in differences*, and show that an affine maximizer satisfies this condition. For any two admissible profiles of preferences, monotonicity in differences requires that if alternative a is selected by the choice function under the first profile and alternative b is chosen under the second profile, then there exists at least one agent for whom the valuation difference between b and a is weakly greater in the second situation than in the first. Like PAD, monotonicity in differences is a collective condition insofar as it considers simultaneous changes of valuations for multiple agents. Unlike PAD, monotonicity in differences is not implied by the dominant strategy incentive constraints in every preference domain, nor does it necessarily imply truthful implementation or the affine maximization property in all domains.

In Section 4 we introduce two conditions on the preference domain, *elevation of pairs* and *flexibility*. These conditions are rather technical, but at this point we can say that they have the following character: given two (or in some cases three) alternatives and two (or in some cases three) individual preferences, there is another admissible preference that emphasizes the given alternatives because other alternatives become less desirable. If the domain of profiles is a cartesian product of spaces of admissible preferences for the various agents, and each agent's space allows elevation of pairs, then truthful implementability implies monotonicity in differences.

Our main result, Theorem 1, states that if the domain of preferences allows elevation of pair and is flexible, and the image of the SCF has at least three elements,

⁴In addition to Roberts (1979), see Lavi, Mu'alem, and Nisan (2009), Dobzinski and Nisan (2009), Mishra and Sen (2011) and Vohra (2011).

then it is truthfully implementable if and only if it is a lexicographic affine maximizer. The main contribution of Robert's theorem is that truthful implementability implies affine maximization, and the key technical result supporting our generalization of this part of the argument is that if the domain of profiles is flexible, then monotonicity in differences implies affine maximization. This result is proved in Section 7; the influence of the arguments of Lavi, Mu'alem, and Nisan (2009) will be evident. As we will see in Section 4, these results combine to imply Theorem 1. We end this section with an application of our result to a private good allocation problems: one-sided matching problem with side transfers in an environment that is a generalization of the dichotomous environment considered by Mishra and Roy (2011) and Babaioff, Lavi, and Pavlov (2005). Incidentally, this will show how flexibility and elevation of pairs can hold in domains that are strict subsets of the unrestricted preference domain considered by Roberts (1979), even when the set of alternatives is finite, so our result is a strict generalization in this case.

In Section 5 we study some of the implications of our main result. To this end we introduce another condition called *comprehensiveness*, and show that comprehensiveness implies that the domain of preferences allows elevations of pairs and is flexible. Recall that a lattice of real valued functions is a vector space of functions that is closed under the pointwise minimum and pointwise maximum operators. A *resolving lattice* for A is a lattice with the additional property that for each $a \in A$ there is a function in the lattice whose value at a is different from its value at every other point in A . The key result of Section 5.2 shows that if a domain of valuations is a resolving lattice, then it is comprehensive. It is easy to see that the space of continuous functions on a compact metric space is a resolving lattice, as is the set of continuous piecewise affine functions on a convex subset of a Euclidean space. The space of C^r functions on a C^r manifold is not a lattice, so at least from the point of view of the proper development of our techniques, one should inquire as to whether Roberts' theorem also holds in this case. It turns out that it does, and we show this in Section 5.3.

Section 6 presents two counter examples intended to clarify which of the various implications that constitute Theorem 1 do not hold in domains that are not comprehensive. Example 6.1 illustrates the fact that, in certain domains, truthful implementation neither implies, nor is implied by, monotonicity in differences. This example also shows that a monotonic SCF is not necessarily an affine maximizer in every domain.⁵ Example 6.2 shows that truthful implementation also does not necessarily imply affine maximization when the image of the SCF has three or more alternatives.⁶ Thus these conditions are not equivalent in every domain.

Section 8 presents some final remarks.

⁵Our example draws inspiration from Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan, and Sen (2006a).

⁶This has also been shown by Mishra and Sen (2011), who characterize *neutral* social choice functions via weighted welfare maximizers in open interval domains, still assuming that the alternative set A is finite. The neutrality of a choice function may be a reasonable assumption for certain environments, but is violated whenever the choice function discriminates a priori among alternatives, as occurs for instance when the social objective function includes a function embodying societal values. On the other hand, as Mishra and Sen (2011) point out, neutrality is an essential component for restricting the domain to open bounded intervals.

2. THE SOCIAL CHOICE SETTING

There is a set A of social *alternatives*, which can be finite or infinite. There is a finite set $N = \{1, \dots, n\}$ of *agents*. For each $i \in N$ there is a space T_i of *types*. The space of *type profiles* is $T = T_1 \times \dots \times T_n$.

A *difference* is a function $d: A \times A \rightarrow \mathbb{R}$ satisfying

$$d(a, b) + d(b, c) + d(c, a) = 0$$

for all $a, b, c \in A$. Note that

$$d(a, a) = 0 \quad \text{and} \quad d(b, a) = -d(a, b).$$

(Setting $a = b = c$ gives the first equation, after which setting $b = c$ gives the second.) Let $\mathcal{D}(A)$ be the space of differences. We assume that for each $t_i \in T_i$ there is an associated difference $d_{t_i} \in \mathcal{D}(A)$, signifying that for all $a, b \in A$ and $\tau_i \in \mathbb{R}$, agent i with type t_i is indifferent between having a implemented and receiving a transfer of τ_i and having b implemented with a transfer of $\tau_i + d_{t_i}(a, b)$. Thus agents have quasi-linear preferences. For $t \in T$, let $d_t = (d_{t_1}, \dots, d_{t_n})$.

Since agents' types are private information, the scope for incentive compatible social choice functions that discriminate between two types with the same difference is quite limited. Nevertheless, in the corrected version of Robert's theorem the additional generality of general type spaces is significant. Specifically, given an affine maximizer, the derived mechanisms for agents with zero weight in the affine functional can depend in an arbitrary way on the additional information, beyond the differences, reported by the agents with positive weight.

A *valuation* is a function $\nu: A \rightarrow \mathbb{R}$. One may derive a difference from a valuation ν by setting $d_\nu(a, b) = \nu(a) - \nu(b)$, so a valuation may be thought of as a pair consisting of a difference and a shift parameter that does not affect incentives. In this sense valuations are intermediate between differences and fully general types. Most of the literature works with valuations, but at least in principle working with differences should be psychologically valuable, disciplining the analysis by systematically excluding certain extraneous information, which tends to result in greater simplicity and clarity. However, we will see some assumptions that are most naturally expressed in terms of differences, and in those contexts we will work with them.

A *social choice function*⁷ (SCF) is a function

$$f: T \rightarrow A.$$

We fix such an f . An SCF will be combined with a *payment scheme*, which is an n -tuple

$$p = (p_1, \dots, p_n): T \rightarrow \mathbb{R}^n$$

of functions specifying taxes imposed on the agents. We follow the customary notational conventions: for $i \in N$, $T_{-i} = \times_{j \neq i} T_j$, and (t_i, t_{-i}) denotes for the type profile with components $t_i \in T_i$ and $t_{-i} \in T_{-i}$.

⁷It is customary to require that f is surjective. This assumption is known as *non imposition* (e.g. Roberts (1979)) and *citizen sovereignty* (e.g. Dasgupta, Hammond, and Maskin (1979)). While surjectivity may be philosophically significant in some settings, in formal analysis it can be restrictive, for example because we will have derived SCF's with smaller images.

Definition 2.1. We say that p *truthfully implements* f if

$$d_{t_i}(f(t), f(t'_i, t_{-i})) \geq p_i(t) - p_i(t'_i, t_{-i})$$

for all $i \in N$, $t \in T$, and $t'_i \in T_i$. We say that f is *truthfully implementable (in dominant strategies)* if there is a payment scheme that truthfully implements it.

Roberts (1979) is concerned with characterizing truthfully implementable SCFs with $|f(T)| \geq 3$, but we cannot restrict the discussion to such SCFs. Note that any constant choice function (i.e., $|f(T)| = 1$) is truthfully implementable. We say that f is a *binary implementable SCF* if it is truthfully implementable in dominant strategies and $|f(T)| = 2$. Such choice functions are characterized by cutoff differences.

Proposition 1. *If $f(T) = \{x, y\} \subset A$, $x \neq y$, then f is truthfully implementable if and only if for all $i \in N$ and all $t_{-i} \in T_{-i}$, there exists $\delta_i^*(t_{-i}) \in \mathbb{R} \cup \{-\infty, +\infty\}$ such that the following conditions hold:*

- (a) $t_i \in T_i$ and $d_{t_i}(x, y) > \delta_i^*(t_{-i})$ imply $f(t) = x$;
- (b) $t_i \in T_i$ and $d_{t_i}(x, y) < \delta_i^*(t_{-i})$ imply $f(t) = y$.

Similar results are available in the literature; for a somewhat more general version, with a proof, see Theorem 9.36 of Nisan (2007).

The following construction shows that the class of binary implementable SCFs is (in comparison with the affine maximizers we will see below) quite rich. Suppose that the function $G: \mathbb{R}^n \rightarrow \mathbb{R}$ is weakly increasing, in the sense that $G(\delta') \geq G(\delta)$ whenever $\delta'_i \geq \delta_i$ for all $i \in N$. We say that G is *strictly increasing at zero* if, for any $\delta \in \mathbb{R}^n$ with $G(\delta) = 0$, one has $G(\delta_i - \epsilon, \delta_{-i}) < 0 < G(\delta_i + \epsilon, \delta_{-i})$ for each i and every $\epsilon > 0$. Suppose that this is the case, that $f(T) = \{x, y\}$, and that $f(t) = x$ for all t such that $G(d_t(x, y)) > 0$ and $f(t) = y$ for all t such that $G(d_t(x, y)) < 0$. For each i and $t_{-i} \in T_{-i}$ let $\delta_i^*(t_{-i})$ be the infimum of the set of δ_i such that $G(\delta_i, d_{t_{-i}}(x, y)) > 0$ if this set is nonempty, and otherwise set $\delta_i^*(t_{-i}) = \infty$. The result above implies that f is binary implementable.

There are also binary implementable choice functions that are not derived from this construction. Consider the following example, in which $\delta_2^*(t_1)$ does not depend solely on the difference d_{t_1} .

Example 2.2. Let $N = \{1, 2\}$, $T_1 = \mathbb{R} \cup \{t'_1\}$ where $t'_1 \notin \mathbb{R}$, and $T_2 = \mathbb{R}$. Suppose that $d_{t_1}(x, y) = t_1$ if $t_1 \neq t'_1$, $d_{t'_1}(x, y) = 0$, and $d_{t_2}(x, y) = t_2$. If $t_1 \neq t'_1$ we set

$$f(t_1, t_2) = \begin{cases} x, & \text{if } d_{t_1}(x, y) \geq 0 \text{ and } d_{t_2}(x, y) \geq 0; \\ y, & \text{otherwise.} \end{cases}$$

Let $\delta_1^*(t_2) = 0$ if $t_2 \geq 0$, and otherwise set $\delta_1^*(t_2) = \infty$. If $t_1 \neq t'_1$ let $\delta_2^*(t_1) = 0$ if $t_1 \geq 0$, and otherwise set $\delta_2^*(t_1) = \infty$. It is easy to see that in the absence of t'_1 the hypotheses of Proposition 1 are satisfied. The definition of $f: T \rightarrow A$ is completed by setting

$$f(t'_1, t_2) = \begin{cases} x, & \text{if } d_{t_2}(x, y) > 1; \\ y, & \text{if } d_{t_2}(x, y) \leq 1. \end{cases}$$

Let $\delta_2^*(t'_1) = 1$. It is never the case that $d_{t'_1}(x, y) > \delta_1^*(t_2)$, and when $d_{t'_1}(x, y) < \delta_1^*(t_2)$, we have $\delta_1^*(t_2) = \infty$ and $f(t'_1, t_2) = y$, so the hypotheses of Proposition 1 continue to hold.

3. AFFINE MAXIMIZATION

Let $\Delta^{n-1} = \{ \sigma \in \mathbb{R}_+^n : \sigma_1 + \dots + \sigma_n = 1 \}$ be the $(n-1)$ -dimensional simplex.

Definition 3.1. We say that the SCF f is an *affine maximizer* if there is $\sigma \in \Delta^{n-1}$ and $q \in \mathcal{D}(A)$ such that

$$\sum_{i=1}^n \sigma_i d_{t_i}(f(t), a) + q(f(t), a) \geq 0, \quad (1)$$

for all $t \in T$ and $a \in f(T)$.

An affine maximizer need not be truthfully implementable:

Example 3.2. Suppose that $n = 2$ and A is finite. Suppose that f always chooses one of agent 1's favorite alternatives, and if the set of favorites has more than one element, then f chooses one of agent 2's least favorite elements of this set. Clearly f is an affine maximizer because it maximizes agent 1's utility, but it is not truthfully implementable because, for any payment scheme, agent 2 is motivated to misreport some of her types.

As this example suggests, in order to obtain an exact characterization of truthful implementability we need to recursively impose conditions when the affine functional has more than one maximizer.

Definition 3.3. Suppose that f is an affine maximizer, with q and σ as in Definition 3.1, and let $S = \{ i \in N : \sigma_i = 0 \}$, $T_S = \times_{i \in S} T_i$, and $T_{N \setminus S} = \times_{i \in N \setminus S} T_i$. We say that f is a *lexicographic affine maximizer* if, for each $t_{N \setminus S} \in T_{N \setminus S}$, one has:

- (a) if $|f(t_{N \setminus S}, T_S)| = 2$, then the function $t_S \mapsto f(t_{N \setminus S}, t_S)$ is a binary implementable SCF;
- (b) if $|f(t_{N \setminus S}, T_S)| \geq 3$, then the function $t_S \mapsto f(t_{N \setminus S}, t_S)$ is a lexicographic affine maximizer.

Of course the appearance of circularity is resolved by working inductively, assuming that lexicographic affine maximization when there are fewer than n agents has already been defined.

Proposition 2. *If f is a lexicographic affine maximizer, then it is truthfully implementable.*

Proof. Let $\sigma \in \Delta^{n-1}$ and $q \in \mathcal{D}(A)$ be as in Definition 3.1. As above, let $S = \{ i \in N : \sigma_i = 0 \}$. For each $i \in N \setminus S$ choose some $t_i^* \in T_i$, and let $p_i : T \rightarrow \mathbb{R}$ be the (generalized VCG) payment function given by

$$p_i(t) = \frac{1}{\sigma_i} \left\{ \sum_{j \in N \setminus S, j \neq i} \sigma_j d_{t_j}(f(t_i^*, t_{-i}), f(t)) + q(f(t_i^*, t_{-i}), f(t)) \right\}.$$

This construction guarantees that

$$d_{t_i}(f(t), f(t'_i, t_{-i})) \geq p_i(t) - p_i(t'_i, t_{-i})$$

for all $i \in N \setminus S$, $t \in T$, and $t'_i \in T_i$. For each type profile $t_{N \setminus S} \in T_{N \setminus S}$, the derived SCF $f(t_{N \setminus S}, \cdot) : T_S \rightarrow A$ is trivially truthfully implementable if it is a constant function, and automatically truthfully implementable if it is instead a binary implementable SCF. Otherwise it is a lexicographic affine maximizer. Therefore this

argument can be repeated inductively to construct a payment scheme such that truth-telling is incentive compatible for all $i \in N$. $\square$

We now introduce another condition that an SCF may satisfy. For all $\delta, \delta' \in \mathbb{R}^n$, we write $\delta \gg \delta'$ to indicate that $\delta_i > \delta'_i$ for all $i \in N$, and $\delta \ll \delta'$ to indicate $\delta_i < \delta'_i$ for all $i \in N$.

Definition 3.4. We say that f is *monotonic in differences* if for any $t^x, t^y \in T$ with $f(t^x) = x$ and $f(t^y) = y$, there is some $i \in N$ such that

$$d_{t^x_i}(x, y) \geq d_{t^y_i}(x, y).$$

Equivalently, for all $t^x, t \in T$ such that $f(t^x) = x$ and all $a \in A \setminus \{x\}$, if $d_t(x, a) \gg d_{t^x}(x, a)$, then $f(t) \neq a$. The latter formulation of the condition is called *negative unanimity*.

Proposition 3. *If f is an affine maximizer, then it is monotonic in differences.*

Proof. Let $\sigma \in \Delta^{n-1}$ and $q \in \mathcal{D}(A)$ be such that (1) holds for all $t \in T$ and $a \in f(T)$. To obtain a contradiction, suppose that negative unanimity fails: there is a pair of type profiles $t^x, t^y \in T$ such that

$$d_{t^x}(x, y) \ll d_{t^y}(x, y)$$

where $x = f(t^x)$ and $y = f(t^y)$. Since $\sigma_i > 0$ for some i this implies that

$$\sum_{i=1}^n \sigma_i d_{t^x_i}(x, y) < \sum_{i=1}^n \sigma_i d_{t^y_i}(x, y).$$

Inequality (1) with $a = y$ and $t = t^x$ is:

$$0 \geq -\sum \sigma_i d_{t^x_i}(x, y) - q(x, y).$$

Combining these two equations and using the fact that $d(b, a) = -d(a, b)$ for any difference gives

$$0 > \sum_{i=1}^n \sigma_i d_{t^y_i}(y, x) + q(y, x)$$

which contradicts (1) with t replaced by t^y and $a = x$. $\square$

Thus, a lexicographic affine maximizer is truthfully implementable and monotonic in differences in every domain T . However, these three conditions are not equivalent in all domains. We provide examples in Section 6.

4. THE MAIN RESULT

Our generalization of the key implication of Robert's theorem—that truthful revelation implies lexicographic affine maximization—depends on the domains being in certain senses “rich.” The two assumptions of this sort introduced in this section assert that, in comparison with a given pair or triple of alternatives, and given pairs or triples of preferences, it is possible to find preferences or profiles that further enhance the attractiveness of the distinguished alternatives.

Definition 4.1. We say that T_i *allows elevation of pairs* if, for all $t_i^x, t_i^y \in T_i$ and $x, y \in A$ such that $d_{t_i^y}(x, y) > d_{t_i^x}(x, y)$, there is a $t_i \in T_i$ such that

$$\begin{aligned} d_{t_i}(x, a) &> d_{t_i^x}(x, a), & \text{for all } a \neq x, \\ d_{t_i}(y, a) &> d_{t_i^y}(y, a), & \text{for all } a \neq y. \end{aligned}$$

We say that T *allows elevation of pairs* if each T_i allows elevation of pairs.

For example, suppose that $d_{t_i^y}(x, y) > d_{t_i^x}(x, y)$, where $d_{t_i^x}$ and $d_{t_i^y}$ are derived from i 's valuations v_i^x and v_i^y , respectively. After adding a suitable constant, we may assume that $v_i^y(x) > v_i^x(x)$ and $v_i^x(y) > v_i^y(y)$. Elevation of pairs requires that there be a t_i such that d_{t_i} is derived from an admissible valuation v_i with $v_i(x) = v_i^x(x)$, $v_i(y) = v_i^y(y)$, and $v_i(a) < \min\{v_i^x(a), v_i^y(a)\}$ for all $a \neq x, y$. Note that if A is a metric space and v_i^x and v_i^y are continuous, then such a continuous v_i exists.

Proposition 4. *If T allows elevation of pairs and f is truthfully implementable, then f is monotonic in differences.*

Proof. By hypothesis there is a payment scheme $p: T \rightarrow \mathbb{R}^n$ that truthfully implements f . Aiming at a contradiction, suppose that f is not monotonic in differences, so there is a failure of negative unanimity: for some $t^x, t^y \in T$ we have

$$d_{t^y}(x, y) \gg d_{t^x}(x, y).$$

where $x = f(t^x)$ and $y = f(t^y)$ respectively. Since each T_i allows elevation of pairs, there is a type profile $t \in T$ such that

$$d_t(x, a) \gg d_{t^x}(x, a), \quad \text{for all } a \neq x, \quad (2)$$

$$d_t(y, a) \gg d_{t^y}(y, a), \quad \text{for all } a \neq y. \quad (3)$$

Let $t^0 = t^x$, and define $t^1, \dots, t^n$ inductively by setting $t^i = (t_i, t_{-i}^{i-1})$. Truthful implementation implies that for each $i \in N$ we have both

$$d_{t_i}(f(t^i), f(t^{i-1})) \geq p_i(t^i) - p_i(t^{i-1})$$

and

$$d_{t_i^{i-1}}(f(t^{i-1}), f(t^i)) \geq p_i(t^{i-1}) - p_i(t^i).$$

Combining these, and recognizing that $t_i^{i-1} = t_i^x$ and $t_i^i = t_i$, gives

$$d_{t_i^x}(f(t^{i-1}), f(t^i)) \geq d_{t_i}(f(t^{i-1}), f(t^i)).$$

Since $t^0 = t^x$ we have $f(t^0) = x$, and if j is the first index such that $f(t^j) \neq x$, this inequality contradicts (2). Therefore we have

$$x = f(t^0) = \dots = f(t^n) = f(t).$$

The same argument with t^y in place of t^x gives $y = f(t)$, but $x = y$ is impossible. $\square$

Allowing elevation of pairs is a condition that is imposed on each T_i individually. In the following condition, on the other hand, the restriction is coordinated across the different T_i .

Definition 4.2. We say that T is *flexible* if:

- (F1) For any distinct $x, y \in A$ there are disjoint sets $B_x, B_y \subset A$ with $x \in B_x$ and $y \in B_y$ such that for all $i \in N$, $t_i^x, t_i^y \in T_i$, and $\delta_{xyi} \in \mathbb{R}$, there is a $t_i \in T_i$ satisfying

$$d_{t_i}(x, y) = \delta_{xyi},$$

and further:

- (a) $d_{t_i}(a, x) < d_{t_i^x}(a, x)$, for all $a \in A \setminus (\{x\} \cup B_y)$;
 - (b) $d_{t_i}(a, y) < d_{t_i^y}(a, y)$, for all $a \in A \setminus (\{y\} \cup B_x)$.
- (F2) For any distinct $x, y, z \in A$ there are pairwise disjoint sets $B'_x, B'_y, B'_z \subset A$ with $x \in B'_x$, $y \in B'_y$, and $z \in B'_z$, such that for all $i \in N$, $t_i^x, t_i^y, t_i^z \in T_i$, and $\delta_{xyi}, \delta_{yzi} \in \mathbb{R}$, there is a $t_i \in T_i$ such that

$$d_{t_i}(x, y) = \delta_{xyi} \quad \text{and} \quad d_{t_i}(y, z) = \delta_{yzi},$$

and further:

- (a) $d_{t_i}(a, x) < d_{t_i^x}(a, x)$, for all $a \in A \setminus (\{x\} \cup B'_y \cup B'_z)$;
- (b) $d_{t_i}(a, y) < d_{t_i^y}(a, y)$, for all $a \in A \setminus (\{y\} \cup B'_x \cup B'_z)$;
- (c) $d_{t_i}(a, z) < d_{t_i^z}(a, z)$, for all $a \in A \setminus (\{z\} \cup B'_x \cup B'_y)$.

These properties abstract the key features of a topological setting that we will employ in Section 5. To develop intuition for them it may help to imagine that each profile d_i is continuous, and that the sets B_x and B'_x are certain neighborhoods of x , and similarly for y and z .

The following result will be proved in Section 7.

Proposition 5. *If T is flexible, f is monotonic in differences, and $|f(T)| \geq 3$, then f is an affine maximizer.*

Our main result is the following characterization theorem.

Theorem 1. *If T allows elevation of pairs and is flexible, then f is truthfully implementable if and only if one of the following hold:*

- (a) f is a constant function;
- (b) f is a binary implementable SCF;
- (c) f is a lexicographic affine maximizer.

Proof of Theorem 1. It is obvious that f is truthfully implementable if (a) holds, truthful implementability is part of the definition of (b), and Proposition 2 states that (c) implies truthful implementability.

Suppose that f is truthfully implementable. Evidently there is nothing to prove unless $|f(T)| \geq 3$, in which case Propositions 4 and 5 imply that f is an affine maximizer, so there are q and σ such that the conditions of Definition 3.1 are satisfied. Let $S = \{i : \sigma_i = 0\}$, $T_S = \times_{i \in S} T_i$, and $T_{N \setminus S} = \times_{i \in N \setminus S} T_i$, and consider $t_{N \setminus S} \in T_{N \setminus S}$. Of course $f(t_{N \setminus S}, \cdot) : T_S \rightarrow A$ is truthfully implementable. By induction we may assume that the result has already been established for the derived SCFs with fewer than n agents, so $f(t_{N \setminus S}, \cdot)$ is constant, a binary implementable SCF, or a lexicographic affine maximizer. Since $t_{N \setminus S}$ was arbitrary, the proof is complete. $\square$

In the next section we study conditions that imply that T allows elevation of pairs and is flexible. When A is finite, however, it can be straightforward to check directly that a restricted domain T allows elevation of pairs and satisfies flexibility.

For instance, let $5 \leq |A| < +\infty$. A valuation function $\nu: A \rightarrow \mathbb{R}$ is called a *split* if there exists a non-empty subset B of A , with $|B| \geq 2$, such that $\nu(b) = 0$ for all $b \in B$ and $\nu(a) > 0$ for all $a \in A \setminus B$. Let $\mathcal{S}(A)$ be the space of differences $d(x, y) = \nu(x) - \nu(y)$ generated by split functions. Clearly $\mathcal{S}(A)$ is neither a linear space nor a convex set. In this context, a type t_i associated with difference d_{t_i} describes a set B of “unacceptable” alternatives together with the positive values of “acceptable” alternatives in $A \setminus B$.

Lemma 4.3. *Let $5 \leq |A| < +\infty$. If for all $i \in N$ and $d \in \mathcal{S}(A)$ there exists $t_i \in T_i$ such that $d_{t_i} = d$, then T allows elevation of pairs and is flexible.*

Proof. To show that T satisfies elevation of pairs, fix $i \in N$, $x, y \in A$, and $t_i^x, t_i^y \in T_i$ such that $d_{t_i^y}(x, y) = v_i^y(x) - v_i^y(y) > v_i^x(x) - v_i^x(y) = d_{t_i^x}(x, y)$. Choose $\epsilon_i, \delta_i > 0$ to satisfy $v_i^y(y) - v_i^x(y) < \epsilon_i - \delta_i < v_i^y(x) - v_i^x(x)$. Let v_i be defined by $v_i(x) = v_i^x(x) + \epsilon_i$, $v_i(y) = v_i^y(y) + \delta_i$ and $v_i(b) = 0$ for all $b \neq x, y$. Clearly, v_i is a split with unacceptable alternatives $B = A \setminus \{x, y\}$, hence the type t_i associated with the difference generated by v_i belongs to T_i . It is now immediate to verify that $d_{t_i}(x, a) > d_{t_i^x}(x, a)$ for all $a \neq x$ and $d_{t_i}(y, a) > d_{t_i^y}(y, a)$ for all $a \neq y$.

To show that T satisfies (F2), fix $x, y, z \in A$, $i \in N$, $t_i^x, t_i^y, t_i^z \in T_i$ and $\delta_{xyi}, \delta_{yzi} \in \mathbb{R}$. Consider the split v_i with $B = A \setminus \{x, y, z\}$ satisfying, in addition, $v_i(x) > v_i^x(x)$, $v_i(y) > v_i^y(y)$, $v_i(z) > v_i^z(z)$ and $d_{t_i}(x, y) = \delta_{xyi}$, $d_{t_i}(y, z) = \delta_{yzi}$. Clearly type t_i associated with the difference generated by v_i belongs to T_i and satisfies (a) to (c) of condition (F2) with $B'_x = \{x\}$, $B'_y = \{y\}$, $B'_z = \{z\}$. Condition (F1) is similarly shown. Thus T is flexible. $\square$

Lemma 4.3 applies to allocation problems in which the set of alternatives is discrete and there are natural restrictions on the domain of valuations, as in matching models with externalities where each agent cares also about other players' assignments. For instance, consider a generalization of the one-sided matching model with transfers studied in Mishra and Roy (2011), where a collection of objects (trading routes) is to be assigned among a group of agents (competing firms). An alternative is then a vector $a = (o_1, \dots, o_n)$ of assigned routes to each firm. A firm has unacceptable alternatives, either because an unacceptable alternative includes a worthless route for the firm or a combination of assignments to other firms that renders the route worthless for the initial firm. If the collection of unacceptable alternatives is private information, then a type t_i for firm i specifies a set B of unacceptable alternatives, so that $v_i(b) = 0$ for all $b \in B$, and (potentially different) values $v_i(a) > 0$ for acceptable alternatives $a \in A \setminus B$. From Lemma 4.3, it follows that in these environments an SCF is implementable if and only if it is a lexicographic affine maximizer.

5. EXTENSIONS

Here we explore some extensions and applications that the additional generality of our result opens up.

As we explained in the introduction, quasi-linear preferences are natural and commonly assumed in connection with public goods, public bads such as pollution, cost sharing, and various other allocation problems. In such models it is common to treat the infinite set of alternatives A as a compact metric or topological space, and the domain of admissible valuation profiles as containing only continuous, piecewise

linear, or even differentiable functions. We show in this section how Theorem 1 can be employed in these cases. In particular, if the space of individual valuations is the space of continuous functions on a compact metric space A , then it will follow from Corollary 5.10 that any truthfully implementable SCF must be a lexicographic affine maximizer. We are able to show, with some additional work, that this result holds even if the space of individual valuations is the space of smooth valuations on a smooth compact manifold.

5.1. COMPREHENSIVE DOMAINS

Insofar as a *valuation* $\nu: A \rightarrow \mathbb{R}$ induces a difference $d_\nu \in \mathcal{D}(A)$ given by $d_\nu(a, b) = \nu(a) - \nu(b)$, we may regard a valuation as a particular sort of type. Let $\mathcal{V}(A)$ be the space of all valuations. We assume now that each agent $i \in N$ is endowed with a preference domain $V_i \subseteq \mathcal{V}(A)$. Let

$$V = V_1 \times \cdots \times V_n \subseteq \mathcal{V}(A)^n.$$

An element v of V is called an *admissible valuation profile*. Our goal is to develop conditions on V that imply that it is flexible and allows elevation of pairs.

Of course $\mathcal{V}(A)$ is a vector space when endowed with the pointwise addition and scalar multiplication operations, and it is partially ordered by the pointwise ordering. For $\nu, \nu' \in \mathcal{V}(A)$ the pointwise max and pointwise min functions, $\nu \vee \nu'$ and $\nu \wedge \nu'$ respectively, are:

$$\nu \vee \nu'(a) = \max\{\nu(a), \nu'(a)\} \quad \text{and} \quad \nu \wedge \nu'(a) = \min\{\nu(a), \nu'(a)\}.$$

These operations extend component-wise to vectors of valuations.

Definition 5.1. For $i \in N$ we say that V_i is *combinative* if:

- (C1) For all distinct $x, y \in A$ and $v_i^x, v_i^y \in V_i$ there is a $v_i \in V_i$ such that $v_i(x) = v_i^x(x)$, $v_i(y) = v_i^y(y)$, and $v_i \leq v_i^x \vee v_i^y$.
- (C2) For all distinct $x, y, z \in A$ and $v_i^x, v_i^y, v_i^z \in V_i$ there is a $v_i \in V_i$ such that $v_i(x) = v_i^x(x)$, $v_i(y) = v_i^y(y)$, $v_i(z) = v_i^z(z)$, and $v_i \leq v_i^x \vee v_i^y \vee v_i^z$.
- (C3) For all $x, y \in A$ and $v_i^x, v_i^y \in V_i$ such that $v_i^x(x) \neq v_i^x(y)$ and $v_i^y(x) \neq v_i^y(y)$, there is a $v_i \in V_i$ such that $v_i(x) = v_i^x \wedge v_i^y(x)$, $v_i(y) = v_i^x \wedge v_i^y(y)$, and $v_i \leq v_i^x \wedge v_i^y$.
- (C4) For any finite set $B \subset A$ and collection of valuations $\{v_i^x\}_{x \in B}$ in V_i , there is a $v_i \in V_i$ such that $v_i \leq \bigwedge_{x \in B} v_i^x$.

We say that V is *combinative* if each V_i is combinative.

We now introduce a collection of functions that will be used to perturb valuations. The *support* of a function $\mu: A \rightarrow \mathbb{R}$ is the set

$$\text{supp}(\mu) = \{a \in A : \mu(a) \neq 0\}.$$

Definition 5.2. A set $\mathcal{U}$ of functions from A to $[0, 1]$ is called a *separating family* if, for any distinct $x, y, z \in A$ there are $\mu^x, \mu^y, \mu^z \in \mathcal{U}$ satisfying $\mu^x(x) = \mu^y(y) = \mu^z(z) = 1$ whose supports are pairwise disjoint:

$$\text{supp}(\mu^x) \cap \text{supp}(\mu^y) = \text{supp}(\mu^x) \cap \text{supp}(\mu^z) = \text{supp}(\mu^y) \cap \text{supp}(\mu^z) = \emptyset.$$

Recall that a topological space is *normal* if any two disjoint closed sets are contained in disjoint open sets. Urysohn's lemma asserts that in a normal space, for any two disjoint closed sets there is a continuous function from the space to $[0, 1]$

that is identically zero on the first set and identically one on the second. If, in addition, the space is T_1 (singletons are closed) and consequently Hausdorff, then the set of continuous real valued functions on the space contains a separating family.

Fix a separating family $\mathcal{U}$.

Definition 5.3. For $i \in N$ we say that V_i is $\mathcal{U}$ -perturbative if for all $x \in A$, $v_i^x \in V_i$, $\mu \in \mathcal{U}$, and $\delta_i \in \mathbb{R}_+$, there is a valuation $v_i \in V_i$ such that

$$v_i(a) \leq v_i^x(a) + \delta_i \mu(a)$$

for all $a \in A$ with strict inequality if and only if $a \neq x$. We say that V is $\mathcal{U}$ -perturbative if each V_i is $\mathcal{U}$ -perturbative.

Definition 5.4. For $i \in N$ we say that V_i is $\mathcal{U}$ -comprehensive if it is combinative and $\mathcal{U}$ -perturbative. We say that V is $\mathcal{U}$ -comprehensive if each V_i is $\mathcal{U}$ -comprehensive.

Lemma 5.5. Suppose that V is $\mathcal{U}$ -comprehensive and $x, y \in A$ are distinct. Then for any $v^x, v^y \in V$ there are disjoint sets $B_x, B_y \subset A$, with $x \in B_x$ and $y \in B_y$, and an admissible profile $v \in V$ such that $v(x) = v^x(x)$, $v(y) = v^y(y)$, and

$$\begin{aligned} v(a) &\ll v^x(a), & \text{for all } a \in B_x \setminus \{x\}, \\ v(a) &\ll v^y(a), & \text{for all } a \in B_y \setminus \{y\}, \\ v(a) &\ll v^x \wedge v^y(a), & \text{for all } a \in A \setminus (B_x \cup B_y). \end{aligned}$$

Proof. Fix functions $\mu^x, \mu^y : A \rightarrow [0, 1]$ in $\mathcal{U}$ with disjoint supports B_x and B_y and $\mu^x(x) = \mu^y(y) = 1$. Fix a particular $i \in N$.

Applying (C4), choose $v_{i1} \in V_i$ such that

$$v_{i1} \leq v_i^x \wedge v_i^y.$$

Choose $\delta_i > 0$ large enough that

$$v_i^x(x) < v_{i1}(x) + \delta_i \quad \text{and} \quad v_i^y(y) < v_{i1}(y) + \delta_i.$$

Since V_i is $\mathcal{U}$ -perturbative there is a $v_{i2}^x \in V_i$ such that

$$v_{i2}^x \leq v_{i1} + \delta_i \mu^x$$

with equality at x and strict inequality at all other $a \in A$. Applying (C3) to v_{i2}^x and v_{i2}^x gives a $v_{i3}^x \in V_i$ such that

$$v_{i3}^x \leq v_{i2}^x \wedge v_{i2}^x$$

with equality at x . Applying the assumption that V is $\mathcal{U}$ -perturbative (with $\delta_i = 0$), there is a $v_{i4}^x \in V_i$ such that

$$v_{i4}^x \leq v_{i3}^x$$

with equality at x and strict inequality at all other $a \in A$.

Similarly, there are $v_{i2}^y, v_{i3}^y, v_{i4}^y \in V_i$ such that

$$v_{i2}^y \leq v_{i1} + \delta_i \mu^y, \quad v_{i3}^y \leq v_{i2}^y \wedge v_{i2}^y, \quad v_{i4}^y \leq v_{i3}^y$$

with equality at y and strict inequality at all other $a \in A$, in all three cases. Finally, applying (C1) to v_{i4}^x and v_{i4}^y gives $v_i \in V_i$ such that

$$v_i(x) = v_{i4}^x(x), \quad v_i(y) = v_{i4}^y(y), \quad \text{and} \quad v_i(a) \leq v_{i4}^x \vee v_{i4}^y.$$

We have

$$v_i(x) = v_{i4}^x(x) = v_{i3}^x(x) = v_{i2}^x(x) = (v_{i1} + \delta_i) \wedge v_i^x(x) = v_i^x(x)$$

and $v_i(y) = v_i^y(y)$ by symmetry. For $a \in B_x \setminus \{x\}$ we have $\mu^y(a) = 0$ and thus

$$v_{i3}^y(a) \leq v_{i2}^y(a) < v_{i1}(a) \leq v_i^x \wedge v_i^y(a) \leq v_i^x(a).$$

Since $v_{i3}^x(a) \leq v_i^x(a)$, we conclude that

$$v_i(a) \leq v_{i4}^x \vee v_{i4}^y(a) < v_{i3}^x \vee v_{i3}^y(a) \leq v_i^x(a).$$

Symmetrically, $v_i(a) < v_i^y(a)$ for all $a \in B_y \setminus \{y\}$. For $a \in A \setminus (B_x \cup B_y)$ we have $v_{i3}^x(a) < v_{i1}(a)$ and $v_{i3}^y(a) < v_{i1}(a)$, so

$$v_i(a) \leq v_{i4}^x \vee v_{i4}^y(a) \leq v_{i3}^x \vee v_{i3}^y(a) < v_{i1}(a) \leq v_i^x \wedge v_i^y(a).$$

This gives us the desired result. $\square$

Proposition 6. *If V is $\mathcal{U}$ -comprehensive, then it allows elevation of pairs.*

Proof. Consider $v^x, v^y \in V$ and $x, y \in A$ such that $d_{v^y}(x, y) \gg d_{v^x}(x, y)$. Assume without loss of generality that $v^y(x) \gg v^x(x)$ and $v^y(y) \ll v^x(y)$. We can apply (C3) to v^x and v^y , at x and y respectively, to obtain a profile v_1 in V with $v_1 \leq v^x \wedge v^y$, $v_1(x) = v^x(x)$, and $v_1(y) = v^y(y)$. Since V is $\mathcal{U}$ -perturbative there are $v_2^x \in V$ such that $v_2^x(x) = v_1(x) = v^x(x)$ and $v_2^x(a) \ll v_1(a)$ for all $a \neq x$, and $v_2^y \in V$ such that $v_2^y(y) = v_1(y) = v^y(y)$ and $v_2^y(a) \ll v_1(a)$ for all $a \neq y$. By Lemma 5.5, there exists an admissible profile $v \in V$ such that $v(x) = v_2^x(x) = v^x(x)$, $v(y) = v_2^y(y) = v^y(y)$, and $v(a) \ll v_2^x \vee v_2^y(a) \leq v^x \wedge v^y(a)$, for each $a \neq x, y$. $\square$

We omit the proof of the following because it is slightly more complex, but otherwise similar to the proof of Lemma 5.5 above.

Lemma 5.6. *Suppose that V is $\mathcal{U}$ -comprehensive and $x, y, z \in A$ are distinct. Then for any $v^x, v^y, v^z \in V$ there are pairwise disjoint sets $B_x, B_y, B_z \subset A$, with $w \in B_w$ for each $w = x, y, z$, and an admissible profile $v \in V$ such that*

$$v(a) \ll v^x \wedge v^y \wedge v^z(a), \quad \text{for all } a \in A \setminus (B_x \cup B_y \cup B_z),$$

and further, for each $w = x, y, z$, $v(w) = v^w(w)$ and $v(a) \ll v^w(a)$ for all $a \in B_w \setminus \{w\}$.

Proposition 7. *If V is $\mathcal{U}$ -comprehensive, then it is flexible.*

Proof. We will only prove (F1); the proof of (F2) follows the same pattern. Fix distinct $x, y \in A$. Choose $\mu^x, \mu^y \in \mathcal{U}$ such that $\mu^x(x) = \mu^y(y) = 1$ and $B_x = \text{supp}(\mu^x)$ and $B_y = \text{supp}(\mu^y)$ are disjoint. Fix $v^x, v^y \in V$ and $\delta_{xy} \in \mathbb{R}^n$. It suffices to prove that there is a profile $v \in V$ such that $d_v(x, y) = \delta_{xy}$, $d_v(a, x) \ll d_{v^x}(a, x)$ for all $a \in A \setminus (\{x\} \cup B_y)$, and $d_v(a, y) \ll d_{v^y}(a, y)$ for all $a \in A \setminus (\{y\} \cup B_x)$.

Choose $\delta_x, \delta_y \in \mathbb{R}_+^n$ such that

$$v^x(x) - v^y(y) + \delta_x - \delta_y = \delta_{xy}.$$

Since V is $\mathcal{U}$ -perturbative there are $v_1^x, v_2^y \in V$ such that $v_1^x(x) = v^x(x) + \delta_x$, $v_2^y(y) = v^y(y) + \delta_y$, $v_1^x(a) < v^x(a) + \mu^x(a)\delta_x$ for all $a \in A \setminus \{x\}$, and $v_2^y(a) < v^y(a) + \mu^y(a)\delta_y$ for all $a \in A \setminus \{y\}$. Lemma 5.5 now gives a $v \in V$ such that

$$v(x) = v_1^x(x) = v^x(x) + \delta_x, \quad v(y) = v_2^y(y) = v^y(y) + \delta_y,$$

and:

$$\begin{aligned} v(a) &\ll v_1^x(a) && \text{for all } a \in B_x \setminus \{x\}; \\ v(a) &\ll v_2^y(a) && \text{for all } a \in B_y \setminus \{y\}; \\ v(a) &\ll v_1^x \wedge v_2^y(a) && \text{for all } a \in A \setminus (B_x \cup B_y). \end{aligned}$$

If $a \in A \setminus (B_x \cup B_y)$, then

$$v(a) \ll v_1^x \wedge v_2^y(a) \ll v^x \wedge v^y(a).$$

If $a \in B_x \setminus \{x\}$, then

$$v(a) \ll v_1^x(a) \ll v^x(a) + \mu^x(a)\delta_x.$$

Of course $v(x) = v^x(x) + \delta_x$, so in either case one has

$$d_v(a, x) \ll d_{v^x}(a, x).$$

Reversing x and y in this argument, we find that $d_v(a, y) \ll d_{v^y}(a, y)$ for all $a \in A \setminus (\{y\} \cup B_x)$, as desired. $\square$

5.2. RESOLVING LATTICES

One of our major concerns is the possibility that A be a compact metric space and each V_i the space of all continuous real valued functions on A . In this subsection we explain the details of the verification $\mathcal{U}$ -comprehensiveness for this case in a way that allows somewhat more generality (in the next subsection we show directly that the space of smooth valuations is $\mathcal{U}$ -comprehensive).

Definition 5.7. A function $\nu^x \in \mathcal{V}(A)$ is a *resolving function* for $x \in A$ if $\nu^x(x) \neq \nu^x(a)$ for all $a \in A \setminus \{x\}$. A subset $\mathcal{R}$ of $\mathcal{V}(A)$ is a *resolving set* if it contains a resolving function for each $x \in A$.

For example, the distance from a point in a metric space to x is a continuous resolving function for x . Recall that $\mathcal{L} \subset \mathcal{V}(A)$ is a *lattice* if $\nu \vee \nu', \nu \wedge \nu' \in \mathcal{L}$ for all $\nu, \nu' \in \mathcal{L}$.

Definition 5.8. A subset $\mathcal{L}$ of $\mathcal{V}(A)$ is called a *resolving lattice* if:

- (a) $\mathcal{L}$ is a linear subspace of $\mathcal{V}(A)$;
- (b) $\mathcal{L}$ contains all constant functions;
- (c) $\mathcal{L}$ is a lattice;
- (d) $\mathcal{L}$ is a resolving set.

Henceforth $\mathcal{L}$ will denote a given resolving lattice. Note that for any subset B of A , the space of restricted functions $\{\nu|_B : \nu \in \mathcal{L}\}$ is also a resolving lattice. We assume that $\mathcal{L}$ contains a separating family $\mathcal{U}$. As we mentioned earlier, this is certainly the case when $\mathcal{L}$ is the set of all continuous functions on a T_1 normal space. In the next two results we consider a fixed $i \in N$.

Proposition 8. *If $V_i = \mathcal{L}$, then V_i is combinative.*

Proof. To prove (C1) suppose that $v_i^x, v_i^y \in V_i$ and that $x, y \in A$ are distinct. Since $\mathcal{U}$ is a separating family there are $\mu^x, \mu^y \in \mathcal{L}$ taking values in $[0, 1]$ with disjoint supports and $\mu^x(x) = \mu^y(y) = 1$. If we let $\delta_i \geq v_i^x(x) - v_i^y(x), v_i^y(y) - v_i^x(y)$, then

$$v_i = [v_i^x + \delta_i \mu^y] \wedge [v_i^y + \delta_i \mu^x]$$

belongs to V_i and has the desired properties.

To prove (C2) suppose that $v_i^x, v_i^y, v_i^z \in V_i$ and that $x, y, z \in A$ are distinct. Since $\mathcal{U}$ is a separating family there are $\mu^x, \mu^y, \mu^z \in \mathcal{L}$ taking values in $[0, 1]$ with pairwise disjoint supports and $\mu^x(x) = \mu^y(y) = \mu^z(z) = 1$. If $\delta_i \geq 0$ is sufficiently large, then

$$v_i = [v_i^x + \delta_i(\mu^y + \mu^z)] \wedge [v_i^y + \delta_i(\mu^x + \mu^z)] \wedge [v_i^z + \delta_i(\mu^x + \mu^y)]$$

belongs to V_i and has the desired properties.

Note that (C3) and (C4) holds simply because V_i is a lattice. $\square$

In the following result and below we abuse notation, letting a scalar also denote the constant function with that particular value. We also let ν^+ and $|\nu|$ denote $\nu \vee 0$ and $\nu^+ + (-\nu)^+$, respectively.

Proposition 9. *If $V_i = \mathcal{L}$, then V_i is $\mathcal{U}$ -perturbative.*

Proof. Consider $x \in A$, $v_i^x \in V_i = \mathcal{L}$, $\mu \in \mathcal{L}$ mapping A to $[0, 1]$, and $\delta_i \in \mathbb{R}_+$. We know there is a resolving function $\nu^x \in \mathcal{L}$ such that $\nu^x(x) \neq \nu^x(a)$ for all $a \in A$, $a \neq x$. Let $\nu = |\nu^x - \nu^x(x)|$. Then i 's valuation

$$v_i = v_i^x + \delta_i \mu - \nu$$

is in V_i and has the desired properties. $\square$

Corollary 5.9. *If $V_i = \mathcal{L}$ for all $i \in N$, then V is flexible and allows elevation of pairs.*

If A is a topological space, $C(A)$ is the space of all continuous functions on A .

Corollary 5.10. *If A is a topological space such that there exists a continuous resolving function ν^x for each $x \in A$ and $C(A)$ contains a separating family, then $V = C(A)^n$ is $C(A)$ -comprehensive.*

If A is a metric space with metric ρ , functions of the form

$$\nu_\alpha^x : a \mapsto \max\{0, 1 - \alpha\rho(x, a)\}$$

show that $C(A)$ is a resolving set, and they can also obviously be used to construct a separating family. The advantage of this indirect approach is that one can also look at proper subsets of $C(A)$.

Example 5.11. Let $\mathcal{R}$ be a resolving set. Let $\mathcal{L}$ be the space of all functions of the form

$$\max_{j=1}^k \min_{j'=1}^{k'} \alpha_{jj'} + \nu_{jj'},$$

where k and k' are two natural numbers, each $\alpha_{jj'}$ is a real number, and each $\nu_{jj'}$ is a linear combination of functions in $\mathcal{R}$. Then $\mathcal{L}$ is a resolving lattice.

For a concrete example, suppose that A is a convex subset of a finite dimensional Euclidean space $\mathbb{R}^m$. A function $\nu : A \rightarrow \mathbb{R}$ is said to be piecewise affine if there is a finite number $\nu^1, \dots, \nu^k$ of affine functions from A to $\mathbb{R}$ such that for each $a \in A$ we have $\nu(a) = \nu^j(a)$ for some $j = 1, \dots, k$. Let $\mathcal{P}(A)$ be the space of continuous piecewise affine functions on A .

Corollary 5.12. *If A is a convex subset of $\mathbb{R}^m$, for some positive integer m , then $V = \mathcal{P}(A)^n$ is $\mathcal{P}(A)$ -comprehensive.*

It turns out that resolving lattices of valuations are uninteresting when A is finite.

Lemma 5.13. *If A is finite, then $\mathcal{L}$ is all of $\mathbb{R}^A$.*

Proof. Fix $x \in A$ and let $\nu^x \in \mathcal{L}$ be a resolving function for x . There are numbers c_1 and c_2 such that $c_1 < \nu^x(x) < c_2$ and for all $a \in A \setminus \{x\}$, either $\nu^x(y) < c_1$ or $c_2 < \nu^x(y)$. The function $(\nu^x - c_1)^+ \wedge (c_2 - \nu^x)^+$ is in $\mathcal{L}$, because $\mathcal{L}$ contains all constant functions and is a lattice, and it is nonzero at x and nowhere else. Since x was arbitrary, $\mathcal{L}$ contains a basis of $\mathbb{R}^A$. $\square$

5.3. SMOOTH VALUATIONS

Fix an order of differentiability $1 \leq r \leq \infty$. We now assume that A is a compact C^r manifold, and as usual we let $C^r(A)$ denote the space of all C^r real valued functions on A . We will often describe elements of $C^r(A)$ as *smooth* functions. The closure of $C^r(A)$ in the topology of uniform convergence is all of $C(A)$, so there is little economic content in assuming that valuations are smooth, but the ability to use such functions might be technically convenient in some contexts. We point out that this is an example in which the set of valuations is *not* a lattice, and it illustrates the additional generality obtained by not collapsing (C3) and (C4) into a single condition asserting that for any $x, y \in A$ and $v_i^x, v_i^y \in V_i$ such that there is a $v_i \in V_i$ such that $v_i \leq v_i^x \wedge v_i^y$ with $v_i(x) = v_i^x \wedge v_i^y(x)$ and $v_i(y) = v_i^x \wedge v_i^y(y)$.

For any $x \in A$ and neighborhood B of x , standard constructions (cf. Section 2.2 of Hirsch (1976)) give an element of $C^r(A)$ whose range is $[0, 1]$ that attains the value 1 uniquely at x and whose support is contained in B . Let $\mathcal{U}$ be a subset of $C^r(A)$ whose elements map into $[0, 1]$, and which contains such a function for each x and U . Of course $\mathcal{U}$ is a separating family.

Lemma 5.14. *If $V_i = C^r(A)$, then V_i is $\mathcal{U}$ -perturbative.*

Proof. Consider $x \in A$, $v_i^x \in V_i$, $\mu \in \mathcal{U}$, and $\delta_i \in \mathbb{R}$. Let ν be an element of $C^r(A)$ with $\nu(x) = 0$ and $\nu(a) < 0$ for all $a \neq x$. It suffices to set $v_i = v_i^x + \delta_i \mu + \nu$. $\square$

Lemma 5.15. *If $V_i = C^r(A)$, then V is combinative.*

Proof. (C1) Fix $v_i^x, v_i^y \in V$ and distinct alternatives $x, y \in A$. Let B_x, B_y be neighborhoods of x and y , respectively, whose closures are disjoint. Let c_i be a constant such that $c_i \leq v_i^x \wedge v_i^y$ for all $a \in A$ (since A is compact and all functions are continuous, such a c_i exists) and let $\nu: A \rightarrow [0, 1]$ be a smooth function such that $\nu(a) = 1$ if $a = x, y, z$, $\nu(a) < 1$ otherwise, and $\nu(a) = 0$ for all $a \notin (B_x \cup B_y \cup B_z)$. Let v_i be defined on A by

$$v_i(a) = \begin{cases} (1 - \nu(a))c_i + \nu(a)v_i^x(a), & \text{if } a \in B_x, \\ (1 - \nu(a))c_i + \nu(a)v_i^y(a), & \text{if } a \in B_y, \\ c_i, & \text{otherwise.} \end{cases}$$

Then v_i agrees with $(1 - \nu(a))c_i + \nu(a)v_i^x(a)$ on $A \setminus \overline{B}_y$, so it is smooth on this set, and it is also smooth on $A \setminus \overline{B}_x$, so it is smooth on all of A . Obviously we have $v_i(x) = v_i^x(x)$, $v_i(y) = v_i^y(y)$, and $v_i \leq v_i^x \vee v_i^y$.

(C2) With obvious modifications, the proof is the same as for (C1).

(C3) Let $v_i^x, v_i^y \in V$ and $x, y \in A$ be such that $v_i^x(x) \neq v_i^y(x)$ and $v_i^x(y) \neq v_i^y(y)$. Let B_x and B_y be disjoint neighborhoods of x and y , respectively, such that the sign of $v_i^x(a) - v_i^y(a)$ coincides with the sign of $v_i^x(x) - v_i^y(x)$ for all a in the closure of B_x , and similarly the sign of $v_i^x(a) - v_i^y(a)$ coincides with that of $v_i^x(y) - v_i^y(y)$ for all a in the closure of B_y . Let δ_i be a constant such that $\delta_i \leq v_i^x \wedge v_i^y$. There is a smooth function $\nu: A \rightarrow [0, 1]$ such that $\nu(x) = \nu(y) = 1$, $\nu(a) < 1$ if $a \neq x, y$,

and $\nu(a) = 0$ if $a \notin B_x \cup B_y$. (One can take a sum of two appropriate elements of $\mathcal{U}$.) Let v_i be the function

$$v_i(a) = (1 - \nu(a))\delta_i + \nu(a) \min\{v_i^x(a), v_i^y(a)\}.$$

Then $v_i \leq v_i^x \wedge v_i^y$ with $v_i(x) = v_i^x \wedge v_i^y(x)$ and $v_i(y) = v_i^x \wedge v_i^y(y)$. This function agrees with either $(1 - \nu(a))\delta_i - v_i^x(a)$ or $(1 - \nu(a))\delta_i - v_i^y(a)$ on a neighborhood of the closure of B_x , and also on a neighborhood of the closure of B_y , and it is constant on $A \setminus (B_x \cup B_y)$, so it is smooth.

(C4) For any finite collection $\{v_i^x\}_{x \in S}$ there is some constant function δ_i such that $\delta_i \leq \bigwedge_{x \in S} v_i^x$. $\square$

Combining the last two results for all i gives:

Proposition 10. *If, for every agent $i \in N$, $V_i = C^r(A)$, then V is $\mathcal{U}$ -comprehensive.*

6. COUNTEREXAMPLES

In this section we discuss examples that clarify the extent to which the various implications that constitute Theorem 1 do not hold in domains that are not comprehensive. Example 6.1, inspired by Example S3 in Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan, and Sen (2006a), considers an auction-like environment without externalities to show that truthful implementation does not imply monotonicity in differences in every domain V . It also illustrates that the affine maximization property of an SCF does not necessarily follow from monotonicity in differences without flexibility. Example 6.2 shows an environment in which the affine maximization property does not follow from truthful implementation, even though the preference domain is partially ordered. This example is somewhat similar to the one in Mishra and Sen (2011), which they attribute to Meyer-ter-Vehn and Moldovanu⁸.

Example 6.1. There are three units of a good to be allocated among agents with unit demands. We use $o_i = \mathbf{y}$ to indicate that $i \in N = \{1, 2, 3\}$ receives one unit of the good, and $o_i = \mathbf{n}$ to indicate otherwise. Thus, an alternative is $a = o_1 o_2 o_3$ and $A = \{o_1 o_2 o_3 : o_i = \mathbf{y}, \mathbf{n}, i \in N\}$. Agent i 's domain V_i contains all vectors $v_i \in \mathbb{R}^{|A|}$ satisfying:

$$v_i(o_1 o_2 o_3) = \begin{cases} \alpha_i, & \text{if } o_i = \mathbf{y}, \\ 0, & \text{if } o_i = \mathbf{n}; \end{cases}$$

where $\alpha_i \in \mathbb{R}$ for each $i = 1, 2, 3$. Note v_i is not a split function (cf. Section 4).

We first give an SCF that is truthfully implementable but not monotonic in differences, which shows that Proposition 4 does not hold without the assumption of elevation of pairs. Define $f_1: V \rightarrow A$ by specifying that $f_1(v) = o_1 o_2 o_3$, with $o_i = \mathbf{y}$ if and only if $\alpha_i > \alpha_j + \alpha_k - 10$, where j and k denote the two other agents. Immediately, f_1 is truthfully implemented using the payment rule $p: V \rightarrow A$ defined by $p_i(v) = \alpha_j + \alpha_k - 10$ if $f_1(v)$ satisfies $o_i = \mathbf{y}$ and $p_i(v) = 0$ otherwise. However, f_1 is not monotonic in differences. To see this, let $v, v' \in V$ be two profiles such that for each i , $\alpha_i = 9$ and $\alpha'_i = 11$. Clearly, $f_1(v) = \mathbf{yyy}$ whereas $f_1(v') = \mathbf{nnn}$, which implies $d_{v_i}(f_1(v), f_1(v')) = 9 < 11 = d_{v'_i}(f_1(v), f_1(v'))$ for all $i \in N$.

Our second SCF is monotonic in differences but not an affine maximizer. Since its image has more than two elements, this shows that Proposition 5 requires the

⁸Private communication.

assumption of flexibility. Define $f_2: V \rightarrow A$ by $f_2(v) = o_1 o_2 o_3$, where $o_i = \mathbf{y}$ if and only if $\alpha_i < \alpha_j + \alpha_k - 10$, for $i = 1, 2$, and $o_3 = \mathbf{y}$ if and only if $\alpha_3 > 0$. It is easy to see that f_2 is not an affine maximizer. Monotonicity in differences is however satisfied. Choose $v, v' \in V$ arbitrarily, and let $f_2(v) = o_1 o_2 o_3$ and $f_2(v') = o'_1 o'_2 o'_3$. If $o_3 = o'_3$, then immediately one has

$$d_{v_3}(f_2(v), f_2(v')) = 0 = d_{v'_3}(f_2(v), f_2(v')).$$

If, on the other hand, $o_3 = \mathbf{y}$ and $o'_3 = \mathbf{n}$, so that $\alpha_3 > 0 \geq \alpha'_3$, then

$$d_{v_3}(f_2(v), f_2(v')) = \alpha_3 > \alpha'_3 = d_{v'_3}(f_2(v), f_2(v')).$$

Example 6.2. The alternative set is $A = \{w, x, y, z\}$. There are two agents, and the domain of valuation profiles is $V = V_1 \times V_2$ where:

$$\begin{aligned} V_1 &= \{v_1 \in \mathbb{R}^4 : 0 < v_1(w) \leq v_1(y), v_1(z) \leq v_1(x)\}, \\ V_2 &= \{v_2 \in \mathbb{R}^4 : 0 < v_2(w) \leq v_2(x) \leq v_2(y), v_2(z)\}. \end{aligned}$$

Thus V_1 is the set of positive valuations consistent with the partial order $x \succeq_1 y, z \succeq_1 w$, while V_2 is the set of positive valuations consistent with $y, z \succeq_2 x \succeq_2 w$. Notice that V is open, convex (hence connected) and unbounded from above, but it is not comprehensive.

Consider now the following subsets of V_1 and V_2 :

$$\begin{aligned} V_1^\circ &= \{v_1 \in V_1 : 0 < v_1(w) < \tfrac{1}{2}, v_1(y) = v_1(z) = v_1(w), v_1(x) = v_1(w) + \tfrac{1}{2}\}, \\ V_2^\circ &= \{v_2 \in V_2 : 0 < v_2(w) < \tfrac{1}{2}, v_2(x) = v_2(w), v_2(y) = v_2(z) = v_2(w) + \tfrac{1}{2}\}. \end{aligned}$$

Define the SCF $f: V \rightarrow A$ by $f(v) = w$ if and only if $v_1 \in V_1^\circ$ and $v_2 \in V_2^\circ$, and $f(v) = \arg \max\{v_1(a) + v_2(a) : a \in A, a \neq w\}$ otherwise. Clearly f is not an affine maximizer. It is however truthfully implementable. Indeed, using the Taxation Principle, we write the payment function p_i in terms of v_j and chosen alternative $a \in A$. Consider the payment scheme given by:

$$p_1(a, v_2) = \begin{cases} v_2(w), & \text{for } v_2 \in V_2^\circ, a = w, y, z, \\ v_2(w) + \tfrac{1}{2}, & \text{for } v_2 \in V_2^\circ, a = x, \\ -v_2(a), & \text{for } v_2 \in V_2 \setminus V_2^\circ, a \in A; \end{cases}$$

and similarly

$$p_2(a, v_1) = \begin{cases} v_1(w), & \text{for } v_1 \in V_1^\circ, a = w, x, \\ v_1(w) + \tfrac{1}{2}, & \text{for } v_1 \in V_1^\circ, a = y, z, \\ -v_1(a), & \text{for } v_1 \in V_1 \setminus V_1^\circ, a \in A. \end{cases}$$

The reader can verify without difficulty that this payment scheme truthfully implements f in dominant strategies.

7. PROOF OF PROPOSITION 5

We now prove Proposition 5, which is the most challenging aspect of Robert's theorem. The argument is modelled on one of the proofs given in Lavi, Mu'alem, and Nisan (2009). Throughout the section we assume that T is flexible, that $|f(T)| \geq 3$, and that f satisfies monotonicity in differences. We begin with some elementary consequences of flexibility.

Lemma 7.1. *For any distinct $x, y \in f(T)$ and any $\delta_{xy} \in \mathbb{R}^n$, there exists an profile $t \in T$ such that $d_t(x, y) = \delta_{xy}$ and $f(t) \in \{x, y\}$.*

Proof. Choose $t^x, t^y \in T$ such that $f(t^x) = x$ and $f(t^y) = y$. Then (F1) provides disjoint sets $B_x, B_y \subset A$ containing x and y , respectively, and a type profile $t \in T$ such that $d_t(x, y) = \delta_{xy}$ and:

- (a) $d_t(a, x) \ll d_{t^x}(a, x)$, for all $a \in A \setminus (\{x\} \cup B_y)$;
- (b) $d_t(a, y) \ll d_{t^y}(a, y)$, for all $a \in A \setminus (\{y\} \cup B_x)$.

If $f(t) \notin \{x, y\}$, then either $f(t) \in A \setminus (\{x\} \cup B_y)$ or $f(t) \in A \setminus (\{y\} \cup B_x)$, so that either (a) or (b) gives a failure of negative unanimity. $\square$

Lemma 7.2. *For any distinct $x, y, z \in f(T)$ and any $\delta_{xy}, \delta_{yz} \in \mathbb{R}^n$ there exists a type profile $t \in T$ such that $d_t(x, y) = \delta_{xy}$, $d_t(y, z) = \delta_{yz}$, and $f(t) \in \{x, y, z\}$.*

Proof. The argument follows the same pattern as the proof of Lemma 7.1, with (F2) in place of (F1), so it is left to the reader. $\square$

For distinct $x, y \in f(T)$, let

$$Q(x, y) = \{ \alpha \in \mathbb{R}^n : d_t(x, y) \ll \alpha \text{ for some } t \in T \text{ with } f(t) = x \}.$$

Lemma 7.3. *For any distinct $x, y \in f(T)$, $Q(x, y) \cap -Q(y, x) = \emptyset$.*

Proof. Suppose that $\alpha \in Q(x, y)$ and $-\alpha \in Q(y, x)$. Choose $t^x, t^y \in T$ such that $f(t^x) = x$, $f(t^y) = y$, $d_{t^x}(x, y) \ll \alpha$ and $d_{t^y}(y, x) \ll -\alpha$. Since $d_{t^y}(y, x) = -d_{t^y}(x, y)$, we have

$$d_{t^y}(x, y) \gg \alpha \gg d_{t^x}(x, y),$$

which is a violation of negative unanimity. $\square$

The next two results develop the pertinent consequences of flexibility.

Lemma 7.4. *For any distinct $x, y \in f(T)$ and $\alpha, \beta \in \mathbb{R}^n$ with $\alpha \gg \beta$, either $\alpha \in Q(x, y)$ or $\beta \in -Q(y, x)$.*

Proof. Lemma 7.1 gives a $t \in T$ such that $d_t(x, y) = \alpha$ and $f(t) \in \{x, y\}$. Therefore either $f(t) = x$ and thus $\alpha \in Q(x, y)$, or $f(t) = y$, in which case $-\beta \in Q(y, x)$ because $d_t(y, x) + (\alpha - \beta) = -\beta$ and $\alpha - \beta \gg 0$. $\square$

Lemma 7.5. *For any distinct $x, y, z \in f(T)$, $Q(x, y) + Q(y, z) \subseteq Q(x, z)$.*

Proof. Consider $\alpha \in Q(x, y)$ and $\beta \in Q(y, z)$. Choose t^x and t^y such that $f(t^x) = x$, $f(t^y) = y$, $d_{t^x}(x, y) \ll \alpha$, $d_{t^y}(y, z) \ll \beta$. Lemma 7.2 gives a profile $t \in T$ such that $d_t(x, y) \ll d_{t^x}(x, y)$, $d_t(y, z) \ll d_{t^y}(y, z)$, and $f(t) \in \{x, y, z\}$. Then negative unanimity implies that $f(t) \neq y$ and $f(t) \neq z$, so $f(t) = x$; since $d_t(x, z) = d_t(x, y) + d_t(y, z) \ll \alpha + \beta$, the result is proven. $\square$

Let $\mathbf{e} = (1, \dots, 1) \in \mathbb{R}^n$. For each distinct $x, y \in f(T)$, let

$$q(x, y) = \sup \{ s \in \mathbb{R} : -s\mathbf{e} \in Q(x, y) \}.$$

Lemma 7.6. *For any distinct $x, y \in f(T)$, $q(x, y) = -q(y, x)$ and*

$$-q(x, y)\mathbf{e} \notin Q(x, y) \cup -Q(y, x).$$

Proof. In view of Lemma 7.3, $-q(x, y)\mathbf{e} \ll q(y, x)\mathbf{e}$ is impossible, so $-q(x, y) \geq q(y, x)$. The inequality cannot be strict because then $(-\frac{1}{3}q(x, y) + \frac{2}{3}q(y, x))\mathbf{e}$ and $(-\frac{2}{3}q(x, y) + \frac{1}{3}q(y, x))\mathbf{e}$ would be in neither $Q(x, y)$ nor $-Q(y, x)$, which would contradict Lemma 7.4. In view of the definition of $Q(x, y)$ it is obvious that $-q(x, y)\mathbf{e} \notin Q(x, y)$ and $-q(y, x)\mathbf{e} \notin Q(y, x)$. $\square$

Lemma 7.7. $q \in \mathcal{D}(f(T))$.

Proof. Since $Q(x, y) + Q(y, z) \subseteq Q(x, z)$ and $Q(x, z) + Q(z, y) \subseteq Q(x, y)$, it must be the case that $q(x, y) + q(y, z) \leq q(x, z)$ and $q(x, z) + q(z, y) \leq q(x, y)$. But $q(z, y) = -q(y, z)$. Thus, $q(x, y) + q(y, z) = q(x, z)$, as desired. $\square$

For distinct $x, y \in f(T)$, let $Q^*(x, y) = Q(x, y) + q(x, y)\mathbf{e}$.

Lemma 7.8. For all distinct x, y, z in $f(T)$ we have:

- (a) $Q^*(x, y) + Q^*(y, z) \subseteq Q^*(x, z)$.
- (b) $Q^*(x, y) \cap -Q^*(y, x) = \emptyset$.
- (c) If $\alpha, \beta \in \mathbb{R}^n$ with $\alpha \gg \beta$, then $\alpha \in Q^*(x, y)$ or $\beta \in -Q^*(y, x)$.
- (d) $Q^*(x, y) = Q^*(x, z) = Q^*(y, z)$.

Proof. We have:

- (a) This follows from $Q(x, y) + Q(y, z) \subseteq Q(x, z)$ and $q(x, y) + q(y, z) = q(x, z)$.
- (b) This follows from $Q(x, y) \cap -Q(y, x) = \emptyset$ and $q(x, y) = -q(y, x)$.
- (c) Since $q(x, y) = -q(y, x)$, Lemma 7.4 implies that $\alpha - q(x, y)\mathbf{e} \in Q(x, y)$ or $\beta + q(y, x)\mathbf{e} \in -Q(y, x)$.
- (d) Since $q(y, z)$ is in the boundary of $Q(y, z)$, the origin is in the boundary of $Q^*(y, z)$, and $Q^*(x, y) + Q^*(y, z) \subseteq Q^*(x, z)$, so $Q^*(x, y) \subseteq \overline{Q^*(x, z)}$. If $\alpha \in Q^*(x, y)$, there is $\beta \in Q^*(x, y)$ with $\beta \ll \alpha$ and $\gamma \in Q^*(x, z)$ arbitrarily close to β . Therefore there is $\gamma \in Q^*(x, z)$ with $\gamma \ll \alpha$, which implies that $\alpha \in Q^*(x, z)$. This implies the first equality, and the same argument with the roles of $Q^*(x, y)$ and $Q^*(y, z)$ reversed gives the second. $\square$

Proof of Proposition 5. If $w, x, y, z \in f(T)$ with $x \neq y$ and $w \neq z$, then (d) implies that $Q^*(x, y) = Q^*(w, z)$ except when the four alternatives are distinct, and in that case applying (d) twice gives $Q^*(x, y) = Q^*(x, z) = Q^*(w, z)$. Therefore there is a single set Q^* such that $Q^*(x, y) = Q^*$ for all distinct x, y .

We claim that Q^* is convex. It suffices to show that $\frac{1}{2}\alpha + \frac{1}{2}\beta \in Q^*$ whenever $\alpha, \beta \in Q^*$ because for any $0 \leq s' \leq 1$ we can find s arbitrarily close to 1 such that

$$(1 - s')\alpha + s'\beta = (1 - r)\alpha + r(\alpha + s(\beta - \alpha))$$

where r is a integer multiple of 2^{-k} for some positive integer k , and $\alpha + s(\beta - \alpha) \in Q^*$ when s is sufficiently close to 1 because Q^* is open. In turn, since $Q^* + Q^* \subset Q^*$ it suffices to show that $\frac{1}{2}\alpha \in Q^*$ whenever $\alpha \in Q^*$. But if $\frac{1}{2}\alpha \notin Q^*$, then for $\beta \ll \frac{1}{2}\alpha$ we have $\beta \in -Q^*$ and $2\beta \in -Q^*$, which implies that α is in the closure of $-Q^*$, and this is impossible because Q^* is open.

Now the separating hyperplane theorem gives a nonzero $\sigma \in \mathbb{R}^n$ such that $\sigma \cdot \alpha > 0$ for all $\alpha \in Q^*$. Since $Q^* + \mathbb{R}_{++}^n \subset Q^*$, all the components of σ are nonnegative, so we can take $\sigma \in \Delta^{n-1}$. It must be the case that $\alpha \in Q^*$ whenever $\sigma \cdot \alpha > 0$ because otherwise $\beta \in -Q^*$ whenever $\beta \ll \alpha$, and for β close to α we would have $\sigma \cdot \beta > 0$, which is impossible. Thus $Q^* = \{\alpha \in \mathbb{R}^n : \sigma \cdot \alpha > 0\}$.

Suppose that $t \in T$ and $a \in f(T) \setminus \{f(t)\}$. Then $d_t(f(t), a)$ is in the closure of $Q(f(t), a)$, so $d_t(f(t), a) + q(f(t), a)\mathbf{e}$ is in the closure of Q^* , and consequently

$$0 \leq \sigma \cdot (d_t(f(t), a) + q(f(t), a)\mathbf{e}) = \sigma \cdot d_t(f(t), a) + q(f(t), a).$$

Thus f is an affine maximizer. $\square$

8. CONCLUDING REMARKS

We have provided a generalization of Roberts theorem: if the space of type profiles T allows elevation of pairs and is flexible, then an SCF f is truthfully implementable if and only if it is a constant function, binary implementable, or a lexicographic affine maximizer. A flexible domain that allows elevation of pairs can be smaller than the unrestricted quasi-linear domain, even with finite alternative sets. Indeed, we show that the related notion of comprehensiveness is satisfied if the set of alternatives A is a topological space and the domain of valuations is the space of continuous functions; or if A is a convex subset of a finite dimensional Euclidean space and the domain of valuations is the space of continuous piecewise affine functions; or if A is a compact differentiable manifold and the domain of valuations is the space of smooth functions. Applications of our results include standard models of public good provision and problems that consider private good allocations with externalities.

To what extent do our results allow economically significant generality beyond the case of all continuous valuations on a metric space? There are at least two directions in which additional generality can be pursued. If $C(A)$ contains a resolving set, then A must be a T_1 space, because continuity of the function ν^x implies that $\{x\}$ is closed. The metrization theorem (e.g., Kelley (1955), p. 127) states that a topological space is metrizable if and only if it is T_1 and regular (that is, every neighborhood of a point contains a closed neighborhood) and it has a base that is a countable union of collections of open sets that are locally finite. Thus there is some room for other sorts of topological spaces, which must be in some sense exotic, and at present there are no obvious applications in view.

Unfortunately we have not been able to fully assess the extent to which our results allow one to go beyond the space of all continuous valuations, in an economically interesting manner, by further restricting the agents' preferences. In particular, we do not know of a proper subset of the space of continuous valuations that is closed in the topology of uniform convergence, contains all constant functions, and is $\mathcal{U}$ -comprehensive, nor have we been able to prove that no such subspace exists. More generally, we do not know whether there are proper subspaces of the space of continuous differences that are closed in the topology of uniform convergence, flexible, and allow elevation of pairs.

Other characterizations of truthful implementability available in the literature are based on cyclic monotonicity à la Rochet (1987) or one of its weaker forms. If the set of alternatives is finite, then weak monotonicity characterizes truthful implementation in order-based, auction-like domains (Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan, and Sen (2006b)), convex domains (Saks and Yu (2005)), and monotone domains (Ashlagi, Braverman, Hassidim, and Monderer (2010)). With infinite allocation sets and quasi-linear preferences parameterized by multi-dimensional types,

weak monotonicity (in addition to an integral path-independence condition) is sufficient for truthful implementation when valuations are linear in types (Archer and Kleinberg (2008)), while integral weak monotonicity (plus the path-integrability condition) suffices when valuations are convex or differentiable in types (Berger, Müller, and Naeemi (2010)), or Lipschitz continuous in types (Carbajal and Ely (2011)).

These results do not provide a description, in terms of a functional form, of implementable choice functions, which is given by monotonicity in differences in comprehensive domains. It is known that non affine maximizer choice functions can be truthfully implemented in settings with single-dimensional type spaces (cf. Nisan (2007) and references therein; see also Section 2). Clearly, single dimensional type domains are not comprehensive. However, the relationship between monotonicity in differences and weak person-by-person monotonicity in multi-dimensional parametric, quasi-linear utility settings remains to be studied.

REFERENCES

- ARCHER, A., AND R. KLEINBERG (2008): “Truthful germs are contagious: a local-to-global characterization of truthfulness,” in *10th ACM Conference on Electronic Commerce*.
- ASHLAGI, I., M. BRAVERMAN, A. HASSIDIM, AND D. MONDERER (2010): “Monotonicity and implementability,” *Econometrica*, 78(5), 1749–1772.
- BABAIOFF, M., R. LAVI, AND E. PAVLOV (2005): “Mechanism design for single-value domains,” in *Proceedings of the 20th National Conference on Artificial Intelligence (AAAI-05)*, pp. 241 – 247.
- BARBERÀ, S., AND B. PELEG (1990): “Strategy-Proof Voting Schemes with Continuous Preferences,” *Social Choice and Welfare*, 7(1), 31–38.
- BERGER, A., R. MÜLLER, AND S. H. NAEEMI (2010): “Path-monotonicity and truthful implementation,” Discussion paper, Maastricht University.
- BIKHCHANDANI, S., S. CHATTERJI, R. LAVI, A. MU’ALEM, N. NISAN, AND A. SEN (2006a): “Supplement to Weak monotonicity characterizes deterministic dominant-strategy implementation,” *Econometrica* Supplement Material.
- (2006b): “Weak monotonicity characterizes deterministic dominant-strategy implementation,” *Econometrica*, 74(4), 1109–1132.
- BLACK, D. (1958): *The Theory of Committees and Elections*. Cambridge University Press, Cambridge, England.
- CARBAJAL, J. C., AND J. C. ELY (2011): “Mechanism design without revenue equivalence,” Discussion paper, University of Queensland and Northwestern University.
- DASGUPTA, P., P. HAMMOND, AND E. MASKIN (1979): “The implementation of social choice rules: some general results on incentive compatibility,” *Review of Economic Studies*, 46(2), 185–216.
- (1980): “On imperfect information and optimal pollution control,” *Review of Economic Studies*, 47(5), 857–860.
- DOBZINSKI, S., AND N. NISAN (2009): “A modular approach to Roberts’ theorem,” in *Algorithmic Game Theory: SAGT 2009 Proceedings*, ed. by M. Mavronicolas, and V. G. Papadopoulos, vol. 5814 of *Lecture Notes in Computer Science*, pp. 14–23, Berlin. Springer.
- DUGGAN, J., AND J. ROBERTS (2002): “Implementing the efficient allocation of pollution,” *American Economic Review*, 92(4), 1070–1078.
- GIBBARD, A. (1973): “Manipulation of voting schemes: a general result,” *Econometrica*, 41(4), 587–601.
- GREEN, J. R., AND J.-J. LAFFONT (1977): “Characterization of satisfactory mechanisms for the revelation of preferences for public goods,” *Econometrica*, 45(2), 427–438.
- (1979): *Incentives in Public Decision-Making*, Studies in Public Economics. North-Holland, New York, NY.

- HIRSCH, M. W. (1976): *Differential Topology*, Graduate Texts in Mathematics. Springer-Verlag, New York.
- JEHIEL, P., AND B. MOLDOVANU (2006): “Allocative and informational externalities in auctions and related mechanisms,” in *Advances in Economics and Econometrics: Theory and Applications, Ninth World Congress*, ed. by R. Blundell, W. K. Newey, and T. Persson, chap. 3. Cambridge University Press, Cambridge, UK.
- JEHIEL, P., B. MOLDOVANU, AND E. STACCHETTI (1999): “Multidimensional mechanism design for auctions with externalities,” *Journal of Economic Theory*, 85(2), 258–294.
- KELLEY, J. L. (1955): *General Topology*. Van Nostrand, New York.
- LAFFONT, J.-J., AND E. MASKIN (1980): “A differential approach to dominant strategy mechanisms,” *Econometrica*, 48(6), 1507–1520.
- LAVI, R., A. MU’ALEM, AND N. NISAN (2009): “Two simplified proofs for Roberts’ theorem,” *Social Choice and Welfare*, 32(3), 407–423.
- MASKIN, E. (1985): “The theory of implementation in Nash equilibrium: a survey,” in *Social Goals and Social Organization*, ed. by L. Hurwicz, D. Schmeidler, and H. Sonnenschein, chap. 6. Cambridge University Press, Cambridge, UK.
- MISHRA, D., AND S. ROY (2011): “Implementation in multidimensional dichotomous domains,” Discussion paper, Indian Statistical Institute.
- MISHRA, D., AND A. SEN (2011): “Roberts’ theorem with neutrality: a social welfare ordering approach,” Forthcoming: *Games and Economic Behavior*.
- MIYAGAWA, E. (2001): “House allocation with transfers,” *Journal of Economic Theory*, 100(2), 329 – 355.
- MONTERO, J.-P. (2008): “A simple auction mechanism for the optimal allocation of the commons,” *American Economic Review*, 98(1), 496–518.
- MOULIN, H. (1988): *Axioms of Cooperative Decision Making*, Econometric Society Monographs. Cambridge University Press, Cambridge, UK.
- (2010): “An efficient and almost budget balanced cost sharing method,” *Games and Economic Behavior*, 70(1), 107–131.
- MULLER, E., AND M. A. SATTERTHWAIT (1977): “The equivalence of strong positive association and strategy-proofness,” *Journal of Economic Theory*, 14(2), 412–418.
- NISAN, N. (2007): “Introduction to mechanism design (for computer scientists),” in *Algorithmic Game Theory*, ed. by N. Nisan, T. Roughgarden, E. Tardos, and V. V. Vazirani, chap. 9. Cambridge University Press, Cambridge, UK.
- ROBERTS, K. (1979): “The characterization of implementable choice rules,” in *Aggregation and Revelation of Preferences*, ed. by J.-J. Laffont, chap. 18. North-Holland, Amsterdam, The Netherlands.
- ROCHET, J.-C. (1987): “A necessary and sufficient condition for rationalizability in a quasi-linear context,” *Journal of Mathematical Economics*, 16(2), 191–200.
- ROZENSHTRUM, I. (1999): “Dominant strategy implementation with quasi-linear preferences,” Master’s thesis, Department of Economics, The Hebrew University of Jerusalem.
- SAKS, M., AND L. YU (2005): “Weak monotonicity suffices for truthfulness on convex domains,” in *7th ACM Conference on Electronic Commerce*.
- SATTERTHWAIT, M. A. (1975): “Strategy-proofness and Arrow’s conditions: existence and correspondence theorems for voting procedures and social welfare functions,” *Journal of Economic Theory*, 10(1), 187–217.
- VOHRA, R. V. (2011): *Mechanism Design: a Linear Programming Approach*, Econometric Society Monographs. Cambridge University Press.