Mechanism Design Without Revenue Equivalence*

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Abstract

We study mechanism design problems in quasi-linear environments where the envelope theorem and revenue equivalence principle fail due to non-convex and non-differentiable valuations. Despite these obstacles, we obtain a characterization of incentive compatibility based on the familiar Mirrlees representation of the indirect utility and a monotonicity condition on the allocation rule. These conditions pin down the *range* of possible payoffs as a function solely of the allocation rule, thus providing a revenue inequality. We illustrate the usefulness of our approach in three economic applications where standard techniques do not apply: we derive the optimal selling mechanism in a buyer-seller situation where the buyer has loss-averse preferences; we find a zero payment (hence budget-balanced) efficient mechanism in a public goods location model; and we consider a principal–agent model with ex post non-contractible actions available to the agent.

Keywords: Incentive Compatibility, Revenue Equivalence, Integral Monotonicity, Revenue Maximization, Loss Aversion, Efficiency, Public Goods, Non-contractible Actions.

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1 Introduction

Revenue equivalence states that two (dominant strategy[1]) incentive compatible mechanisms with the same allocation rule generate utilities for the agent and payments to the planner

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[1] We study single-agent, dominant strategy incentive compatible mechanisms. This is for expositional clarity. It is straightforward to generalize our results to many agents and to Bayesian incentive compatibility.

that differ at most by a constant. The revenue equivalence principle is very useful in simplifying mechanism design problems in a variety of settings, as first shown by Myerson (1981), and a large literature has studied the breadth of conditions under which it holds.[2] Instead, in this paper we are interested in situations where revenue equivalence, as traditionally formulated, may fail due to the absence of linearity, convexity and differentiability assumptions on the valuation function with respect to types. The following example serves as a simple illustration.

Example 1. The allocation set is $\mathscr{X} = [0, 1]$. The agent has a quasi-linear utility function $u = v(x, \theta) - \rho$ defined over alternatives $x \in \mathscr{X}$ and monetary payments $\rho \in \mathbb{R}$. Types are private information and lie in $\Theta = [0, 1]$. The agent's valuation function $v\colon \mathscr{X} \times \Theta \to \mathbb{R}$ is given by

$$v(x, \theta) \;=\; \begin{cases} \theta x, & \text{if } x \le \theta; \\ 2\theta^2 \,-\, \theta x, & \text{if } x > \theta. \end{cases}$$

Think of x as the quantity traded of some good. The agent has positive marginal utility for the first θ units, and negative marginal utility for additional amounts. Note that for each x, the function $\theta \mapsto v(x, \theta)$ is neither convex nor fully differentiable on Θ. In particular, its partial derivative with respect to θ fails to exist whenever $\theta = x$.

The efficient allocation rule $X^*\colon \Theta \to \mathscr{X}$ selects quantity $X^*(\theta) = \theta$. Clearly, X^* is implementable by a constant payment rule $p \equiv 0$. In this case, the direct mechanism (X^*, p) generates zero revenue and an indirect utility function for the agent equal to $U(\theta) \equiv v(X^*(\theta), \theta) - p(\theta) = \theta^2$. However, revenue equivalence fails. Consider for instance the alternative payment rule p' defined by $p'(\theta) = \theta^2/2$, for all $\theta \in \Theta$. To see that (X^*, p') is incentive compatible, fix a type $\theta \in \Theta$. Reporting $\hat{\theta} \in \Theta$ generates payoffs equal to

$$v(X^*(\hat{\theta}), \theta) \,-\, p'(\hat{\theta}) \;=\; \begin{cases} \theta\hat{\theta} \,-\, \hat{\theta}^2/2, & \text{if } \hat{\theta} \le \theta; \\ 2\theta^2 \,-\, \theta\hat{\theta} \,-\, \hat{\theta}^2/2, & \text{if } \hat{\theta} > \theta. \end{cases}$$

This expression is maximized when $\hat{\theta} = \theta$. In this mechanism, type $\theta = 0$ obtains zero utility, just as in (X^*, p), and yet (X^*, p') generates strictly positive revenues and an indirect utility function U' equal to $U'(\theta) = \theta^2/2$. □

The problem in the above example is caused by the downward kink exhibited by the valuation function at $\theta = x$. Since the outcome being implemented is $X^*(\theta) = \theta$, there is a continuum of types at which the agent is placed at the kink point of his valuation. This causes the failure of revenue equivalence. At an abstract level, if downward kinks occur for a small number of types, it would seem that Example 1 shall be treated as a mathematical pathology, which can be safely assumed away in practical models. We argue that since the

[2] Berger, Müller, and Naeemi (2010) build on the envelope theorem of Milgrom and Segal (2002) to establish revenue equivalence when the type space is a convex subset of a multi-dimensional Euclidean space and valuations are convex or differentiable in types. See also Krishna and Maenner (2001), Jehiel, Moldovanu, and Stacchetti (1996, 1999), Williams (1999), Heydenreich, Müller, Uetz, and Vohra (2009) and Chung and Olszewski (2007). Holmström (1979) and Carbajal (2010) develop an alternative method to revenue equivalence for efficient allocation rules.

allocation rule is a choice of the planner, economic considerations often suggest otherwise. When valuations have downward kinks, precisely the violations of convexity and differentiability that impede the standard version of revenue equivalence, clusters of allocations at the kinks can occur because of efficiency or optimality considerations: that is often how payoffs or revenue are maximized. On the other hand, kinks in the valuation function are not just modeling pathologies but may capture crucial considerations. They may appear if the valuation is itself the value function of a decision problem that the agent faces after the social allocation has been resolved. Also, in many important behavioral models, kinks express the essential property under focus: in Kőszegi and Rabin's (2006) theory of endogenous reference points, in Gul's (1991) model of disappointment aversion, and in Fehr and Schmidt's (1999) model of inequality aversion, a kink in preferences represents a qualitative difference between gains and losses and differential treatment of small-stakes variation versus large-stakes variation.

One wishes to extend mechanism design theories of efficient public good provision, optimal selling mechanisms, insurance, etc., to these behavioral models. Yet without revenue equivalence one may lose the analytical convenience of a tractable representation of the set of incentive feasible mechanisms. The main goal of this paper is to show how to characterize incentive compatibility when the standard version of revenue equivalence and the envelope theorem fail. We also provide economic applications to demonstrate how this characterization can be employed analogously to the traditional one to study a variety of mechanism design problems. Applied to Example 1, we can show that U is an indirect utility function arising from an incentive compatible mechanism with allocation rule X^* if, and only if,

$$U(\theta) \;=\; U(0) + \int_0^\theta s(\tilde{\theta})\, d\tilde{\theta}, \qquad \text{for all } \theta \in \Theta, \tag{1}$$

for some function $\theta \mapsto s(\theta)$ such that

$$s \text{ is an integrable selection from the correspondence } S(\theta) = [\theta, 3\theta] \tag{2}$$

satisfying, for all $\theta, \hat{\theta} \in \Theta$, the following inequalities

$$v(X^*(\theta), \theta) - v(X^*(\theta), \hat{\theta}) \;\geq\; \int_{\hat{\theta}}^{\theta} s(\tilde{\theta})\, d\tilde{\theta} \;\geq\; v(X^*(\hat{\theta}), \theta) - v(X^*(\hat{\theta}), \hat{\theta}). \tag{3}$$

Equation 1 is the familiar *Mirrlees representation* of incentive compatible payoffs. Equation 2 generalizes the envelope theorem derivation of the integrand $\theta \mapsto s(\theta)$. In models with differentiable valuations, $s(\theta)$ would be given by the partial derivative of $v(X^*(\theta), \tilde{\theta})$ with respect to $\tilde{\theta}$, evaluated at $\tilde{\theta} = \theta$. This requirement is modified to allow $s(\theta)$ to range between the right and left subderivatives[3] of $v(X^*(\theta), \tilde{\theta})$ at $\tilde{\theta} = \theta$, which in this case are θ and 3θ, respectively. Finally, as is always the case, some form of monotonicity of the allocation rule is necessary for incentive compatibility. Equation 3, the *integral monotonicity condition*, is necessary and, together with the previous two conditions, also sufficient.

[3]Mathematical concepts and results used in this paper are presented in Section 2.2.

The theoretical developments are presented in Section 3. Our first result, Theorem 1 in Section 3.1, extends the above characterization of incentive compatibility to a mechanism design setting with quasi-linear preferences, a convex, possibly multi-dimensional type space, and a measurable space of allocations. In place of differentiability, convexity, or absolute continuity, we assume that the valuation function satisfies a uniform Lipschitz condition with respect to types. This condition, along with an additional technical restriction, allows us to employ integrable selections of the subderivative correspondence (cf. Equation 2) in place of the usual envelope derivation. In multi-dimensional settings, a closed-path integrability condition appears in our characterization of incentive compatible mechanisms, in addition to the three conditions mentioned above. In Section 3.2 we derive a *revenue inequality* in the spirit of the standard revenue equivalence: Theorem 2 shows that, after normalizing the utility of the "lowest type", the difference in equilibrium payoffs generated by two incentive compatible mechanisms with the same allocation rule is bounded, with bounds depending solely on the allocation rule. Using our version of the Mirrlees representation of indirect utilities, we also show that *any* incentive compatible payment rule has a simple representation based on the subderivative correspondence. It follows that the set of normalized equilibrium payoffs is convex: if two payment rules implement an allocation rule and generate indirect utility functions U and U', respectively, then for every convex combination U'' of U and U', there exists a payment function that implements the allocation rule and generates an indirect utility equal to U''.

In Section 4 we apply our results to three relevant economic settings: the design of an optimal selling mechanism when the buyer is loss-averse; the design of efficient mechanisms in a public goods setting where, in addition to allocative efficiency, minimal external transfers are desirable; and a principal–agent model enriched with a posterior stage in which the agent makes discrete, non-contractible choices. In the buyer-seller model, the buyer has loss-averse preferences à la Kőszegi and Rabin (2006) which, due to a kink on the valuation at the reference point, fail standard requirements of differentiability or convexity. Nonetheless, we expand Myerson's (1981) techniques to this situation, thus reformulating the seller's problem in terms of virtual surplus. The fact that, at the optimal selling rule, kinks occur for a interval of types plays an important role in providing extra flexibility for the construction of optimal pricing scheme. Compared to the case without loss aversion, high intermediate types have their quantities distorted downward to exploit the loss aversion of even higher types, while the seller exploits the loss aversion of lower intermediate types by making them pay a premium to increase their quantity from zero to their reference points.

Section 4.2 considers a two-agent public good provision problem. It is well understood that, when revenue equivalence holds, an efficient allocation rule is implementable if and only if it is implementable via a Vickrey-Clarke-Groves (VCG) payment rule. We consider a location model for a public good in a square city where the residents, whose location is non-verifiable, face linear transportation costs. In this model, any allocation rule that selects a location with coordinates that are convex combinations of the coordinates of the residents' locations is efficient. One can show that, regardless of which efficient allocation rule is chosen by the central planner, no VCG payment scheme balances the budget ex post. However, revenue equivalence fails for efficient allocation rules that select "corner locations" (i.e., extreme points of the line segment formed by the residents location in each coordinate of the square city), so that in these cases there exist non-VCG incentive compatible payment rules. We characterize all incentive compatible payment rules for one of these corner allocation rules, and show that there exists one that specifies zero payments from the agents ex post. The corresponding VCG payments, in contrast, require the largest external subsidies. As before, the extra flexibility is afforded by the fact that, at the efficient locations selected by corner allocation rules, one agent has kinks in her valuation.

In Section 4.3 we expand the basic principal–agent framework by allowing the agent to make discrete choices after interaction with the principal has concluded. This may happen for instance if the principal, a wholesale supplier, is selling a productive input to the agent, a retail firm, who then has the option to enter one of several new consumer markets. When these posterior actions are non-contractible, the contracts offered by the principal cannot be contingent on the agent's choices in the ex post stage. The agent's induced valuation function, which is the value function coming from the ex post decision problem, may fail to be linear, convex or differentiable without further assumptions on the ex post payoff function for the agent. Nevertheless, we show that the induced valuation satisfies all the requirements of our characterization theorem, and in this class of problems, the standard version of revenue equivalence holds for any implementable allocation rule considered by the principal.

Our results rely on a generalization of the envelope theorem of Milgrom and Segal (2002), where the differentiability of the valuation with respect to types is replaced with a uniform Lipschitz condition. We make an additional technical assumption to obtain sufficiency, as well as necessity, of the integral representation of the indirect utility under incentive compatible allocation rules. The role of this assumption is discussed after the proof of Theorem 1. In Section 5 we show that, with the inclusion of a minor measurability hypothesis on the right and left subderivatives of the valuation function, it is satisfied whenever the valuation is either concave, convex, or differentiable with respect to types. We also prove that revenue equivalence (and the standard envelope theorem) is restored when valuations are convex or differentiable, or whenever the allocation set is finite. As the first two applications of Section 4 illustrate, concavity of the valuation with respect to types does not necessarily restore revenue equivalence. Section 6 ends this paper with some concluding remarks.

Related literature on incentive compatibility

Integral monotonicity is connected to the variety of monotonicity conditions studied in the literature since Rochet (1987), who characterized incentive compatibility via cyclic monotonicity. Saks and Yu (2005), Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan, and Sen (2006), and Ashlagi, Braverman, Hassidim, and Monderer (2010) investigate conditions under which a weaker form of cyclic monotonicity, namely 2-cycle monotonicity, is sufficient and necessary for incentive compatibility in environments with finitely many allocations. For infinite allocation sets, Archer and Kleinberg (2008) extend the insights of Jehiel, Moldovanu, and Stacchetti (1999) and characterize implementable allocation rules via 2-cycle monotonicity (plus a standard closed-path integrability condition) in environments with multi-dimensional convex type spaces and valuations that are linear in types. Independently of us, Berger, Müller, and Naeemi (2010) study environments with convex or differentiable valuations, where revenue equivalence holds, and obtain a characterization of incentive compatibility using integral monotonicity (which they call path monotonicity).

In related but independent work, Kos and Messner (2011) and Rahman (2011) study incentive compatibility in more general settings. Rahman (2011) uses linear programming and duality to characterize incentive compatible allocations in terms of detectable deviations. Kos and Messner (2011) extend Rochet's (1987) cyclic monotonicity condition to characterize the (pointwise) largest and (pointwise) smallest payment rules that implement a given allocation function, which differ when revenue equivalence fails, and apply these "extremal transfers" to study problems with endogenous participation constraints, and with a budget-balancing requirement. The explicit formulas for the boundaries of incentive compatible payment schemes obtained by Kos and Messner (2011) do not rely on any topological or ordering structure on the environment. Relative to these papers, we impose additional structure on the type space (notably convexity) and the valuation function (notably uniform Lipschitz continuity) and in return obtain sharper characterizations of both incentive compatible allocation rules and payment rules (including extremal payment rules), which are more immediately suitable for applications.

2 Preliminaries

2.1 The design setting

We consider a single-agent mechanism design setting; extensions to multi-agent settings are straightforward (see Section 4.2). An outcome (x, ρ) is composed of an alternative x that belongs to the *allocation set* $\mathscr{X}$, and a real number ρ representing some quantity of a perfectly divisible commodity (money). Our agent has *quasi-linear preferences*, so that

$$u(x, \theta, \rho) \;=\; v(x, \theta) \,-\, \rho$$

represents the agent's utility when $x \in \mathscr{X}$ is selected and amount $\rho \in \mathbb{R}$ is paid, given her privately known type θ. We denote the agent's *type space* by Θ and refer to $v\colon \mathscr{X} \times \Theta \to \mathbb{R}$ as the *valuation function.*

The following assumptions are made throughout this paper.

(A1) The pair $(\mathscr{X}, \mathcal{M})$ is a measurable space ($\mathcal{M}$ denotes a σ-algebra of subsets of $\mathscr{X}$).

(A2) The type space Θ is a convex, bounded subset of $\mathbb{R}^k$ $(k \geq 1)$.

(A3) The family of functions $\{\theta \mapsto v(x, \theta) \mid x \in \mathscr{X}\}$ is equi-Lipschitz on Θ: for every $x \in \mathscr{X}$, there is a positive number $\ell(x)$ such that $|v(x,\theta) - v(x,\hat{\theta})| \,\leq\, \ell(x)\, \|\theta - \hat{\theta}\|$, for all $\theta, \hat{\theta} \in \Theta$, and further $\sup\{\ell(x) \mid x \in \mathscr{X}\} = \ell < +\infty$.

(A1) does not impose any burdensome restriction on the allocation set, which could be finite or infinite. The convexity of the type space in (A2) is a standard assumption and is satisfied in several economic applications.[4] (A3) is used to prove the Lipschitz continuity of the indirect utility function. It is not possible to dispense with the boundedness assumption on the set of Lipschitz constants when $\mathscr{X}$ is infinite.[5] In addition, (A3) allows us to work with the right and left subderivatives of the valuation function with respect to types.

We consider direct mechanisms of the form (X,p), where $X\colon \Theta \to \mathscr{X}$ is called the *allocation rule* and $p\colon \Theta \to \mathbb{R}$ is the *payment rule*. X is said to be *implementable* if there exists a payment rule p such that truth-telling is a dominant strategy for the agent; i.e.,

$$v(X(\theta),\theta) \,-\, p(\theta) \;\geq\; v(X(\hat\theta),\theta) \,-\, p(\hat\theta), \qquad \text{for all } \theta,\hat\theta \in \Theta. \tag{4}$$

In that case the direct mechanism (X,p) is said to be *incentive compatible* and the function $U\colon \Theta \to \mathbb{R}$ defined by

$$U(\theta) \;\equiv\; v(X(\theta),\theta) \,-\, p(\theta), \qquad \text{all } \theta \in \Theta, \tag{5}$$

is called the *indirect utility generated by* (X,p). We shall restrict our analysis to measurable allocation rules.

2.2 Subderivatives and the integral of a correspondence

We introduce some mathematical concepts and results for later use. The reader is referred to Aubin and Frankowska (1990), Hildenbrand (1974) and Rockafellar and Wets (1998) for details.

Fix $x \in \mathscr{X}$, $\hat\theta \in \Theta$. Let $\delta \in \mathbb{R}^k, \delta \neq \mathbf{0}$, be a directional vector for which $\hat\theta + r\delta \in \Theta$ for a sufficiently small scalar r. The right and left subderivatives of the function $\theta \mapsto v(x,\theta)$ evaluated at $\hat\theta \in \Theta$ in the direction δ are defined, respectively, as the following lower and upper limits:

$$\begin{aligned}
\overline{\mathrm{d}}v(x,\hat\theta;\delta) \;&\equiv\; \liminf_{r\downarrow 0} \frac{v(x,\hat\theta + r\delta) \,-\, v(x,\hat\theta)}{r},\\
\underline{\mathrm{d}}v(x,\hat\theta;\delta) \;&\equiv\; \limsup_{r\uparrow 0} \frac{v(x,\hat\theta + r\delta) \,-\, v(x,\hat\theta)}{r}.
\end{aligned}$$

By (A3), these subderivatives exist and are finite. Note that $\underline{\mathrm{d}}v(x,\hat\theta;\delta) = -\,\overline{\mathrm{d}}v(x,\hat\theta;-\delta)$. In models where $\theta \mapsto v(x,\theta)$ admits one-sided directional derivatives at $\hat\theta$, one can replace the lower and upper limits in the above expressions with the usual one-sided limits, although the previous notation is maintained.

Given types θ_1,θ_2 in Θ, denote their vector difference by $\delta_1^2 \equiv \theta_2 - \theta_1$. The open line segment connecting θ_1 to θ_2 is the set $L(\theta_1,\theta_2) = \{\theta_1 + \alpha\delta_1^2 \mid \alpha \in (0,1)\}$. By (A2),

[4] At some notational cost, one could instead assume that Θ is polygonally connected. The boundedness of the type space plays a technical role in some of our results.

[5] The following example is well-known. Let $\mathscr{X} = \mathbb{N}$, $\Theta = [0,1]$, and $f\colon \mathbb{N}\times\Theta \to \mathbb{R}$ be defined by $f(n,\theta) = 1$ if $0 \leq \theta \leq 1 - (1/n)$ and $f(n,\theta) = n(1-\theta)$ if $1-(1/n) < \theta \leq 1$. For each $n \in \mathbb{N}$, $\theta \mapsto f(n,\theta)$ is Lipschitz on Θ with $\ell(n) = n$. However, the function $g\colon \Theta \to \mathbb{R}$ defined by $g(\theta) = \sup_{n\in\mathbb{N}} f(n,\theta)$, all $\theta \in \Theta$, is clearly discontinuous at $\theta = 1$.

one has $L(\theta_1, \theta_2) \subseteq \Theta$ for all $\theta_1, \theta_2 \in \Theta$. We shall make implicit use of the function $\alpha \mapsto \theta_1^2(\alpha) \equiv \theta_1 + \alpha\, \delta_1^2$ mapping $(0,1)$ onto $L(\theta_1, \theta_2)$.

Let $\mathcal{B}(0,1)$ denote the Borel σ-algebra of subsets of $(0,1)$. Let $S\colon (0,1) \rightrightarrows \mathbb{R}$ be a correspondence with closed images. Then S is said to be a measurable correspondence if for every open set O in $\mathbb{R}$, the inverse image $S^{-1}(O) = \{\alpha \in (0,1) \mid S(\alpha) \cap O \neq \emptyset\}$ belongs to $\mathcal{B}(0,1)$; in particular, $\operatorname{dom} S = \{\alpha \in (0,1) \mid S(\alpha) \neq \emptyset\}$ and its complement are measurable sets. S is said to be integrably bounded if there exists a non-negative (Lebesgue) integrable function g defined on $(0,1)$ such that $S(\alpha) \subseteq [-\,g(\alpha), g(\alpha)]$ for almost all α in $(0,1)$. A selection s of the correspondence S is a function $\alpha \mapsto s(\alpha)$ such that $s(\alpha) \in S(\alpha)$ a.e. in $(0,1)$. By the measurable selection theorem, a measurable correspondence $S\colon (0,1) \rightrightarrows \mathbb{R}$ admits a measurable selection $s\colon (0,1) \to \mathbb{R}$. If in addition S is integrably bounded, then it admits (Lebesgue) integrable selections. Let $\mathcal{L}_S$ be the set of all integrable selections of the correspondence S. Aumann's (1965) integral of S is the set

$$\int S \;\equiv\; \left\{ \int_0^1 s(\alpha)\, d\alpha \;\middle|\; s \in \mathcal{L}_S \right\}.$$

By the Lyapunov's convexity theorem, the integral $\int S$ of a closed-valued, measurable and integrably bounded correspondence $S\colon (0,1) \rightrightarrows \mathbb{R}$ is a non-empty, closed interval.

3 Incentive compatible mechanisms

A characterization theorem and a revenue inequality are now provided for the mechanism design setting developed in Section 2.1. We shall next employ the results of this section to study relevant economic design problems in Section 4.

3.1 A characterization theorem

Consider a direct mechanism (X, p). We start with the following observation: if (X, p) is incentive compatible, then its associated indirect utility is Lipschitz on Θ and has two-sided directional derivatives a.e. in the line segment connecting any two types.

Lemma 1. *Assume that (A1) to (A3) are satisfied, and let $p\colon \Theta \to \mathbb{R}$ implement the allocation rule $X\colon \Theta \to \mathscr{X}$. The following statements hold:*

(a) The indirect utility function U generated by (X, p) is Lipschitz continuous on Θ.

(b) For every pair θ_1, θ_2 of distinct types in Θ, U admits two-sided directional derivatives in the direction δ_1^2 a.e. in $L(\theta_1, \theta_2)$.

Proof (a) Consider any $\theta_1, \theta_2 \in \Theta$. From (4) and (5) one sees that

$$\begin{aligned} U(\theta_2) - U(\theta_1) \;&\leq\; \big(v(X(\theta_2), \theta_2) - p(\theta_2)\big) \;-\; \big(v(X(\theta_2), \theta_1) - p(\theta_2)\big) \\ &=\; v(X(\theta_2), \theta_2) \;-\; v(X(\theta_2), \theta_1) \;\leq\; \ell(X(\theta_2))\, \|\theta_2 - \theta_1\| \;\leq\; \ell\, \|\theta_2 - \theta_1\|; \end{aligned}$$

where the last two inequalities follow from (A3). Reversing the roles of θ_1 and θ_2, one readily concludes that $|U(\theta_2) - U(\theta_1)| \leq \ell \|\theta_2 - \theta_1\|$, as desired.

(b) Fix distinct types θ_1, θ_2 in Θ. Define the function μ on $(0,1)$ by $\mu(\alpha) = U(\theta_1^2(\alpha))$, where $\theta_1^2(\alpha) \in L(\theta_1, \theta_2)$. Since for any $0 < \alpha, \alpha' < 1$, one has

$$|\mu(\alpha) - \mu(\alpha')| \;=\; |U(\theta_1^2(\alpha)) - U(\theta_1^2(\alpha'))| \;\leq\; \ell \, \|\theta_1^2(\alpha) - \theta_1^2(\alpha')\| \;=\; \ell \, \|\delta_1^2\| \, |\alpha - \alpha'|,$$

it follows that μ is Lipschitz on $(0,1)$ and therefore absolutely continuous and differentiable a.e. in $(0,1)$. In particular, if μ is differentiable at $\alpha \in (0,1)$, then we deduce:

$$D\mu(\alpha) \;=\; \lim_{r \to 0} \frac{\mu(\alpha + r) - \mu(\alpha)}{r} \;=\; \lim_{r \to 0} \frac{U(\theta_1^2(\alpha) + r\,\delta_1^2) - U(\theta_1^2(\alpha))}{r} \;=\; DU(\theta_1^2(\alpha); \delta_1^2),$$

with last term above denoting the two-sided directional derivative of U at $\theta_1^2(\alpha)$ in the direction δ_1^2. ■

Milgrom and Segal (2002) obtained an analogue of Lemma 1-(a) under the alternative hypothesis of absolute continuity of $v(x, \cdot)$ on a one-dimensional type space, plus an integral bound condition on the derivative of v with respect to types. The extension of the absolute continuity concept to multi-dimensional settings is not straightforward.[6] Lipschitz continuity extends naturally to multi-dimensional domains and allows us to work with right and left subderivatives. We also mention that in multi-dimensional settings, Lemma 1-(b) does not imply that U is fully differentiable a.e. in the line connecting θ_1 and θ_2.[7] What Lemma 1-(b) states is that the indirect utility generated by an incentive compatible mechanism admits two-sided directional derivatives in the direction δ_1^2 a.e. in $L(\theta_1, \theta_2)$.

These observations are used to derive a relationship between the right and left subderivatives of the valuation function at equilibrium points. Define the *right* and *left subderivative functions between* θ_1 *and* θ_2, respectively, by

$$\overline{s}(\theta_1^2(\alpha)) \;\equiv\; \overline{\mathrm{d}}v(X(\theta_1^2(\alpha)), \theta_1^2(\alpha); \delta_1^2), \qquad \text{for all } \alpha \in (0,1), \tag{6}$$

and

$$\underline{s}(\theta_1^2(\alpha)) \;\equiv\; \underline{\mathrm{d}}v(X(\theta_1^2(\alpha)), \theta_1^2(\alpha); \delta_1^2), \qquad \text{for all } \alpha \in (0,1). \tag{7}$$

The following technical assumption is required in our characterization theorem.

(M) Given $X\colon \Theta \to \mathscr{X}$*, for every pair of distinct types* $\theta_1, \theta_2 \in \Theta$*, the right and left subderivative functions* $\alpha \mapsto \overline{s}(\theta_1^2(\alpha))$ *and* $\alpha \mapsto \underline{s}(\theta_1^2(\alpha))$ *are* $\mathcal{B}(0,1)$*-measurable.*

Assumption (M), which is stated in terms of the valuation function and the allocation rule, may be easily verified in some circumstances — cf. Example 1, where for types $\theta \in \Theta = [0,1]$, $\overline{\mathrm{d}}v(X^*(\theta), \theta) = \theta$ and $\underline{\mathrm{d}}v(X^*(\theta), \theta) = 3\theta$. A further discussion of the role played by (M) is presented after Theorem 1.

[6] For instance, it is possible to construct a convex function on a plane that fails to be absolutely continuous; see Friedman (1940) for details.

[7] In fact, there may be many (piecewise) smooth paths between θ_1 and θ_2 for which U is nowhere fully differentiable in such paths. See Krishna and Maenner (2001) for an example.

We use Equation 6 and Equation 7 to define, for any pair of types $\theta_1 \neq \theta_2$, the *subderivative correspondence* $S\left(\theta_1^2(\cdot)\right) : (0,1) \rightrightarrows \mathbb{R}$ *between* θ_1 *and* θ_2 as follows:

$$S\left(\theta_1^2(\alpha)\right) \;\equiv\; \left\{ r \in \mathbb{R} \;\middle|\; \overline{s}(\theta_1^2(\alpha)) \;\leq\; r \;\leq\; \underline{s}(\theta_1^2(\alpha)) \right\}, \qquad \text{for all } \alpha \in (0,1). \tag{8}$$

$S\left(\theta_1^2(\alpha)\right)$ is empty-valued if $\overline{s}(\theta_1^2(\alpha)) > \underline{s}(\theta_1^2(\alpha))$. Whenever the opposite inequality holds, it contains all real numbers between the right and left subderivative of v with respect to types in the direction δ_1^2, evaluated at $(X(\theta_1^2(\alpha)), \theta_1^2(\alpha))$. Observe that for any $\alpha \in (0,1)$, one can write $\theta_2^1(1-\alpha) = \theta_1^2(\alpha)$. In view of $\underline{\mathrm{d}}v(x,\theta;\delta) = -\overline{\mathrm{d}}v(x,\theta;-\delta)$, one has

$$S\left(\theta_2^1(1-\alpha)\right) \;=\; -\,S\left(\theta_1^2(\alpha)\right) \qquad \text{for all } \theta_1, \theta_2 \in \Theta, \text{ all } \alpha \in (0,1). \tag{9}$$

The subderivative correspondence between θ_1 and θ_2 is said to be *regular* if it is non empty-valued a.e. in $(0,1)$, closed-valued, measurable and integrably bounded.

Lemma 2. *Assume that (A1) to (A3) and (M) are satisfied. If X is implementable, then for all types $\theta_1, \theta_2 \in \Theta$, $\theta_1 \neq \theta_2$, the subderivative correspondence $S\left(\theta_1^2(\cdot)\right)$ is regular.*

Proof Suppose that $p\colon \Theta \to \mathbb{R}$ implements X. Fix distinct types θ_1, θ_2. Then for every $\theta_1^2(\alpha) \in L(\theta_1, \theta_2)$, for small scalar r, the indirect utility U generated by (X,p) satisfies

$$\begin{aligned} U(\theta_1^2(\alpha) + r\,\delta_1^2) \;-\; U(\theta_1^2(\alpha)) \;&\geq\; v(X(\theta_1^2(\alpha)), \theta_1^2(\alpha) + r\,\delta_1^2) \;-\; p(\theta_1^2(\alpha)) \\ &\quad\;-\; v(X(\theta_1^2(\alpha)), \theta_1^2(\alpha)) \;+\; p(\theta_1^2(\alpha)) \\ &=\; v(X(\theta_1^2(\alpha)), \theta_1^2(\alpha) + r\,\delta_1^2) \;-\; v(X(\theta_1^2(\alpha)), \theta_1^2(\alpha)). \end{aligned}$$

If $r > 0$ then it follows from the above expression that

$$\frac{v(X(\theta_1^2(\alpha)), \theta_1^2(\alpha) + r\,\delta_1^2) - v(X(\theta_1^2(\alpha)), \theta_1^2(\alpha))}{r} \;\leq\; \frac{U(\theta_1^2(\alpha) + r\,\delta_1^2) - U(\theta_1^2(\alpha))}{r}. \tag{10}$$

The reverse inequality is obtained if instead $r < 0$. By Lemma 1-(b), U admits two-sided directional derivatives in the direction δ_1^2 a.e. in $L(\theta_1, \theta_2)$. Thus, taking the lower limit as $r \downarrow 0$ in (10) and the upper limit as $r \uparrow 0$ in the corresponding expression with the reverse inequality, we infer that a.e. in $(0,1)$ the following holds:

$$\overline{s}(\theta_1^2(\alpha)) \;\leq\; DU(\theta_1^2(\alpha); \delta_1^2) \;\leq\; \underline{s}(\theta_1^2(\alpha)). \tag{11}$$

This shows that $S\left(\theta_1^2(\alpha)\right) \neq \emptyset$ a.e. in $(0,1)$, as desired.

$S\left(\theta_1^2(\cdot)\right)$ is closed-valued by definition. To show that it is measurable, define the correspondence $T\colon (0,1) \rightrightarrows \mathbb{R}$ by $T(\alpha) = \{\overline{s}(\theta_1^2(\alpha))\} \cup \{\underline{s}(\theta_1^2(\alpha))\}$ when (11) is satisfied, and $T(\alpha) = \emptyset$, otherwise. Since the set $\{\alpha \in (0,1) \mid T(\alpha) = \emptyset\}$ has zero measure, we deduce from (M) that T is a measurable correspondence, and so is its convex hull $\mathrm{conv}(T) = S\left(\theta_1^2(\cdot)\right)$.[8] Further, notice that by (A2) Θ is bounded and that for every $x \in \mathscr{X}$, (A3) implies that

[8] Here we use two facts: (i) the union of measurable correspondences is measurable; (ii) the convex hull of a measurable correspondence is also measurable. See references given in Section 2.2.

$|\overline{\mathrm{d}}v(x,\theta_1^2(\alpha);\delta_1^2)| \leq \ell\|\delta_1^2\|$ and similarly $|\underline{\mathrm{d}}v(x,\theta_1^2(\alpha);\delta_1^2)| \leq \ell\|\delta_1^2\|$, for all $\alpha \in (0,1)$. Hence, $S\left(\theta_1^2(\cdot)\right)$ is integrably bounded. ■

The integral of the regular correspondence $S\left(\theta_1^2(\cdot)\right)$ is a non-empty, closed interval. In particular, $\overline{s}(\theta_1^2(\cdot))$ and $\underline{s}(\theta_1^2(\cdot))$ are integrable selections, with the inequalities

$$\int_0^1 \overline{s}(\theta_1^2(\alpha))\,d\alpha \;\leq\; \int_0^1 s(\theta_1^2(\alpha))\,d\alpha \;\leq\; \int_0^1 \underline{s}(\theta_1^2(\alpha))\,d\alpha$$

valid for any integrable selection $s(\theta_1^2(\cdot))$ of $S\left(\theta_1^2(\cdot)\right)$. In light of Equation 9, we can choose an integrable selection $s(\theta_2^1(\cdot))$ from the regular correspondence $S\left(\theta_2^1(\cdot)\right)$ such that $\int_0^1 s(\theta_1^2(\alpha))d\alpha + \int_0^1 s(\theta_2^1(\alpha))d\alpha \;=\; 0$. We use these facts in our characterization result. Given types $\theta_n, \theta_m \in \Theta$, denote $\delta_n^m \equiv \theta_m - \theta_n$ and let $S\big(\theta_n^m(\cdot)\big)$ be the subderivative correspondence between θ_n and θ_m. It is understood that $\overline{s}(\theta_n^m(\cdot)) = \underline{s}(\theta_n^m(\cdot)) \equiv 0$ whenever $\theta_n = \theta_m$.

Theorem 1. *Assume (A1) to (A3) are satisfied. Suppose that (M) holds for $X\colon \Theta \to \mathscr{X}$. The following statements are then equivalent.*

(a) The allocation rule X is implementable, with indirect utility function U.

(b) For all pairs of types θ_n, θ_m in Θ, the subderivative correspondence $S\big(\theta_n^m(\cdot)\big)$ between θ_n and θ_m is regular and admits integrable selections. Moreover, for each $\theta_0 \in \Theta$, there exists a family of integrable selections $\mathscr{S}_0 = \{s(\theta_0^m(\cdot)) \in S\big(\theta_0^m(\cdot)\big) \mid \theta_m \in \Theta\}$ for which the following properties are satisfied for all $\theta_1, \theta_2 \in \Theta$.

(b.1) The closed-path integrability condition:

$$\int_0^1 s(\theta_0^2(\alpha))\,d\alpha \;-\; \int_0^1 s(\theta_0^1(\alpha))\,d\alpha \;=\; \int_0^1 s(\theta_1^2(\alpha))\,d\alpha.$$

(b.2) The integral monotonicity condition:

$$v(X(\theta_2),\theta_2) - v(X(\theta_2),\theta_1) \;\geq\; \int_0^1 s(\theta_1^2(\alpha))\,d\alpha \;\geq\; v(X(\theta_1),\theta_2) - v(X(\theta_1),\theta_1).$$

(b.3) The Mirrlees representation of the indirect utility function U:

$$U(\theta_2) \;-\; U(\theta_1) \;=\; \int_0^1 s(\theta_1^2(\alpha))\,d\alpha.$$

Note that the family of integrable selections $\mathscr{S}_0$ and (b.1) are combined to obtain integrable selections $s(\theta_1^2(\cdot))$ from $S\big(\theta_1^2(\cdot)\big)$ for which (b.2) and (b.3) hold. The closed-path integrability condition is trivially satisfied in models with one-dimensional type spaces.

Proof $(a) \implies (b)$ From Lemma 2, one has that $S\big(\theta_n^m(\cdot)\big)$ is regular and admits integrable selections, for all $\theta_n, \theta_m \in \Theta$. Fix an arbitrary type θ_0, and let $\theta_0^m(\alpha) = \theta_0 + \alpha\,\delta_0^m$ for all $\alpha \in (0,1)$, for any $\theta_m \in \Theta$. From Lemma 1-(b), the function μ_0^m defined on the unit interval by $\mu_0^m(\alpha) = U(\theta_0^m(\alpha))$ is absolutely continuous, with $D\mu_0^m(\alpha) = DU(\theta_0^m(\alpha); \delta_0^m)$ for almost all $\alpha \in (0,1)$. Therefore,

$$\mu_0^m(1) - \mu_0^m(0) \;=\; U(\theta_m) - U(\theta_0) \;=\; \int_0^1 DU(\theta_0^m(\alpha); \delta_0^m)\, d\alpha.$$

This expression is combined with (11) to obtain

$$\int_0^1 \overline{s}(\theta_0^m(\alpha))\, d\alpha \;\le\; U(\theta_m) \,-\, U(\theta_0) \;\le\; \int_0^1 \underline{s}(\theta_0^m(\alpha))\, d\alpha. \tag{12}$$

Using the convexity of the integral $\int S\big(\theta_0^m(\cdot)\big)$, we obtain a family of integrable selections $\mathscr{S}_0 = \{s(\theta_0^m(\cdot) \in S\big(\theta_0^m(\cdot)\big) \mid \theta_m \in \Theta\}$, each of whose members satisfies

$$\int_0^1 s(\theta_0^m(\alpha))\, d\alpha \;=\; U(\theta_m) \,-\, U(\theta_0). \tag{13}$$

We shall verify that (b.1) to (b.3) are satisfied for $\mathscr{S}_0$ and any $\theta_1, \theta_2 \in \Theta$.

Since X is implementable with indirect utility U, one immediately sees that (12) remains valid if we replace θ_0 and θ_m with θ_1 and θ_2, respectively. Thus, there exists a selection $s(\theta_1^2(\cdot))$ such that $U(\theta_2) - U(\theta_1) = \int s(\theta_1^2(\alpha)) d\alpha$. We combine this expression with (13) for $m = 1, 2$ to infer that both properties (b.1) and (b.3) are satisfied. Further, from the proof of Lemma 1-(a), we notice that

$$v(X(\theta_2), \theta_2) \,-\, v(X(\theta_2), \theta_1) \;\ge\; U(\theta_2) \,-\, U(\theta_1) \;\ge\; v(X(\theta_1), \theta_2) \,-\, v(X(\theta_1), \theta_1).$$

This expression is combined with (b.3) to obtain (b.2).

$(b) \Longrightarrow (a)$ Suppose the subderivative correspondence $S\big(\theta_n^m(\cdot)\big)$ is regular for all θ_n, θ_m. Fix a type $\theta_0 \in \Theta$ and let $\mathscr{S}_0 = \{s(\theta_0^m(\cdot)) \in S\big(\theta_0^m(\cdot)\big) \mid \theta_m \in \Theta\}$ be a family of integrable selections for which properties (b.1) to (b.3) are satisfied. Define $p \colon \Theta \to \mathbb{R}$ by

$$p(\theta_m) \;=\; v(X(\theta_m), \theta_m) \,-\, \int_0^1 s(\theta_0^m(\alpha))\, d\alpha \,-\, U(\theta_0), \qquad \text{for all } \theta_m \in \Theta,$$

where the integrable selection $s(\theta_0^m(\cdot))$ belongs to $\mathscr{S}_0$ for each θ_m. It is clear that if p implements X then it generates an indirect utility U compatible with (b.3). We claim that (X, p) is indeed incentive compatible. For any $\theta_1, \theta_2 \in \Theta$, the payment difference is

$$\begin{aligned} p(\theta_2) \,-\, p(\theta_1) \;&=\; v(X(\theta_2), \theta_2) \,-\, v(X(\theta_1), \theta_1) \,-\, \int_0^1 s(\theta_0^2(\alpha))\, d\alpha \,+\, \int_0^1 s(\theta_0^1(\alpha))\, d\alpha \\ &=\; v(X(\theta_2), \theta_2) \,-\, v(X(\theta_1), \theta_1) \,-\, \int_0^1 s(\theta_1^2(\alpha))\, d\alpha, \end{aligned}$$

where the last equality follows from (b.1). Using this expression, we deduce from the integral monotonicity condition (b.2) that

$$\begin{aligned}\big(v(X(\theta_1),\theta_1) \;-\; p(\theta_1)\big) \;&-\; \big(v(X(\theta_2),\theta_1) \;-\; p(\theta_2)\big)\\ &= \; v(X(\theta_1),\theta_1) \;-\; v(X(\theta_2),\theta_1) \;+\; p(\theta_2) \;-\; p(\theta_1)\\ &= \; v(X(\theta_2),\theta_2) \;-\; v(X(\theta_2),\theta_1) \;-\; \int_0^1 s(\theta_1^2(\alpha))\,d\alpha \;\geq\; 0.\end{aligned}$$

Hence, it follows that $v(X(\theta_1),\theta_1) - p(\theta_1) \geq v(X(\theta_2),\theta_1) - p(\theta_2)$. Since θ_1 and θ_2 were arbitrarily chosen, this shows that p implements X, as desired. ■

Remark. The inequalities in (11) follow directly from the implementability of X and do not require assumption (M). Thus, under (A1) to (A3) alone, if X is implementable with associated indirect utility U, then for any pair of types θ_1, θ_2 in Θ there exists an integrable function $\tilde{s}$, defined on $(0,1)$, for which the Mirrlees representation of U holds: $U(\theta_2) - U(\theta_1) = \int \tilde{s}(\alpha)d\alpha$. A similar result is contained in Theorem 2 of Milgrom and Segal (2002) under the assumption of differentiability of the valuation with respect to types. On the other hand, to demonstrate that the implementability of the allocation rule X follows from integral monotonicity and the Mirrlees representation, one has to make sure that there actually is an integrable function consistent with Equation 11. Assumption (M) not only guarantees that such a function exists, but in addition provides the regularity of the subderivative correspondence $S\big(\theta_1^2(\cdot)\big)$, for all $\theta_1 \neq \theta_2$. In what follows we exploit this regularity to derive certain important properties of incentive compatible mechanisms. In Section 5 we show that (M) is satisfied in settings commonly employed in economic applications.

Example 1 (continued). The regular subderivative correspondence for $X^*(\theta) = \theta$ is given by $S(\theta) = [\theta, 3\theta]$, all $\theta \in \Theta = [0,1]$. Integral monotonicity is expressed as

$$\theta^2 + \theta\hat{\theta} - 2\hat{\theta}^2 \;\geq\; \int_{\hat{\theta}}^{\theta} s(\tilde{\theta})\,d\tilde{\theta} \;\geq\; \theta\hat{\theta} - \hat{\theta}^2, \qquad \text{for } 0 \leq \hat{\theta} < \theta \leq 1.$$

Letting $\hat{\theta} = 0$ in the above expression, one sees that for each type $\theta \in \Theta$, it must be that $\theta^2 \geq \int_0^\theta s(\tilde{\theta})\,d\tilde{\theta} \geq 0$. It follows that here *any* integrable selection $\theta \mapsto s(\theta)$ of the subderivative correspondence S satisfying $\theta \leq s(\theta) \leq 2\theta$, all $\theta \in \Theta$, can be employed to construct a payment rule that implements X^*. Consider instead the selection $\theta \mapsto s''(\theta) = 3\theta$, which violates integral monotonicity. The payment rule p'' associated with this selection is $p''(\theta) = -\frac{1}{2}\theta^2$, all $\theta \in \Theta$ (cf. Proposition 2). But p'' does not implement X^*: for any $\theta, \hat{\theta}$, the payoffs of reporting $\hat{\theta}$ when the agent's true type is θ are

$$v(X^*(\hat{\theta}),\theta) \;-\; p''(\hat{\theta}) \;=\; \begin{cases}\theta\hat{\theta} \;+\; \frac{1}{2}\hat{\theta}^2, & \text{if } \hat{\theta} \leq \theta;\\ 2\theta^2 \;-\; \theta\hat{\theta} \;+\; \frac{1}{2}\hat{\theta}^2, & \text{if } \hat{\theta} > \hat{\theta}.\end{cases}$$

These payoffs are increasing in $\hat{\theta}$ everywhere on Θ, thus our agent has incentives to overstate her type. □

One can replace the global conditions of Theorem 1-(b) with their local versions, an approach introduced by Archer and Kleinberg (2008) for the linear valuation case. This relies on the fact that the convex hull of the closed line segments between θ_n and θ_m ($n, m = 1, 2, 3$) is compact. Thus, an open cover of any such set can be replaced with a finite subcover to obtain the desired conditions. We state this formally in the next proposition, whose proof can be adapted from the arguments of Lemmas 3.2 and 3.5 in Archer and Kleinberg (2008).

Proposition 1. *Assume (A1) to (A3) are satisfied. Assume in addition that (M) is satisfied for the allocation rule $X\colon \Theta \to \mathscr{X}$. The following are equivalent:*

(a) *The allocation rule X is implementable, with indirect utility function U.*

(b) *For each $\theta_0 \in \Theta$, there exists an open neighborhood O of θ_0 such that for all $\theta_n, \theta_m \in O \cap \Theta$, the subderivative correspondence $S\big(\theta_n^m(\cdot)\big)$ is regular and admits integrable selections, and further there exists a family of integrable selections $\mathscr{S}_0 = \{s(\theta_0^m(\cdot)) \in S\big(\theta_0^m(\cdot)\big) \mid \theta_m \in O \cap \Theta\}$ for which the following local properties are satisfied for all $\theta_1, \theta_2 \in O \cap \Theta$:*

(b.1) $$\int_0^1 s(\theta_0^2(\alpha))\, d\alpha \;-\; \int_0^1 s(\theta_0^1(\alpha))\, d\alpha \;=\; \int_0^1 s(\theta_1^2(\alpha))\, d\alpha;$$

(b.2) $$v(X(\theta_2), \theta_2) - v(X(\theta_2), \theta_1) \;\geq\; \int_0^1 s(\theta_1^2(\alpha))\, d\alpha \;\geq\; v(X(\theta_1), \theta_2) - v(X(\theta_1), \theta_1);$$

(b.3) $$U(\theta_2) \;-\; U(\theta_1) \;=\; \int_0^1 s(\theta_1^2(\alpha))\, d\alpha.$$

3.2 A revenue inequality

In the mechanism design setting considered in this paper, the revenue associated with a given allocation rule may not be uniquely determined up to a constant. However, one important feature that resembles the revenue equivalence principle is preserved: from Theorem 1, it follows that the *range* of indirect utilities is determined by the allocation rule alone. We focus on normalized payment rules that generate an indirect utility of $u_0 \in \mathbb{R}$ for the "lowest type" $\theta_0 \in \Theta$ to derive the following *revenue inequality.*

Theorem 2. *Assume (A1) to (A3) hold and let $X\colon \Theta \to \mathscr{X}$ be an allocation rule for which (M) is satisfied. Suppose that $p\colon \Theta \to \mathbb{R}$ and $p'\colon \Theta \to \mathbb{R}$ implement X and let U and U' denote the indirect utility functions generated by (X, p) and (X, p'), respectively, with $U(\theta_0) = U'(\theta_0) = u_0$. Then for all $\theta_1 \in \Theta$:*

$$|\, U(\theta_1) \;-\; U'(\theta_1)\,| \;\leq\; \int_0^1 \big[\underline{s}(\theta_0^1(\alpha)) \;-\; \overline{s}(\theta_0^1(\alpha))\big]\, d\alpha. \tag{14}$$

Proof By Theorem 1, both indirect utilities U and U' satisfy (12) for types θ_1 and θ_0.

We deduce the following inequalities:

$$\int_0^1 \overline{s}(\theta_0^1(\alpha))\, d\alpha \;\leq\; U(\theta_1) - U(\theta_0) \;\leq\; \int_0^1 \underline{s}(\theta_0^1(\alpha))\, d\alpha, \qquad \text{and}$$
$$-\int_0^1 \underline{s}(\theta_0^1(\alpha))\, d\alpha \;\leq\; U'(\theta_0) - U'(\theta_1) \;\leq\; -\int_0^1 \overline{s}(\theta_0^1(\alpha))\, d\alpha.$$

Since $U(\theta_0) = U'(\theta_0) = u_0$, (14) follows by adding up these expressions. ■

Theorem 2 implies that, given two incentive compatible mechanisms that share an allocation rule and generate equilibrium payoff of u_0 to θ_0, the difference in the indirect utility functions is pinned down by solely the allocation rule, since X alone determines the right and left subderivative functions of Equation 14. Notice that Theorem 2 provides a bound on the difference between equilibrium utilities without the need for computing incentive compatible payments. We later show how to improve the bounds provided in (14), using the restrictions imposed by the integral monotonicity condition.

Example 1 (continued). The constant payment rule $p \equiv 0$ implements X^* and generates an indirect utility U with $U(\theta) = \theta^2$, all $\theta \in \Theta$. In addition, the payment rule p' defined by $p'(\theta) = \theta^2/2$ implements X^* and generates an indirect utility U' with $U'(\theta) = \theta^2/2$, all $\theta \in \Theta$. Immediately, for every θ in Θ,

$$\tfrac{1}{2}\theta^2 \;=\; \left|U(\theta) - U'(\theta)\right| \;\leq\; \int_0^\theta \left[\underline{s}(\tilde{\theta}) - \overline{s}(\tilde{\theta})\right] d\tilde{\theta} \;=\; \theta^2. \qquad \square$$

Suppose that (X, p) is incentive compatible with associated indirect utility $U\colon \Theta \to \mathbb{R}$. Using the Mirrlees representation of U given in Theorem 1, we infer that for all $\theta_1 \in \Theta$:

$$U(\theta_1) \;\equiv\; v(X(\theta_1), \theta_1) \;-\; p(\theta_1) \;=\; \int_0^1 s(\theta_0^1(\alpha))\, d\alpha \;+\; U(\theta_0),$$

where $s(\theta_0^1(\cdot))$ is an integrable selection of $S\left(\theta_0^1(\cdot)\right)$ satisfying integral monotonicity and closed-path integrability. Regularity of the subderivative correspondence is used to show that the converse of this argument also holds. We obtain the following.

Proposition 2. *Assume (A1) to (A3) and (M) hold for $X\colon \Theta \to \mathscr{X}$. The direct mechanism (X, p) is incentive compatible, with associated indirect utility function $U\colon \Theta \to \mathbb{R}$ and $U(\theta_0) = u_0$ if, and only if, there exists a family of integrable selections $\mathscr{S}_0 = \{s(\theta_0^m(\cdot)) \in S\big(\theta_0^m(\cdot)\big) \mid \theta_m \in \Theta\}$ satisfying (b.1) and (b.2) of Theorem 1 such that*

$$p(\theta_1) \;=\; v(X(\theta_1), \theta_1) \;-\; \int_0^1 s(\theta_0^1(\alpha))\, d\alpha \;-\; u_0, \qquad \textit{for all } \theta_1 \in \Theta. \tag{15}$$

Consider two payment rules that implement X, namely p' and p'', with respective selections $s'(\theta_0^1(\cdot))$ and $s''(\theta_0^1(\cdot))$ from subderivative correspondence $S\left(\theta_0^1(\cdot)\right)$, for all $\theta_1 \in \Theta$. Fix $0 \leq \lambda \leq 1$. Since for every $\theta_1 \neq \theta_0$, the integral $\int S\left(\theta_0^1(\cdot)\right)$ is a closed interval (cf. Section 2.2), there exists an integrable selection $\alpha \mapsto s^{\lambda}(\theta_0^1(\alpha))$ such that

$$\int_0^1 s^{\lambda}(\theta_0^1(\alpha))\, d\alpha \;=\; \lambda \int_0^1 s'(\theta_0^1(\alpha))\, d\alpha \;+\; (1-\lambda) \int_0^1 s''(\theta_0^1(\alpha))\, d\alpha.$$

Clearly, $s^{\lambda}(\theta_0^1(\cdot))$ satisfies the integral monotonicity and closed-path integrability conditions. Thus, the payment rule $p^{\lambda}\colon \Theta \to \mathbb{R}$ defined by $p^{\lambda}(\theta) = \lambda p'(\theta) + (1-\lambda)p''(\theta)$, all $\theta \in \Theta$, implements X as well.

Corollary 1. *If U' and U'' are normalized indirect utility functions generated by two incentive compatible mechanisms sharing the allocation rule X, then for every $\lambda \in [0,1]$ there exists an incentive compatible mechanism (X, p^{λ}) such that $U^{\lambda} = \lambda U' + (1-\lambda)U''$.*

Extending the cyclic monotonicity condition of Rochet (1987), Kos and Messner (2011) derive a formula for the upper and lower bounds of the set of payment rules implementing a given allocation rule X in general mechanism design settings. Proposition 2 above provides an expression for all incentive compatible payment rules under (A1)-(A3) and (M), including the extremal payment rules. To derive an explicit representation for these payments, fix any $\theta_1 \neq \theta_0$ and consider the half-closed line

$$I^*(\theta_1) \;=\; [v(X(\theta_0), \theta_1) - v(X(\theta_0), \theta_0)\,,\, +\infty)\,.$$

Clearly, the set $\int S\left(\theta_0^1(\cdot)\right) \cap I^*(\theta_1)$ is a non-empty, closed bounded interval. Let $s^*(\theta_0^1(\cdot))$ be an integrable selection of the subderivative correspondence $S\left(\theta_0^1(\cdot)\right)$ such that

$$\int_0^1 s^*(\theta_0^1(\alpha))\, d\alpha \;=\; \min\left\{ r \mid r \in \int S\left(\theta_0^1(\cdot)\right) \cap I^*(\theta_1) \right\}.$$

We now can replace the selection $s^*(\theta_0^1(\cdot))$ in Equation 15 to obtain the upper bound of the set of incentive compatible payment rules:

$$p^*(\theta_1) \;=\; v(X(\theta_1), \theta_1) \;-\; \int_0^1 s^*(\theta_0^1(\alpha))\, d\alpha \;-\; u_0, \qquad \text{for all } \theta_1 \in \Theta.$$

With the obvious modifications to the above argument, one can obtain an analogous expression for the lower extremal payment rule $\theta_1 \mapsto p_*(\theta_1)$.

Corollary 2. *Under (A1) to (A3) and (M), let $X\colon \Theta \to \mathscr{X}$ be an implementable allocation rule. Then the extremal payment rules $p^*\colon \Theta \to \mathbb{R}$ and $p_*\colon \Theta \to \mathbb{R}$ both implement X. Moreover, if (X, p) is incentive compatible, then $p_*(\theta_1) \leq p(\theta) \leq p^*(\theta_1)$ for all $\theta_1 \in \Theta$.*

4 Applications

We apply our results to three relevant economic settings: a buyer-seller situation with a loss-averse buyer, a public goods problem where we focus on efficiency, and a principle–agent framework with ex post non-contractible actions.

In the buyer-seller model, the buyer has reference-dependent preferences for some good. In particular, the buyer exhibits loss aversion à la Kőszegi and Rabin (2006) which, due to a kink of the valuation at the reference point, fail standard requirements of differentiability or convexity. Our characterization allows us to follow Myerson's (1981) approach and reformulate the seller's revenue maximization problem in terms of virtual surplus. In the optimal selling mechanism, a range of intermediate types purchase their reference quantity, so that the flexibility of pricing rules afforded by our characterization plays an important role. Compared to the optimal selling mechanism without loss aversion, some of these intermediate types have their quantities distorted downward to exploit the loss aversion of higher types, whose incentive constraints are thereby slackened and whose payments are correspondingly increased. The loss aversion of the lower end of this intermediate type range is exploited by making them pay a premium to increase their quantity from zero to their reference point. It follows that expected revenue generated by the the optimal selling mechanism under loss aversion is higher than expected revenue generated by its counterpart for the case without loss aversion.

Turning our attention to efficient design in public goods problems, we state that a direct implication of revenue equivalence is that efficient allocation rules are implementable solely via Vickrey-Clarke-Groves (VCG) payment schemes. It is well known that with sufficiently rich type spaces, generally VCG payments do not balance the budget ex post.[9] We consider a location model for a public good in a square city where residents, possibly due to restrictions on the transportation network, exhibit linear transportation costs. While there are many efficient allocation rules, we show that regardless of which is chosen there is no VCG payment rule for which payments from the agents vanish everywhere. However, for efficient allocation rules that select one of the available corner solutions, we show that revenue equivalence fails and thus efficient allocations can be implemented via non-VCG payments. We characterize all incentive compatible payments for one of the corner solutions and show that, in particular, there exists a payment scheme with ex post zero payments from the agents.

In our final application we consider a principal–agent framework in which, after participating in the mechanism, the agent can take a private, non-contractible action. For example, the principal is supplying a productive input to the agent, a firm, who then decides whether to enter a new market, or which of several different product lines to operate, etc. When the contract offered by the principal cannot be made contingent upon the agent's posterior actions, we show that the agent's induced utility for a given allocation received from the principal exhibits non-differentiabilities which preclude the use of existing methods of characterizing incentive compatibility. Nevertheless, our generalized Mirrlees condition applies and, by contrast with our previous two applications, we use it to show that the conventional revenue equivalence holds in this class of problems irrespective of the allocation rule used by the principal.[10]

[9] The classic reference is Green and Laffont (1979); see also recent work by Yenmez (2010).

[10] This revenue equivalence principle does not follow from previous results, as we consider an arbitrary allocation set (as opposed to Chung and Olszewski (2007)), non-convex, non-differentiable valuations (as opposed to Berger, Müller, and Naeemi (2010) and Milgrom and Segal (2002)), and arbitrary (measurable) allocation rules (as opposed to Krishna and Maenner (2001)). Moreover, it holds in this class of problems

4.1 Selling to a buyer with reference-dependent preferences

A seller and a buyer are negotiating the quantity and price of a homogenous good. The seller produces $x \in \mathscr{X} = [0, \bar{x}]$ at a constant zero marginal cost. The buyer has a privately known type $\theta \in \Theta = [0, 1]$, and his gross valuation for x is given by θx. The buyer has reference-dependent preferences: he enters negotiations with a reference point $y \in \mathscr{X}$ and evaluates deviations from y differently, depending on whether they are gains or losses. Following Kőszegi and Rabin (2006), we model the buyer's valuation for x, given θ and y, as

$$\nu(x, \theta, y) \; = \; \theta x \, + \, \mu(x, y),$$

where the gain-loss utility $\mu(x, y)$ is given by

$$\mu(x, y) \; = \; \gamma \, [x - y]^{+} \, + \, (\gamma + \lambda) \, [x - y]^{-}$$

and $\gamma \geq 0$ and $\lambda > 0$. Here $[\alpha]^{+} = \max\{\alpha, 0\}$ and $[\alpha]^{-} = \min\{\alpha, 0\}$ denote the positive and negative parts, respectively. Thus, the buyer is loss-averse: losses relative to the reference point have a larger marginal impact than gains.[11] We assume that y is a linear function of the buyer's type, so that $\theta \mapsto y(\theta) = \theta \bar{x}$.[12] In this formulation, the buyer's valuation function $v \colon \mathscr{X} \times \Theta \to \mathbb{R}$ is given by

$$v(x, \theta) \; = \; \begin{cases} \theta x \, + \, \gamma(x - \theta \bar{x}), & \text{if } x \geq \theta \bar{x}; \\ \theta x \, + \, (\gamma + \lambda)(x - \theta \bar{x}), & \text{if } x \leq \theta \bar{x}. \end{cases}$$

The buyer has the option not to participate in the transaction and thereby obtaining zero quantity and making no payment; this gives type θ a reservation value of $v(0, \theta) = -(\gamma + \lambda)\theta \bar{x}$.[13] The seller has a prior belief about θ represented by the distribution function F over domain Θ, with continuous, strictly positive density f. The seller chooses an incentive compatible and individually rational selling mechanism (X, p) to maximize expected payments from the buyer.

Let $X \colon \Theta \to \mathscr{X}$ be a selling rule. One has $\overline{\mathrm{d}}v(X(\theta), \theta) = \underline{\mathrm{d}}v(X(\theta), \theta) = X(\theta) - \gamma \bar{x}$ when $\theta \bar{x} < X(\theta)$, $\overline{\mathrm{d}}v(X(\theta), \theta) = \underline{\mathrm{d}}v(X(\theta), \theta) = X(\theta) - (\gamma + \lambda)\bar{x}$ when $\theta \bar{x} > X(\theta)$, and $\overline{\mathrm{d}}v(X(\theta), \theta) = X(\theta) - (\gamma + \lambda)\bar{x} < X(\theta) - \gamma \bar{x} = \underline{\mathrm{d}}v(X(\theta), \theta)$ when $\theta \bar{x} = X(\theta)$. In this case, (M) is satisfied whenever X is measurable (cf. Section 5.1). The subderivative correspondence

irrespective of the allocation rule used (as opposed to Heydenreich, Müller, Uetz, and Vohra (2009)).

[11] Eisenhuth (2010) considers the optimal auction to many loss-averse buyers. Unlike selling to a single buyer, auctions naturally generate random allocations to individual bidders, and sufficient smoothness in the distribution of types restores differentiability and allows the use of standard techniques.

[12] In this model, a constant reference point will have no effect in the formulation or solution of the monopolist problem, other than a fixed transfer from the buyer to the seller due to a negative value of the outside option. In richer settings, it is possible that constant reference point will affect the optimal contract design. For illustrative purposes, we choose a simple functional form for the reference point as a function of types.

[13] Thus, the reservation value is type-dependent. Equivalently, one could express the buyer's preferences in terms of the net valuation $w(x, \theta) = v(x, \theta) - v(0, \theta)$ and a zero reservation utility for all θ.

$S\colon \Theta \rightrightarrows \mathbb{R}$ is defined by

$$S(\theta) \;=\; \begin{cases} X(\theta) - \gamma\bar{x}, & \text{if } X(\theta) > \theta\bar{x};\\ X(\theta) - (\gamma+\lambda)\bar{x}, & \text{if } X(\theta) < \theta\bar{x};\\ \big[X(\theta) - (\gamma+\lambda)\bar{x}, X(\theta) - \gamma\bar{x}\big], & \text{if } X(\theta) = \theta\bar{x}. \end{cases} \tag{16}$$

From Proposition 2, we restrict the analysis to payment rules of the form $p(\theta) = v(X(\theta),\theta) - \int_0^\theta s(\tilde{\theta})\,d\tilde{\theta} - u_0$, with u_0 indicating the payoff of type $\theta = 0$ and $\theta \mapsto s(\theta)$ is an integrable selection of S. Clearly, an optimal payment rule chooses $u_0 = 0$. Thus, the seller's expected profits are

$$\Pi^E \;=\; \int_0^1 v(X(\theta),\theta) f(\theta)\,d\theta \;-\; \int_0^1\!\int_0^\theta s(\tilde{\theta})\,d\tilde{\theta}\, f(\theta)\,d\theta.$$

Integrating by parts the second term of the right-hand side of the above expression and rearranging, we obtain

$$\Pi^E \;=\; \int_0^1 \left[v(X(\theta),\theta) - \frac{1-F(\theta)}{f(\theta)} s(\theta)\right] f(\theta)\,d\theta. \tag{17}$$

The seller's problem is to choose $X\colon \Theta \to \mathscr{X}$ together with an integrable selection of the subderivative correspondence in Equation 16 to maximize expected revenue Π^E, subject to the incentive compatibility and participation constraints. Let us ignore these constraints for the moment. Pointwise maximization of Equation 17 requires choosing the selection $\overline{s}$ with values $\overline{s}(\theta) = X(\theta) - \gamma\bar{x}$ for $X(\theta) > \theta\bar{x}$, and $\overline{s}(\theta) = X(\theta) - (\gamma+\lambda)\bar{x}$ for $X(\theta) \le \theta\bar{x}$. To simplify notation, express the inverse hazard rate as $\phi(\theta) = [1-F(\theta)]/f(\theta)$; as usual, we assume that ϕ is a decreasing function. Thus, for every $\theta \in \Theta$, virtual surplus $VS(\theta) \equiv v(X(\theta),\theta) - \phi(\theta)\overline{s}(\theta)$ takes the form

$$VS(\theta) \;=\; \begin{cases} \big(\theta - \phi(\theta) + \gamma\big)X(\theta) \;-\; \gamma\big(\theta - \phi(\theta)\big)\bar{x}, & \text{if } X(\theta) > \theta\bar{x};\\ \big(\theta - \phi(\theta) + \gamma + \lambda\big)X(\theta) \;-\; (\gamma+\lambda)\big(\theta - \phi(\theta)\big)\bar{x}, & \text{if } X(\theta) \le \theta\bar{x}. \end{cases}$$

It follows that $VS(\theta)$ is maximized by choosing a quantity equal to 0, to $\theta\bar{x}$, or $\bar{x}$; which is the case will vary with θ and ϕ. For suitable parameters, i.e., for $1/f(0) > \gamma + \lambda$, there exist cutoff types θ' and θ'', with $0 < \theta' < \theta'' < 1$, such that $\theta - \phi(\theta) + \gamma + \lambda \ge 0$ for all $\theta \ge \theta'$ and $\theta - \phi(\theta) + \gamma + \lambda < 0$ for all $\theta < \theta'$, and similarly $\theta - \phi(\theta) + \gamma \ge 0$ for all $\theta \ge \theta''$ and $\theta - \phi(\theta) + \gamma < 0$ for all $\theta < \theta''$. We distinguish three cases. In the first case, $0 < \theta < \theta'$ and the optimal solution is therefore $X^o(\theta) = 0$. In the second case, $\theta' \le \theta < \theta''$ and the optimal solution is $X^o(\theta) = \theta\bar{x}$. The analysis of the third case, when $\theta'' \le \theta \le 1$, is slightly more complicated as the optimal selling rule depends on which of the two feasible solutions, $\bar{x}$ and $\theta\bar{x}$, generates higher virtual surplus. Taking differences, we obtain

$$VS(\theta|\bar{x}) - VS(\theta|\theta\bar{x}) \;=\; (\theta - \phi(\theta) + \gamma)(1-\theta)\bar{x} \;-\; \lambda\phi(\theta)\bar{x}. \tag{18}$$

For purposes of exposition we specialize to an example where this difference can easily be signed and a simple expression for the optimal allocation rule can be derived.

Example 2. Let $\gamma = 0$, $0 < \lambda < 1$ so that only losses relative to the reference point enter μ. Suppose that types are distributed uniformly on $\Theta = [0,1]$, so that $\theta' = \frac{1-\lambda}{2}$, $\theta'' = \frac{1}{2}$, and the difference in Equation 18 equals $(2\theta - 1 - \lambda)(1-\theta)\bar{x}$. Thus, the optimal selling rule is

$$X^o(\theta) \;=\; \begin{cases} 0, & \text{if } 0 \le \theta < \frac{1-\lambda}{2}; \\ \theta\bar{x}, & \text{if } \frac{1-\lambda}{2} \le \theta < \frac{1+\lambda}{2}; \\ \bar{x}, & \text{if } \frac{1+\lambda}{2} \le \theta \le 1. \end{cases} \tag{19}$$

Notice that X^o is strictly increasing for intermediate types. Contrast this with the case of no loss aversion ($\lambda = 0$), where the optimal selling rule gives 0 to every type $\theta < \frac{1}{2}$, and $\bar{x}$ to every type $\theta \ge \frac{1}{2}$. The integrable selection $\overline{s}$ is therefore

$$\overline{s}(\theta) \;=\; \begin{cases} -\lambda\bar{x}, & \text{if } 0 \le \theta < \frac{1-\lambda}{2}; \\ (\theta - \lambda)\bar{x}, & \text{if } \frac{1-\lambda}{2} \le \theta < \frac{1+\lambda}{2}; \\ \bar{x}, & \text{if } \frac{1+\lambda}{2} \le \theta \le 1. \end{cases}$$

It is not difficult to verify that the integral monotonicity condition holds. For instance, for $0 \le \hat{\theta} < \frac{1-\lambda}{2} \le \theta < \frac{1+\lambda}{2}$, we have $v(X^o(\theta),\theta) - v(X^o(\theta),\hat{\theta}) \;=\; \theta(\theta - \hat{\theta})\bar{x}$, while $v(X^o(\hat{\theta}),\theta) - v(X^o(\hat{\theta}),\hat{\theta}) = -\,\lambda(\theta - \hat{\theta})\bar{x}$, and

$$\int_{\hat{\theta}}^{\theta} \overline{s}(\tilde{\theta})\, d\tilde{\theta} \;=\; -\lambda(\theta - \hat{\theta})\bar{x} \,+\, \tfrac{1}{2}\left(\theta^2 - \left(\tfrac{1-\lambda}{2}\right)^2\right)\bar{x}.$$

From these expressions the integral monotonicity follows readily. Using Theorem 1 we deduce that X^o is implementable. The optimal selling mechanism is (X^o, p^o), where

$$p^o(\theta) \;=\; \begin{cases} 0, & \text{if } 0 \le \theta < \frac{1-\lambda}{2}; \\ \frac{\bar{x}}{2}\left(\theta^2 + 2\lambda\theta + (\frac{1-\lambda}{2})^2\right), & \text{if } \frac{1-\lambda}{2} \le \theta < \frac{1+\lambda}{2}; \\ \frac{\bar{x}}{2}\left(\lambda^2 + \lambda + 1\right), & \text{if } \frac{1+\lambda}{2} \le \theta \le 1. \end{cases}$$

The reader can verify using the Mirrlees representation that the optimal selling mechanism (X^o, p^o) generates an indirect utility function U^o equal to

$$U^o(\theta) \;=\; \begin{cases} -\,\lambda\theta\bar{x}, & \text{if } 0 \le \theta < \frac{1-\lambda}{2}; \\ -\,\lambda\theta\bar{x} + \frac{\bar{x}}{2}\left(\theta^2 - (\frac{1-\lambda}{2})^2\right), & \text{if } \frac{1-\lambda}{2} \le \theta < \frac{1+\lambda}{2}; \\ -\,\lambda\theta\bar{x} + \frac{\bar{x}}{2}\left(2\theta(1+\lambda) - \lambda^2 - \lambda - 1\right), & \text{if } \frac{1+\lambda}{2} \le \theta \le 1. \end{cases}$$

It is immediate to conclude that the participation constraints are satisfied. □

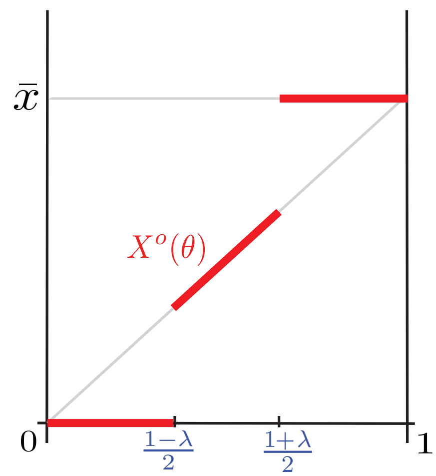

Figure 1: the optimal selling rule X^o

4.2 Locating a public good with zero transfers

A central planner is considering the problem of locating a public good, such as a hospital, a library, or any other public facility, in a square city where the exact location of residents is unobservable or unverifiable. For simplicity, we assume a fixed cost of provision and normalize it to zero. Thus, the allocation set is the square $\mathscr{X} = [0,1]^2$, with coordinates $x = (x_1, x_2)$ denoting the chosen location for the public good.

There are two (ex ante identical) residents, A and B. To avoid confusion with previous notation, write A's type space as $\Theta = [0,1]^2$ and B's type space as $\Xi = [0,1]^2$, with $\theta_1 = (\theta_{11}, \theta_{12})$ and $\xi_1 = (\xi_{11}, \xi_{12})$ representing locations for A and B in the square city, respectively. Each agent derives a net benefit from accessing the public good that depends positively on her type — for instance, think of the horizontal and vertical axes urban areas; a higher type is associated with less access to potential substitutes. In addition, each resident pays a linear transportation cost to access the public facility. Thus, we model the valuation functions for A and B as defined by:

$$\begin{aligned} v_A(x, \theta_1) &= \theta_{11}\theta_{12} - |x_1 - \theta_{11}| - |x_2 - \theta_{12}|, \qquad \text{all } \theta_1 \in \Theta, \\ v_B(x, \xi_1) &= \xi_{11}\xi_{12} - |x_1 - \xi_{11}| - |x_2 - \xi_{12}|, \qquad \text{all } \xi_1 \in \Xi. \end{aligned}$$

It is clear that our assumptions (A1) to (A3) are satisfied.

The central planner is interested in allocation rules that maximize $v_A(x, \theta_1) + v_B(x, \xi_1)$ for each reported type profile and do not require external transfers. Clearly, every efficient allocation rule $(\theta_1, \xi_1) \mapsto X^e(\theta_1, \xi_1) = (X^e_1(\theta_1, \xi_1), X^e_2(\theta_1, \xi_1))$ can be written as $X^e_k(\theta_1, \xi_1) = X^e_k(\theta_{1k}, \xi_{1k})$, for $k = 1, 2$. In particular, any convex combination of θ_{1k} and ξ_{1k} constitutes an efficient location for the public good on the k-th coordinate. Thus, all possible efficient allocation rules are determined by the choice of $\lambda_k(\theta_{1k}, \xi_{1k}) \in [0,1]$ in the expression

$$X_k^e(\theta_{1k}, \xi_{1k}) \;=\; \theta_{1k} \,+\, \lambda_k(\theta_{1k}, \xi_{1k})[\xi_{1k} - \theta_{1k}]. \tag{20}$$

For instance, to obtain the allocation rule that always selects $\min\{\theta_{1k}, \xi_{1k}\}$, we choose $\lambda_k(\theta_{1k}, \xi_{1k}) = 0$ for all $\theta_{1k} \le \xi_{1k}$ and $\lambda_k(\theta_{1k}, \xi_{1k}) = 1$ for all $\theta_{1k} > \xi_{1k}$. It is clear that social welfare at any efficient allocation x^e is

$$v_A(x^e, \theta_1) \,+\, v_B(x^e, \xi_1) \;=\; \theta_{11}\theta_{12} \,+\, \xi_{11}\xi_{12} \,-\, |\theta_{11} - \xi_{11}| \,-\, |\theta_{12} - \xi_{12}|.$$

The reader can verify that there exists no VCG payment scheme that balances the budget ex post regardless of the chosen efficient allocation rule. Instead, we show that for any $0 \le \lambda_k(\theta_{1k}, \xi_{1k}) \le 1$, for $k = 1, 2$ in Equation 20, there is no VCG payment scheme for which payments from the agents vanish everywhere. Indeed, VCG payments satisfy

$$p_A^{VCG}(\theta_1, \xi_1) \;=\; h_A(\xi_1) \,-\, \xi_{11}\xi_{12} \,+\, \sum_{k=1,2} (1 - \lambda_k(\theta_{1k}, \xi_{1k}))|\theta_{1k} - \xi_{1k}|,$$
$$p_B^{VCG}(\theta_1, \xi_1) \;=\; h_B(\theta_1) \,-\, \theta_{11}\theta_{12} \,+\, \sum_{k=1,2} \lambda_k(\theta_{1k}, \xi_{1k})|\theta_{1k} - \xi_{1k}|.$$

Suppose $p_A^{VCG} = p_B^{VCG} = 0$ everywhere, and fix $\xi_1 = \theta_1$. From the above equations, we see that $h_A(\xi_1) = \xi_{11}\xi_{12}$ and $h_B(\theta_1) = \theta_{11}\theta_{12}$. Since the function h_A has domain Ξ and the function h_B has domain Θ, these last two equalities are maintained for any $\theta_2 \ge \xi_1$, $\theta_2 \ne \xi_1$, and any $\xi_2 \le \theta_1$, $\xi_2 \ne \theta_1$. But then one has that either $p_A^{VCG}(\theta_2, \xi_1) \ne 0$ or $p_B^{VCG}(\theta_1, \xi_2) \ne 0$, regardless of how the efficient allocation is specified.

Consider an allocation rule for which $0 < \lambda_k(\theta_{1k}, \xi_{1k}) < 1$ for all type profiles $(\theta_1, \xi_1) \in \Theta \times \Xi$, for $k = 1, 2$. This yields to valuation functions that are differentiable almost everywhere in their respective type domains. It follows that revenue equivalence is in place (cf. Section 5.2) and thus only VCG payment rules can be employed to provide appropriate incentives. We focus on allocation rules that select corner solutions, so that $\lambda_k(\theta_{1k}, \xi_{1k}) \in \{0, 1\}$ in Equation 20 for $k = 1, 2$, for all $(\theta_1, \xi_1) \in \Theta \times \Xi$. To be more concrete, we consider the allocation rule $X^*\colon \Theta \times \Xi \to \mathscr{X}$ defined for all (θ_1, ξ_1) by

$$X_k^*(\theta_{1k}, \xi_{1k}) \;=\; \min\{\theta_{1k}, \xi_{1k}\}, \qquad k = 1, 2,$$

and characterize all incentive compatible payment rules for X^*.

Fix a type ξ_1 for B and consider arbitrary types $\theta_1, \theta_2 \in \Theta$ for agent A. The right and left subderivative functions between θ_1 and θ_2, for given ξ_1 are respectively,

$$\overline{s}(\theta_1^2(\alpha), \xi_1) \;\equiv\; \overline{\mathrm{d}} v_A\big(X^*(\theta_1^2(\alpha), \xi_1), \theta_1^2(\alpha); \delta_1^2\big), \qquad \text{for all } \alpha \in (0,1),$$
$$\underline{s}(\theta_1^2(\alpha), \xi_1) \;\equiv\; \underline{\mathrm{d}} v_A\big(X^*(\theta_1^2(\alpha), \xi_1), \theta_1^2(\alpha); \delta_1^2\big), \qquad \text{for all } \alpha \in (0,1).$$

Write $\delta_1^2 \equiv \theta_2 - \theta_1$ as $\delta_1^2 = (\delta_{11}^2, \delta_{12}^2)$ and $\theta_1^2(\alpha) \equiv \theta_1 + \alpha\delta_1^2$ as $\theta_1^2(\alpha) = (\theta_{11}^2(\alpha), \theta_{12}^2(\alpha))$. Using this notation, we compute from the previous expressions the right and left subderivatives between θ_1 and θ_2, respectively:

$$\begin{aligned}\overline{s}(\theta_1^2(\alpha),\xi_1) \;=\;& \theta_{12}^2(\alpha)\delta_{11}^2 \,+\, \theta_{11}^2(\alpha)\delta_{12}^2 \\ & + \left\{\begin{array}{ll} -|\delta_{11}^2| & : \theta_{11}^2(\alpha) \le \xi_{11} \\ -\delta_{11}^2 & : \theta_{11}^2(\alpha) > \xi_{11} \end{array}\right\} + \left\{\begin{array}{ll} -|\delta_{12}^2| & : \theta_{12}^2(\alpha) \le \xi_{12} \\ -\delta_{12}^2 & : \theta_{12}^2(\alpha) > \xi_{12} \end{array}\right\};\end{aligned}$$

and

$$\begin{aligned}\underline{s}(\theta_1^2(\alpha),\xi_1) \;=\;& \theta_{12}^2(\alpha)\delta_{11}^2 \,+\, \theta_{11}^2(\alpha)\delta_{12}^2 \\ & + \left\{\begin{array}{ll} |\delta_{11}^2| & : \theta_{11}^2(\alpha) \le \xi_{11} \\ -\delta_{11}^2 & : \theta_{11}^2(\alpha) > \xi_{11} \end{array}\right\} + \left\{\begin{array}{ll} |\delta_{12}^2| & : \theta_{12}^2(\alpha) \le \xi_{12} \\ -\delta_{12}^2 & : \theta_{12}^2(\alpha) > \xi_{12} \end{array}\right\}.\end{aligned}$$

The above subderivative functions determine the subderivative correspondence $S\big(\theta_1^2(\cdot),\xi_1\big)$, where for each $\alpha \in (0,1)$ one has $S\big(\theta_1^2(\alpha),\xi_1\big) = [\overline{s}(\theta_1^2(\alpha),\xi_1), \underline{s}(\theta_1^2(\alpha),\xi_1)]$.

Let $\theta_0 = (0,0)$ and normalize payoffs so that $U_A(\theta_0,\xi_1) = 0$. Since $\theta_0^1(\alpha) = \alpha\theta_1$ and $\delta_0^1 = \theta_1$, immediately one obtains:

$$\overline{s}(\theta_0^1(\alpha),\xi_1) \;=\; 2\alpha\theta_{11}\theta_{12} \,+\, \left\{\begin{array}{ll} -\theta_{11} & : \alpha\theta_{11} \le \xi_{11} \\ -\theta_{11} & : \alpha\theta_{11} > \xi_{11} \end{array}\right\} + \left\{\begin{array}{ll} -\theta_{12} & : \alpha\theta_{12} \le \xi_{12} \\ -\theta_{12} & : \alpha\theta_{12} > \xi_{12} \end{array}\right\}, \tag{21}$$

$$\underline{s}(\theta_0^1(\alpha),\xi_1) \;=\; 2\alpha\theta_{11}\theta_{12} \,+\, \left\{\begin{array}{ll} \theta_{11} & : \alpha\theta_{11} \le \xi_{11} \\ -\theta_{11} & : \alpha\theta_{11} > \xi_{11} \end{array}\right\} + \left\{\begin{array}{ll} \theta_{12} & : \alpha\theta_{12} \le \xi_{12} \\ -\theta_{12} & : \alpha\theta_{12} > \xi_{12} \end{array}\right\}. \tag{22}$$

We use Proposition 2 to express any incentive compatible payment rule $\theta_1 \mapsto p_A(\theta_1,\xi_1)$ as

$$p_A(\theta_1,\xi_1) \;=\; v_A(X^*(\theta_1,\xi_1),\theta_1) \,-\, \int_0^1 s(\theta_0^1(\alpha),\xi_1)\,d\alpha,$$

where the family of integrable selections $\mathscr{S}_0(\xi_1) = \{s(\theta_0^1(\cdot),\xi_1) \in S\big(\theta_0^1(\cdot),\xi_1\big) \mid \theta_1 \in \Theta\}$ satisfies closed-path integrability (b.1) and integral monotonicity (b.2).

Routine calculations show that, for all types θ_1,θ_2, the integrals

$$\begin{aligned} I' \;&\equiv\; \int_0^1 s'(\theta_1^2(\alpha),\xi_1)\,d\alpha \;=\; \int_0^1 \overline{s}(\theta_0^2(\alpha),\xi_1)\,d\alpha \,-\, \int_0^1 \overline{s}(\theta_0^1(\alpha),\xi_1)\,d\alpha, \\ I'' \;&\equiv\; \int_0^1 s''(\theta_1^2(\alpha),\xi_1)\,d\alpha \;=\; \int_0^1 \underline{s}(\theta_0^2(\alpha),\xi_1)\,d\alpha \,-\, \int_0^1 \underline{s}(\theta_0^1(\alpha),\xi_1)\,d\alpha, \end{aligned}$$

are such that

$$\int_0^1 \overline{s}(\theta_1^2(\alpha,\xi_1)\,d\alpha \;\le\; I', I'' \;\le\; \int_0^1 \underline{s}(\theta_1^2(\alpha),\xi_1)\,d\alpha.$$

Moreover, since

$$\begin{aligned} v_A(X^*(\theta_2,\xi_1),\theta_2) - v_A(X^*(\theta_2,\xi_1),\theta_1) &\geq \int_0^1 \underline{s}(\theta_1^2(\alpha),\xi_1)\,d\alpha \quad \text{and} \\ v_A(X^*(\theta_1,\xi_1),\theta_2) - v_A(X^*(\theta_1,\xi_1),\theta_1) &\leq \int_0^1 \overline{s}(\theta_1^2(\alpha),\xi_1)\,d\alpha, \end{aligned}$$

it follows that conditions (b.1) and (b.2) of our Theorem 1 are satisfied for the selections defined in Equation 21 and Equation 22.

From the previous discussion and Corollary 1, we conclude that any convex combination of the selections in (21) and (22) can be used to construct incentive compatible payment rules for agent A, which we express as

$$p_A(\theta_1,\xi_1) = \sum_{k=1,2}\left[-\,|\min\{\theta_{1k},\xi_{1k}\} - \theta_{1k}| - \int_0^{\min\{1,\xi_{1k}/\theta_{1k}\}} s_{1k}\,d\alpha + \int_{\min\{1,\xi_{1k}/\theta_{1k}\}}^1 \theta_{1k}\,d\alpha\right]$$

where the constant s_{1k} belongs to the interval $[-\theta_{1k},\theta_{1k}]$, depending on which selection one deploys. It follows that any incentive compatible payment scheme that implements X^* generates payments for agent A that satisfy

$$\sum_{k=1,2} -\min\{\theta_{1k},\xi_{1k}\} \;\leq\; p_A(\theta_1,\xi_1) \;\leq\; \sum_{k=1,2}\min\{\theta_{1k},\xi_{1k}\}.$$

Analogous expression holds for agent B's payment rule $p_B(\theta_1,\xi_1)$. Note that by choosing the selection $s^*(\theta_0^1(\cdot),\xi_1) = \frac{1}{2}\overline{s}(\theta_0^1(\alpha),\xi_1) + \frac{1}{2}\underline{s}(\theta_0^1(\alpha),\xi_1)$ for A and its counterpart for agent B, we generate an incentive compatible payment scheme for which

$$p_A^*(\theta_1,\xi_1) = p_B^*(\theta_1,\xi_1) = 0, \qquad \text{all } (\theta_1,\xi_1)\in\Theta\times\Xi.$$

In contrast, the VCG payment scheme that implements X^* and ensures zero indirect utility for the lowest type of both agents must have $h_A(\xi_1) = \xi_{11}\xi_{12} - \xi_{11} - \xi_{12}$ and $h_B(\theta_1) = \theta_{11}\theta_{12} - \theta_{11} - \theta_{12}$, from which one derives that in this case

$$p_i^{VCG}(\theta_1,\xi_1) = \sum_{k=1,2} -\min\{\theta_{1k},\xi_{1k}\}, \qquad \text{all } (\theta_1,\xi_1)\in\Theta\times\Xi,\ i=A,B.$$

Thus, implementing the allocation rule X^* via VCG payments requires the largest possible subsidies among all incentive compatible payment schemes.

4.3 A principal–agent problem with non-contractible actions

Consider a principal–agent setting where the principal designs a contract specifying an allocation $x \in \mathscr{X}$ (where $\mathscr{X}$ is a measurable set) in exchange for a payment $\rho \in \mathbb{R}$, and the payoffs of the agent, net of transfers, depend on alternative x and a privately observed type $\theta \in \Theta = [0,1]$. We extend this basic framework by adding an *ex post* stage in which the agent selects a non-contractible, private action a from a finite set A. Let

$$f(x, \theta, a)$$

denote the agent's ex post payoff function (excluding payments to the principal). To sharpen the focus on the non-differentiability that arises from the presence of non-contractible ex post actions, we assume that for every allocation x and action a, the function $\theta \to f(x, \theta, a)$ is differentiable, with uniformly bounded (partial) derivative $\theta \to f_\theta(x, \theta, a)$. Further, we assume that for every type θ and action a, the function $x \mapsto f_\theta(x, \theta, a)$ is measurable.

Since an action is chosen by the agent after the resolution of the contract, we can derive the agent's induced valuation, as a function of x and θ alone, as follows:

$$v(x, \theta) \;=\; \max_{a \in A} f(x, \theta, a), \qquad\qquad \text{for all } x \text{ and } \theta.$$

Let $B(x, \theta) = \text{argmax}_{a \in A} f(x, \theta, a)$ denote the set of optimal actions for an agent of type θ who receives x from the principal. Although the ex post payoff function $f(x, \cdot, a)$ is differentiable, the induced valuation $v(x, \theta)$ will typically fail to be differentiable with respect to θ at any point (x, θ) at which $B(x, \theta)$ is not a singleton — see Figure 2. Notice also that without further assumptions on the ex post payoff function, convexity of the induced valuation is not guaranteed. When posterior choices by the agent are non-contractible, the principal's contract cannot make the allocation or payment contingent on the choice of a. Therefore, the principal takes the induced valuation as given and designs a direct contract (X, p) as a function of θ alone. We apply the results of Section 3 to characterize incentive compatible contracts in this setting.

First we note that the conditions of Theorem 1 are satisfied. Indeed, we have assumed (A1) and (A2) directly and (A3) follows from the fact that the family of functions $\{f_\theta(x, \cdot, a) \mid x \in \mathscr{X}, a \in A\}$ is uniformly bounded. We use the following lemma to show that condition (M) is satisfied by any measurable allocation rule.

Lemma 3. *In the principal–agent model with non-contractible ex post actions, the right and left subderivatives of the induced valuation function are derived from the ex post payoff function as follows:*

$$\overline{\mathrm{d}}v(x, \theta) \;=\; \max_{a \in B(x,\theta)} f_\theta(x, \theta, a) \qquad \textit{and} \qquad \underline{\mathrm{d}}v(x, \theta) \;=\; \min_{a \in B(x,\theta)} f_\theta(x, \theta, a).$$

Proof For the first equality, note that by definition we have

$$\begin{aligned} \overline{\mathrm{d}}v(x, \theta) \;&=\; \liminf_{\hat{\theta} \downarrow \theta} \frac{v(x, \hat{\theta}) - v(x, \theta)}{\hat{\theta} - \theta} \\ &=\; \liminf_{\hat{\theta} \downarrow \theta} \frac{\max_{a \in B(x,\hat{\theta})} f(x, \hat{\theta}, a) \;-\; \max_{a \in B(x,\theta)} f(x, \theta, a)}{\hat{\theta} \;-\; \theta}. \end{aligned}$$

Since the set of private actions A is finite and the ex post payoff function $f(x, \cdot, a)$ is

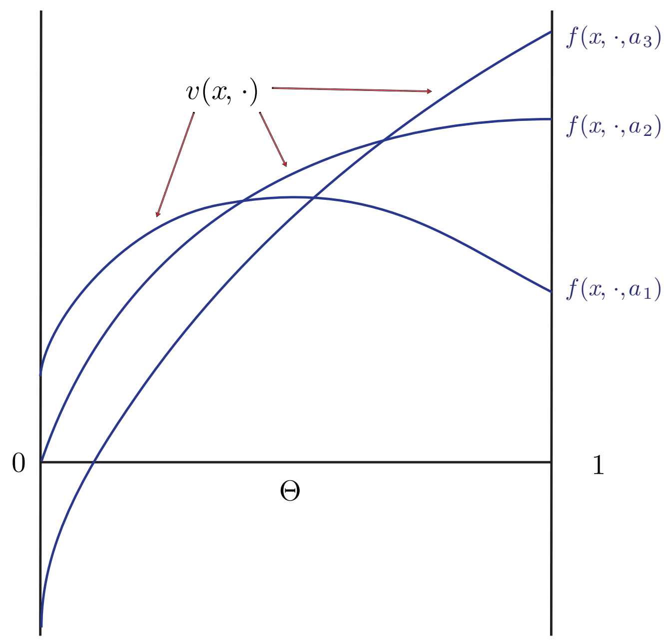

Figure 2: induced valuation $\theta \mapsto v(x,\theta)$

continuous on Θ, there exists $\epsilon > 0$ such that when $0 < \hat{\theta} - \theta < \epsilon$, $B(x,\hat{\theta}) \subset B(x,\theta)$. Thus,

$$\begin{aligned} \overline{\mathrm{d}}v(x,\theta) &= \liminf_{\hat{\theta}\downarrow\theta} \max_{a\in B(x,\hat{\theta})} \frac{f(x,\hat{\theta},a) - f(x,\theta,a)}{\hat{\theta}-\theta} \\ &= \max_{a\in B(x,\theta)} \liminf_{\hat{\theta}\downarrow\theta} \frac{f(x,\hat{\theta},a) - f(x,\theta,a)}{\hat{\theta}-\theta} = \max_{a\in B(x,\theta)} f_\theta(x,\theta,a). \end{aligned}$$

To see why it is valid to interchange the max and lim inf operators in the above expression, we argue that since $B(x,\hat{\theta}) \subset B(x,\theta)$ for $\hat{\theta} > \theta$ sufficiently close to θ, it follows that for any $a' \in B(x,\theta)$,

$$\liminf_{\hat{\theta}\downarrow\theta} \max_{a\in B(x,\hat{\theta})} \frac{f(x,\hat{\theta},a) - f(x,\theta,a)}{\hat{\theta}-\theta} \geq \liminf_{\hat{\theta}\downarrow\theta} \frac{f(x,\hat{\theta},a') - f(x,\theta,a')}{\hat{\theta}-\theta},$$

from where, immediately, one obtains:

$$\liminf_{\hat{\theta}\downarrow\theta} \max_{a\in B(x,\hat{\theta})} \frac{f(x,\hat{\theta},a) - f(x,\theta,a)}{\hat{\theta}-\theta} \geq \max_{a\in B(x,\theta)} \liminf_{\hat{\theta}\downarrow\theta} \frac{f(x,\hat{\theta},a) - f(x,\theta,a)}{\hat{\theta}-\theta}. \tag{23}$$

In the other direction, for $\hat{\theta} > \theta$ sufficiently closed to θ, for any $a'' \in B(x,\hat{\theta})$, one has

$$\liminf_{\hat{\theta}\downarrow\theta} \frac{f(x,\hat{\theta},a'') - f(x,\theta,a'')}{\hat{\theta}-\theta} \geq \liminf_{\hat{\theta}\downarrow\theta} \max_{a\in B(x,\hat{\theta})} \frac{f(x,\hat{\theta},a) - f(x,\theta,a)}{\hat{\theta}-\theta},$$

and since $B(x,\hat{\theta}) \subset B(x,\theta)$, one readily obtains:

$$\max_{a \in B(x,\theta)} \liminf_{\hat{\theta} \downarrow \theta} \frac{f(x,\hat{\theta},a) - f(x,\theta,a)}{\hat{\theta} - \theta} \geq \liminf_{\hat{\theta} \downarrow \theta} \max_{a \in B(x,\hat{\theta})} \frac{f(x,\hat{\theta},a) - f(x,\theta,a)}{\hat{\theta} - \theta}. \tag{24}$$

Combining (23) and (24) yields to the validity of the expression for the right subderivative $\overline{\mathrm{d}}v(x,\theta)$. The corresponding expression for the left subderivative is derived in a similar fashion. ■

From Lemma 3 we conclude that condition (M) will be satisfied for any measurable allocation rule. Indeed, by the measurable maximum theorem, the subderivatives $\overline{\mathrm{d}}v(x,\theta)$ and $\underline{\mathrm{d}}v(x,\theta)$, as the pointwise maximum and minimum of a finite family of measurable functions, respectively, are measurable. Thus, for any measurable allocation rule X, both $\overline{s}(\theta)$ and $\underline{s}(\theta)$, as the composition of measurable functions — see (6) and (7) — are also measurable. Now consider any incentive compatible mechanism (X,p). Then:

$$\overline{s}(\theta) = \overline{\mathrm{d}}v(X(\theta),\theta) = \max_{a \in B(X(\theta),\theta)} f_\theta(X(\theta),\theta,a)$$

and

$$\underline{s}(\theta) = \underline{\mathrm{d}}v(X(\theta),\theta) = \min_{a \in B(X(\theta),\theta)} f_\theta(X(\theta),\theta,a).$$

It follows that at any type $\theta \in [0,1]$, the subderivative correspondence $S(\theta)$ is either empty (when $\overline{s}(\theta) > \underline{s}(\theta)$) or single-valued (for example, when $B(X(\theta),\theta)$ is a singleton). Applying Theorem 1, we have that $\theta \rightrightarrows S(\theta)$ is regular and therefore non empty-valued almost everywhere. In particular, for this class of problems the subderivative correspondence is single-valued a.e., and therefore admits a unique selection $s(\theta)$. As we shall see in Section 5.2, in such cases the standard revenue equivalence is back in place.

5 Discussion

We return to the single agent environment described in Section 2.1 to relate our techniques to recent developments in the literature.

5.1 When is (M) satisfied?

In the applications considered in Section 4, our assumption (M) is verified without difficulty. More importantly, (M) will follow directly from the structure of the design problem in commonly used environments. Consider, as Archer and Kleinberg (2008), a setting with a bounded allocation set $\mathscr{X} \subseteq \mathbb{R}^k$, a bounded convex type space $\Theta \subseteq \mathbb{R}^k$, and valuation that are linear in types, so that $\theta \mapsto v(x,\theta) = x \cdot \theta$ for each allocation x. Notice that (A1) to (A3) are satisfied. Let $X \colon \Theta \to \mathscr{X}$ be any measurable allocation rule. One readily sees that the derivative of v with respect to θ evaluated at $(X(\theta),\theta)$ is given by $Dv(X(\theta),\theta) = X(\theta)$, therefore (M) is satisfied and our characterization theorem applies. Berger, Müller, and Naeemi (2010) deal with an arbitrary allocation set, a convex type space in $\mathbb{R}^k$, and valuations that are convex functions of types. Directional derivatives of convex and concave functions are sufficiently well-behaved for our characterization result to be applicable. We require the following preliminary result, which is taken from Hildenbrand (1974, p. 42). Here $\mathcal{M} \otimes \mathcal{B}([0,1])$ denotes the product σ-algebra of $\mathcal{M}$ and $\mathcal{B}([0,1])$.

Proposition 3. *Let $(\mathscr{X}, \mathcal{M})$ be a measurable space and $f\colon \mathscr{X} \times [0,1] \to \mathbb{R}$ be a bounded real-valued function. Suppose the following conditions hold:*

(a) For every $\alpha \in [0,1]$, the function $x \mapsto f(x,\alpha)$ is $\mathcal{M}$-measurable.

(b) For every $x \in \mathscr{X}$, the function $\alpha \mapsto f(x,\alpha)$ is right (or left) continuous.

Then the function f is $\mathcal{M} \otimes \mathcal{B}([0,1])$-measurable.

Under a mild measurability requirement on the right and left subderivative functions, Theorem 1 also covers relevant cases previously studied in the literature (see Section 1 for references to the convex and differentiable case).

Proposition 4. *Let (A1) to (A3) be satisfied. Assume that for every type $\theta \in \Theta$ and every directional vector $\delta \in \mathbb{R}^k$, the functions $x \mapsto \overline{\mathrm{d}}v(x,\theta;\delta)$ and $x \mapsto \underline{\mathrm{d}}v(x,\theta;\delta)$ are $\mathcal{M}$-measurable. In addition, assume one of the following conditions hold:*

(a) For every $x \in \mathscr{X}$, $\theta \mapsto v(x,\theta)$ is convex in Θ.

(b) For every $x \in \mathscr{X}$, $\theta \mapsto v(x,\theta)$ is concave in Θ.

(c) For every $x \in \mathscr{X}$, $\theta \mapsto v(x,\theta)$ is continuously differentiable in Θ.

If $X\colon \Theta \to \mathscr{X}$ is a measurable allocation rule, then (M) is satisfied.

Proof We deal with convex valuations; the remaining cases are similarly handled. Fix $\theta_1, \theta_2 \in \Theta$. From (A3) and the convexity of $\theta \mapsto v(x,\theta)$ on Θ, it follows that for every $x \in \mathscr{X}$ the right subderivative $\overline{\mathrm{d}}v(x,\theta_1^2(\alpha);\delta_1^2)$ is equal to the right derivative of the convex real-valued function $\alpha \mapsto w(\alpha;x,\delta_1^2) = v(x,\theta_1^2 + \alpha\,\delta_1^2)$ defined on $(0,1)$. Thus, the function $\alpha \mapsto \overline{\mathrm{d}}v(x,\theta_1^2(\alpha);\delta_1^2)$ is right continuous on $(0,1)$. By assumption, the function $x \mapsto \overline{\mathrm{d}}v(x,\theta_1^2(\alpha);\delta_1^2)$ is $\mathcal{M}$-measurable for every $\alpha \in (0,1)$. Thus, we infer from Proposition 3 that the function $\overline{\mathrm{d}}(\cdot,\cdot;\delta_1^2)$ is $\mathcal{M} \otimes \mathcal{B}(0,1)$-measurable. Since X is a measurable allocation rule, it is seen that the function $\alpha \mapsto (X(\theta_1^2(\alpha)),\theta_1^2(\alpha))$ is $\mathcal{B}(0,1)$-measurable, hence we obtain the measurability of $\alpha \mapsto \overline{s}(\theta_1^2(\alpha)) = \overline{\mathrm{d}}v(X(\theta_1^2(\alpha)),\theta_1^2(\alpha);\delta_1^2)$ by noticing that the composition of measurable functions is also measurable. The argument for $\underline{s}(\theta_1^2(\cdot))$ is similar, except that in this case one uses the left derivative of the respective convex scalar function, which is left continuous on $(0,1)$. ■

Suppose that we partition $\mathscr{X}$ into measurable sets $\{\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3\}$, such that for every $x \in \mathscr{X}_1$ (respectively, $\mathscr{X}_2, \mathscr{X}_3$), the function $\theta \mapsto v(x,\theta)$ is convex in Θ (respectively, concave, continuously differentiable). One can use Proposition 4 to show that (M) is also satisfied in this type of mixed models.

5.2 Back to Revenue Equivalence

Our revenue inequality (Theorem 2) establishes a precise range for the value of the difference between two indirect utilities associated with the allocation rule X. If for all types θ_1, θ_2, the subderivative correspondence $S\left(\theta_1^2(\cdot)\right)$ is single-valued almost everywhere, then revenue equivalence is restored.

Proposition 5. *Assume that (A1) to (A3) and (M) are satisfied. Let $p\colon \Theta \to \mathbb{R}$ and $p'\colon \Theta \to \mathbb{R}$ be two payment rules that implement $X\colon \Theta \to \mathscr{X}$ and generate indirect utility functions U and U', respectively, satisfying $U(\theta_0) = U'(\theta_0) = u_0$. If for every pair of types $\theta_1, \theta_2 \in \Theta$, one has $\underline{s}(\theta_1^2(\alpha)) = \overline{s}(\theta_1^2(\alpha))$ for almost all $\alpha \in (0,1)$, then $U = U'$.*

Proof Suppose that for all θ_1, θ_2 in Θ, it is the case that $\underline{s}(\theta_1^2(\alpha)) = \overline{s}(\theta_1^2(\alpha))$ a.e. in $(0,1)$. Readily from Equation 14, it follows $U(\theta_1) = U'(\theta_1)$, for all $\theta_1 \in \Theta$. ■

We obtain the following corollary.

Corollary 3. *Under the assumptions of Proposition 5, $U = U'$ whenever one of the following conditions hold.*

(a) For every $x \in \mathscr{X}$, $\theta \mapsto v(x, \theta)$ is convex in Θ.

(b) For every $x \in \mathscr{X}$, $\theta \mapsto v(x, \theta)$ is differentiable in Θ.

(c) The allocation set $\mathscr{X}$ is finite.

Proof (a) Suppose $\theta \mapsto v(x, \theta)$ is convex. Then, for all $\hat{\theta} \in \Theta$ and all $\delta \in \mathbb{R}^k$,

$$\underline{\mathrm{d}}v(x, \hat{\theta}; \delta) \;=\; \lim_{r \uparrow 0} \frac{v(x, \hat{\theta} + r\delta) - v(x, \hat{\theta})}{r} \;\leq\; \lim_{r \downarrow 0} \frac{v(x, \hat{\theta} + r\delta) - v(x, \hat{\theta})}{r} \;=\; \overline{\mathrm{d}}v(x, \hat{\theta}; \delta).$$

Since Equation 11 requires that at almost all equilibrium points $(X(\theta), \theta)$ the reverse inequality holds, one sees that the condition of Proposition 5 is in place.

(b) Immediate.

(c) Let θ_1, θ_2 be arbitrary and define the function $\alpha \mapsto \hat{X}(\alpha) \equiv X(\theta_1^2(\alpha))$ on $(0,1)$. Since $\mathscr{X}$ is finite, $\hat{X}((0,1)) = \{x_1, \ldots, x_N\}$, for some $N \in \mathbb{N}$. Thus, the sets $E_1, \ldots, E_N$, with $E_n = \hat{X}^{-1}(\{x_n\})$ for each $n = 1, \ldots, N$, constitute a collection of pairwise disjoint, measurable subsets of $(0,1)$ whose union is $(0,1)$. Without loss of generality, we assume that each E_n has non zero measure. Then, for each $n = 1, \ldots, N$, the function $\alpha \mapsto \nu_n(\alpha) \equiv v(x_n, \theta_1^2(\alpha))$ is Lipschitz continuous on E_n and therefore differentiable a.e. in E_n (by the Rademacher-Stepanoff theorem). It follows that $\underline{s}(\theta_1^2(\alpha)) = \overline{s}(\theta_1^2(\alpha))$ almost everywhere on the unit interval $(0,1)$, as desired. ■

Note that Corollary 3-(c) does not require the valuation function to be linear, convex or differentiable in types: for the finite allocation sets, assumptions (A1) to (A3) and (M) suffice to obtain revenue equivalence. On the other hand, the concavity of $\theta \mapsto v(x, \theta)$, while yielding to (M), does not imply that revenue equivalence is back, as the first two applications of Section 4 demonstrate.

5.3 Weak monotonicity versus integral monotonicity

An allocation rule $X\colon \Theta \to \mathscr{X}$ satisfies the *weak monotonicity* condition if for every pair of types $\theta, \hat{\theta}$ in Θ, one has

$$v(X(\theta),\theta) - v(X(\theta),\hat{\theta}) \;\geq\; v(X(\hat{\theta}),\theta) - v(X(\hat{\theta}),\hat{\theta}).$$

Weak monotonicity has been shown to characterize implementation when the type space is convex, the valuation function linear in types, and the allocation set either finite or infinite (the latter case requiring the inclusion of the closed-path integrability condition).[14] Clearly, integral monotonicity implies weak monotonicity. However, there are examples where weak monotonicity does not imply integral monotonicity, even when $\mathscr{X}$ is finite and the valuation function $v(x,\cdot)$ is piecewise linear in types (Berger, Müller, and Naeemi, 2010).

On the other hand, Berger, Müller, and Naeemi (2010) discovered that a single-crossing property is sufficient to obtain the equivalence between weak monotonicity and integral monotonicity, under the assumption that valuations are either convex or differentiable in types. As the next proposition shows, their insight extends to our general setting. Say that the valuation function $v\colon \mathscr{X} \times \Theta \to \mathbb{R}$ satisfies *increasing differences* if for all $x, y \in \mathscr{X}$, all $\theta_1, \theta_2 \in \Theta$, and every $\theta_1^2(\alpha) \in L(\theta_1,\theta_2)$, $v(x,\theta_2) - v(y,\theta_2) \geq v(x,\theta_1^2(\alpha)) - v(y,\theta_1^2(\alpha))$ implies that $v(x,\theta_1^2(\alpha)) - v(y,\theta_1^2(\alpha)) \geq v(x,\theta_1) - v(y,\theta_1)$.

Proposition 6. *Assume that (A1) to (A3) and (M) are satisfied for the allocation rule $X\colon \Theta \to \mathscr{X}$. Suppose in addition that $v\colon \mathscr{X} \times \Theta \to \mathbb{R}$ satisfies increasing differences. Then for every θ_1, θ_2 in Θ, the subderivative correspondence $S\left(\theta_1^2(\cdot)\right)$ is regular and admits an integrable selection $\alpha \mapsto s(\theta_1^2(\alpha))$ such that X satisfies integral monotonicity if and only if X satisfies weak monotonicity.*

Proof Clearly the integral monotonicity condition implies that X is weakly monotone. Conversely, suppose that X satisfies weak monotonicity. Fix arbitrary types θ_1, θ_2. From Lemmas 3 and 5 of Berger, Müller, and Naeemi (2010) it follows that the increasing differences property of the valuation function together with weak monotonicity imply that X is cyclically monotone on the closure of $L(\theta_1,\theta_2)$. Therefore, the restriction of X to the closure of $L(\theta_1,\theta_2)$ is implementable. From Theorem 1, we conclude that the subderivative correspondence $S\left(\theta_1^2(\cdot)\right)$ is regular and admits an integrable selection $\alpha \mapsto s(\theta_1^2(\alpha))$ for which integral monotonicity is satisfied. ■

We stress the fact that, without additional assumptions, Proposition 6 does not imply that there exists a unique selection for which integral monotonicity holds. Thus, while weak monotonicity may be more readily verified in applications, we are nonetheless interested in finding the subderivative correspondence between types and selections consistent with the integral monotonicity condition, since these objects provide the (possibly many) payment rules that can be used for implementation.

[14] See Jehiel, Moldovanu, and Stacchetti (1999), Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan, and Sen (2006), Saks and Yu (2005), Archer and Kleinberg (2008), and Vohra (2011).

6 Concluding remarks

In this paper we study (dominant strategy) mechanism design in quasi-linear settings where the envelope theorem and the revenue equivalence principle fail, due to non-convexity and non-differentiability of the valuation function with respect to types. We obtain a characterization of incentive compatibility based on integral monotonicity and the Mirrlees representation of the indirect utilities (plus the standard closed-path integrability condition). Our framework allows for the allocation set to be finite or infinite and the type space of the agent to be a convex subset of a multi-dimensional Euclidean space. We work with uniformly Lipschitz valuations and impose a measurability requirement on the left and right subderivatives of the valuation function with respect to types.

These conditions pin down the range of payoff differences generated by two incentive compatible mechanisms with the same allocation rule. As a consequence, we obtain a revenue inequality in place of the standard revenue equivalence principle: given an implementable allocation rule and normalized payments that assign the same utility to the "lowest type", the difference in equilibrium payoffs generated by any two payment rules is bounded by the allocation rule alone, even though it may be not vanish.

Our results contribute to the already extensive literature on implementation in several ways. First, our environment imposes few restrictions on the type space and the valuation function. In particular, we do not assume linearity, convexity or differentiability of the valuation with respect to types, but any such hypothesis, in addition to a mild measurability condition, will imply our assumption (M). Second, our results exploit techniques that expand the approaches currently employed in the literature, thus allowing us to handle a broader set of problems with richer payment schemes: we can incorporate models from behavioral economics, which may represent important behavioral considerations via downward kinks in preferences, and we can study situations where there exist non contractible, ex post actions available to the informed party. We have provided three applications to illustrate this point. Clearly, this is a starting point and we leave the study of richer, more realistic models for the future. Third, our approach opens new aspects of institutional design. In particular, in certain economic environments of general interest it may be possible to use several payment rules (which differ by more than just a constant) to implement an efficient or revenue maximizing allocation rule. While such schemes may satisfy desirable properties (e.g., budget balancedness, individual rationality), other (un)desirable features could be introduced. We also leave a thorough study of these aspects of institutional design for future research.

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