A SYMMETRIC INFORMATION-THEORETIC INDEX FOR THE MEASUREMENT OF INEQUALITY

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Abstract

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The application of information theory to the measurement of income inequality has yielded an impressive array of measurement techniques known as Generalized Entropy (GE) measures. Special cases of this class of index include Theil's T and L measures which are considered axiomatically superior to other types of metrics including the popular Gini coefficient. In this paper we show that the parallel between information theory and inequality measurement has not yet been fully explored and propose a new inequality measure based upon this concept. The proposed measure is already established as a tool for use in statistical classification and signal processing problems and is known in these fields as the J-Divergence or Symmetric Kullback-Leibler Divergence. As an inequality metric the measure is shown to be axiomatically complete and is in possession of an additional property allowing for an alternate type of decomposition analysis. The new type of decomposition makes the contribution of any individual or subgroup to the inequality metric directly observable such that the overall index may be reconciled with a weighted sum of each group contribution. We illustrate with an example using income micro-data from the United States where we evaluate the contributions of various racial groups to overall inequality. We also provide a standard decomposition of the inequalities between and within the racial groups to contrast the techniques.

1. INTRODUCTION

Much of the groundwork for our understanding of the relationship between entropy and the measurement of inequality was laid out in a famous book by Theil (1967). The work applied concepts from information theory to the distribution of income to develop two intuitive indices for the measurement of inequality. The more widely used measure became known as Theil's 'T measure', the less common index is referred to as the 'L measure'. The intuition behind the T measure is of particular interest and is well illustrated with the following thought experiment: Suppose we have a small group of people where each person has some non-negative quantity of money, and at least one person has a positive quantity. If one dollar were to be randomly selected from the total holdings of the group, we may ask 'how surprised do we expect to be, on average, when we learn that the selected dollar belongs to a particular person?'

We can see that this thinking can lead us to an intuitive method for measuring equality. If one individual earns all the money we would not be at all surprised to learn that that person owned the randomly selected dollar. Thus it may be said that the expected *information content* (or level of surprise) contained in a draw from such a distribution is zero as we could have anticipated that individual's selection with certainty. Now suppose that each person has the same amount of money. If a dollar were to be randomly selected from this distribution we would have no *a priori* inclination as to whom it would likely belong to. Thus prior to the draw we would expect a high level of surprise when we learn the identity of the owner of the mystery dollar and the information content of this distribution is at its extremum. It should be clear that the first example (where the information content of the distribution is zero) coincides with our notion of perfect inequality, while in the second case, where information content is maximized, the incomes are perfectly equal. Thus the information content of such a distribution gives us a sense of the *equality* of money within the distribution, which may be converted to a measure of *inequality* with some simple algebraic manipulations.

The insight into this parallel between information theory and inequality measurement and the usefulness of the resultant techniques encouraged a flurry of related research. Cowell (1977) in particular formalized the use of a broad class of information theoretic techniques known as Generalized Entropy indices for the measurement of inequality, showing that several prominent measures including Theil's indices could be expressed as special cases of a broader class of measure. Furthermore the decomposability (Cowell and Kuga, 1981a), (Bourguignon, 1979) and axiomatic superiority of these measures (Cowell and Kuga, 1981b), (Foster, 1983) (Shorrocks, 1982) led to their widespread implementation in empirical works. Prominent applied pieces using these techniques and their ordinal equivalents (Atkinson, 1978) include studies by Sala-i-Martin (2006), Sala-i-Martin (2002), Milanovic (2002) and Chotikapanich et al. (1997).

The purpose of this paper is to plot a similar course taken by the above authors for the development of a new measure of inequality based on the concept of entropy/information theory. We set out in section 2 by reviewing some concepts in information theory and show how they may be used to develop a new inequality index. In section 3 the axioms of inequality measurement are presented and it is shown that the new index is axiomatically complete. In section 4 we discuss an alternative method for decomposition analysis which allows the researcher to measure the extent to which any arbitrarily defined subgroup contributes to the overall inequality metric. We investigate the relationship between the proposed method and the standard decomposition technique for separating inequality into between-group and within-group estimates in section 5. Lastly in section 6 we demonstrate the new measure and decomposition technique using income data for various U.S. racial groups.

2. INFORMATION THEORY AND INEQUALITY

To review some basic principles from information theory we consider a set of *n* possible events $E_1, E_2 ... E_n$ with associated probabilities of occurrence $(p_1, p_2, ... p_n)$. The events are mutually exclusive and collectively exhaustive such that exactly one will occur. This implies

$$\sum_{i=1}^{n} p_i = 1 \qquad p_i \geq 0 \qquad i = 1...n \tag{1}$$

Suppose event E_i occurs and we wish to assign an information (surprise) level (*h)* to this occurrence based on the probability p_i. To do so we must specify an 'information function' for $h(p)$ to map the probability of each event to the real line. Generally we would require any candidate for this function to be non-negative and downward sloping over the unit interval, such that only positive information contents exist and that the more probable an event, the less information we gain by observing it. We may also insist that $h(0) = \infty$ and $h(1) = 0$ (impossible events bring infinite information and certain events yield no information) and that there is additivity in the case of independent events i.e. $h(p_1, p_2) = h(p_1) + h(p_2)$. Such conditions impose a logarithmic specification for the information function of the form $h(p) = -\ln(p)$ (see Theil, 1967). This implies that the information content of our specific event E_i is $h(p_i) = -\ln(p_i)$ and that the expected, *a priori* information content of the distribution may be written as the probability-weighted sum of the individual contents

$$H(x) = -\sum_{i=1}^{n} p_i \ln(p_i) \tag{2}$$

Now suppose we have the same set of events $E_1, E_2 ... E_n$ and their corresponding probabilities $(p_1, p_2, ... p_n)$. Instead of anticipating the gain in information we will receive when we learn which event has occurred, this time we receive a message that merely causes us to revise our set of probabilities. We denote these new probabilities $(q_1, q_{2,} ... q_n)$. The information gained from this indirect message is simply the summed gain (or loss) in the information content of each event $h(p_i) - h(q_i)$ weighted by the new (posterior) probabilities. We can therefore write the expected information of the (indirect) message as

$$I(p:q) = \sum_{i=1}^{n} q_i \ln\left(\frac{q_i}{p_i}\right) \tag{3}$$

where $I(p:q)$ denotes the information content of the message transforming prior probabilities *p* into posterior probabilities *q*. Here we may draw our parallel between information content and inequality measurement. Expression (3) is analogous to Theil's T inequality measure, where the posterior probability q_i represents the proportion of total

income accruing to group *i*, and prior probability p_i represents the proportion of the total population in group *i*. We emphasize that this definition for the T measure is compatible with the thought experiment in section (1), only in this instance the result is a little more general as we allow for the population shares p_i to vary in size, rather than being restricted to being equal to $1/n$. The index functions extremely well as an inequality measure (Cowell, 1981); taking on a value of zero when all income shares are equal to their corresponding population shares, and is equal to $\ln(p_i)$ in the perfectly unequal case when one group (p_i) controls all the income.

This derivation of Theil's T index however contains the implicit assumption that our indirect message transforms the population shares p_i into income shares q_i. That is, we treat the population shares as the prior probabilities (before the message is received) and our income shares as the posterior probabilities (after the message) and thus Theil's T index is interpretable as the information gain as we travel from *p* to *q*. In the realm of inequality measurement there is no particular reason why this assignment of 'prior' and 'posterior' labels should be so and hence the 'direction' of the message is arbitrarily chosen. If the income shares are reclassified as prior probabilities and the population shares as posterior probabilities we can evaluate the information content of a message running the opposite direction. This can be calculated as

$$I(q:p) = \sum_{i=1}^{n} p_i \ln\left(\frac{p_i}{q_i}\right) \tag{4}$$

which is an expression for Theil's L inequality index. There is a distinct asymmetry or non-commutativity here, as the information content of a message transforming population shares into income shares is notably different from the information content of a message transforming the income shares into population shares. Thus the choice of how we assign the labels of prior and posterior to the income and population shares determines the measure of inequality to be used. This may be seen as somewhat unsatisfactory as information theory requires us to make a choice on the direction of the message on an arbitrary basis, with the specification of the inequality measure as a non-trivial consequence.

At this point common practice for the economist is to choose a measure based on some other criteria outside the domain of information theory such as the sensitivity of a measure to transfers at higher or lower incomes. There is, however, an alternative solution to this problem provided within information theory that is used in diverse fields such as statistical classification and signal processing. When it is unclear which set of probabilities should be labeled as prior and posterior, a solution is to calculate the information content of indirect messages running both ways rather than arbitrarily selecting one or the other. The sum of the expected information content of the two messages may then be used as a measure of the distance between the two distributions. This metric is variously known as the Symmetrized Kullback-Leibler Divergence,

(Kullback and Leibler, 1951), (Kullback, 1983), Symmetric Relative Entropy or the J-Divergence (Jeffreys, 1946). The statistic is

$$I = I(p:q) + I(q:p) = \sum_{i=1}^{n} (q_i - p_i) \ln\left(\frac{q_i}{p_i}\right) \tag{5}$$

where I is the inequality index proposed in this paper. This statistic provides a measure of the divergence between p and q and does not depend on the assignment of prior or posterior probabilities. In the context of inequality measurement it is simply the sum of Theil's T and L statistics from equations (3) and (4).

Alternative specifications of this index applicable to unit record data and continuous distributions are easily determined and can be given as

$$I = \frac{1}{n} \sum_{i=1}^{n} \frac{x_i - \mu}{\mu} \ln\left(\frac{x_i}{\mu}\right) \tag{6}$$

where Theil's T and L measures are

$$T = \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{\mu} \ln\left(\frac{x_i}{\mu}\right) \qquad L = \frac{1}{n} \sum_{i=1}^{n} -\ln\left(\frac{x_i}{\mu}\right)$$

In the continuous case

$$I = \int_0^\infty \frac{x}{\mu} \ln\left(\frac{x}{\mu}\right) f(x)dx + \int_0^\infty \ln\left(\frac{\mu}{x}\right) f(x)dx \tag{7}$$

where

$$T = \int_0^\infty \frac{x}{\mu} \ln\left(\frac{x}{\mu}\right) f(x)dx \qquad L = \int_0^\infty -\ln\left(\frac{x}{\mu}\right) f(x)dx$$

This new technique for the measurement of inequality has some desirable properties which are outlined in the next section. As the measure is simply the sum of two axiomatically sound inequality indices the proposed measure inherits much of the desirable characteristics from Theil's measures.

3. PROPERTIES OF THE NEW MEASURE

The axioms of inequality measurement provide a strict set of criteria for assessing the capability of a particular inequality index and are reviewed in this section. It is generally accepted that an effective measure must satisfy most of the listed axioms, while the most sophisticated techniques are entirely axiomatically sound. Here we review the axioms and test our new measure against them. Definitions are taken from Fields and Fei (1978),

Foster (1983), Sen (1973), Cowell and Kuga (1981) and Kolm (1976). To illustrate the axioms consider the set of distributions $D = \{x : x_i \in \Re \wedge x_i > 0\}$ where x is income or some other economic variable of interest. Let $I : x \to \Re_+$ be an inequality measure. The definitions are:

Axiom (1) Symmetry

A symmetric measure considers only the incomes of the individuals being measured such that any rearrangements of incomes amongst individuals will leave the measure unchanged. Take the distribution $x \in D$. Inequality measure I is symmetric if $I(x) = I(xP)$ where P is any $n \times n$ permutation matrix[1]. This is analogous to stating $I(x_1, x_2 ... x_n) = I(x_2, x_1, .. x_n)$ for any possible regrouping of x.

Axiom (2) Scale independence

A measure should be insensitive to proportional changes in the underlying variable including changes in the units of measurement. The inequality measure I satisfies the relativity or scale independence axiom if $I(x) = I(\lambda x) \ \forall \lambda \in \Re_+$ where λ is an arbitrary scaling factor.

Axiom (3) Population replication

The inequality measure should be invariant to proportional changes in the underlying population. Consider distribution x' which is a proportional replicate of distribution x where each element occurs with frequency f i.e. $x' = \{x_{11}, ... x_{1f}, x_{21,} ... x_{2f}, x_{n1}, ... x_{nf}\}$. I will satisfy the principle of population replication if $I(x; f) = I(x) \ \forall f \in \mathrm{N}_+$.

Axiom (4) Pigou-Dalton transfer principle

The Pigou-Dalton transfer principle requires that any order preserving transfer of income from a higher to a lower income earner must reduce the inequality measure. $I(x)$ will satisfy the transfer principle if $T_{ji}(x) < 0$ for $x_i < x_j$ where it exists and

$$T_{ji}(x) \equiv \lim_{\varepsilon \to 0} \frac{\{I(x_1, x_2, ... x_i + \varepsilon, x_j - \varepsilon, x_n) - I(x_1, x_2, ... x_n)\}}{\varepsilon}.$$

Axiom (5) Decomposability

Consider a partition of x into k subgroups such that each subgroup x_j has $n_j \geq 1$ elements, $x_j \cap x_k = \varnothing$ for $x_j \neq x_k$ and $\sum_{j=1}^{k} n_j = n$. We also define $p_j = n_j / n$ and refer

[1] A matrix which may be constructed from any permutation of the rows and columns of an $n \times n$ identity matrix.

to this as the population share of group *j*, while q_j is the proportion of total income earned by group *j*. The inequality index I is decomposable if it may be written as a function of the within-subgroup and between-subgroup inequalities. Here $I = f(I_1, I_2, ...I_k, \mu_1, \mu_2, ...\mu_k, n_1, n_2, ...n_k)$ where μ_j is the mean income level of partition *j* and I_j is the inequality measured within x_j.

Axiom (6) Principle of diminishing transfers

The principle of diminishing transfers requires that a progressive transfer (from a higher to a lower income earner) should have a diminishing effect at higher income levels. The example given by Kolm (1976) says a small transfer from an individual with an income of 900 units to an individual with 500 should reduce inequality less than an equivalent transfer from the individual with the income of 500 units to another earning 100 units. Using the definition of the transfer principle from axiom (4) the diminishing transfer principle may be stated as $T_{ji}(x) < T_{lk}(x)$ where $x_j - x_i = x_l - x_k$ and $x_i < x_k$.

Proposition 1: The proposed measure satisfies measurement axioms 1-6.

Proof: We consider each axiom in turn.

Axiom (1) Symmetry: The measure is symmetric as $f(x_i) = f(x_j) \ \forall \ x_i = x_j$

Axiom (2) Scale independence: The measure is insensitive to income scaling factor λ since

$$(8) \qquad I(\lambda x) = \frac{1}{n}\sum_{i=1}^{n} \frac{\lambda x_i - \lambda\mu}{\lambda\mu} \ln\left(\frac{\lambda x_i}{\lambda\mu}\right) = \frac{1}{n}\sum_{i=n}^{n} \frac{x_i - \mu}{\mu} \ln\left(\frac{x}{\mu}\right) = I(x) \ \forall \ \lambda \in \Re_+$$

Axiom (3) Population replication: Consider a replication of underlying distribution *x* where each x_i occurs with frequency $f \in N_+$. The inequality measure is

$$(9) \qquad I(x; f) = \sum_{t=1}^{f}\sum_{i=1}^{n} \frac{1}{nf} \frac{x_i - \mu}{\mu} \ln\left(\frac{x}{\mu}\right)$$

which equals

$$(15) \qquad I(x; f) = f\sum_{i=1}^{n} \frac{1}{nf} \frac{x_i - \mu}{\mu} \ln\left(\frac{x}{\mu}\right) = I(x)$$

Axiom (4) Pigou-Dalton transfer principle: The transfer principle will be satisfied if $f''(x_i) > 0 \ \ i = 1...n$. To see this, consider a progressive transfer of size dx from higher income earner *j* to lower income earner *i*. The effect of the transfer on the inequality index *I* with evaluation function $f(x)$ is

(10) $$T_{ji} = -\frac{1}{n}\frac{df(x)}{dx_j} + \frac{1}{n}\frac{df(x)}{dx_i} < 0 \qquad \text{if} \quad \frac{df(x)}{dx_j} > \frac{df(x)}{dx_i} \text{ for } j > i$$

We can ensure $\frac{df(x)}{dx_j} > \frac{df(x)}{dx_i}$ by requiring $\frac{d^2 f(x)}{dx^2} > 0$ for all x_i. The evaluation function $f(x)$ and first and second derivatives are:

(11) $$f(x) = \frac{x-\mu}{\mu}\ln\left(\frac{x}{\mu}\right)$$

(12) $$\frac{df(x)}{dx} = \frac{1}{\mu}\left(\ln\left(\frac{x}{\mu}\right)+1\right) - \frac{1}{x}$$

(13) $$\frac{d^2 f(x)}{dx^2} = \frac{1}{x^2} + \frac{1}{\mu x} > 0 \qquad x > 0$$

Axiom (5) Decomposability: Various proofs and demonstrations of the decomposition of Theil's T and L measures are given by Theil (1967), Cowell and Kuga (1980), Foster (1983), Shorrocks (1980) and Bourguignon (1979). Here we show that the combined index is decomposable. We take the index as defined in equation (6) and show it may be written as a weighted sum of between-group and within-group inequalities. The expression

$$I = \frac{1}{n}\sum_{i=1}^{n}\frac{x_i - \mu}{\mu}\ln\left(\frac{x_i}{\mu}\right)$$

may be written as a summation across the set of *k* subgroups. Here x_{ij} refers to the *i*th income in group *j*. There are n_j individuals in group *j* with group mean μ_k and $\sum_{j=1}^{k} n_j = n$. This is

(14) $$I = \frac{1}{n}\sum_{j=1}^{k}\sum_{i_j=1}^{n_j}\frac{x_{ij} - \mu}{\mu}\ln\left(\frac{x_{ij}}{\mu}\right)$$

Expanding gives

(15) $$I = \frac{1}{n}\sum_{j=1}^{k}\sum_{i_j=1}^{n_j}\left[\frac{x_{ij}}{\mu}\ln\left(\frac{x_{ij}}{\mu}\right) - \ln\left(\frac{x_{ij}}{\mu}\right)\right]$$

This may be written as

(16) $$I = \frac{1}{n}\sum_{j=1}^{k}\sum_{i_j=1}^{n_j}\left[\frac{x_{ij}}{\mu}\ln\left(\frac{x_{ij}}{\mu_j}\frac{\mu_j}{\mu}\right) - \ln\left(\frac{x_{ij}}{\mu_j}\frac{\mu_j}{\mu}\right)\right]$$

(17) $$I = \frac{1}{n}\sum_{j=1}^{k}\sum_{i_j=1}^{n_j}\left[\frac{x_{ij}}{\mu}\ln\left(\frac{x_{ij}}{\mu_j}\right)+\frac{x_{ij}}{\mu}\ln\left(\frac{\mu_j}{\mu}\right)-\ln\left(\frac{x_{ij}}{\mu_j}\right)-\ln\left(\frac{\mu_j}{\mu}\right)\right]$$

(18) $$I = \frac{1}{n}\sum_{j=1}^{k}\sum_{i_j=1}^{n_j}\left[\frac{x_{ij}}{\mu}\ln\left(\frac{x_{ij}}{\mu_j}\right)+\frac{x_{ij}}{\mu}\ln\left(\frac{\mu_j}{\mu}\right)\right]+\frac{1}{n}\sum_{j=1}^{k}\sum_{i_j=1}^{n_j}\left[-\ln\left(\frac{x_{ij}}{\mu_j}\right)-\ln\left(\frac{\mu_j}{\mu}\right)\right]$$

(19) $$I = \frac{1}{n}\sum_{j=1}^{k}\sum_{i_j=1}^{n_j}\left[\frac{x_{ij}}{\mu}\ln\left(\frac{x_{ij}}{\mu_j}\right)+\frac{x_{ij}}{\mu}\ln\left(\frac{\mu_j}{\mu}\right)\right]+\frac{1}{n}\sum_{j=1}^{k}\sum_{i_j=1}^{n_j}\left[-\ln\left(\frac{x_{ij}}{\mu_j}\right)\right]+\sum_{j=1}^{k}-\frac{n_j}{n}\ln\left(\frac{\mu_j}{\mu}\right)$$

Multiplying the second term by n_j / n_j gives

(20) $$I = \frac{1}{n}\sum_{j=1}^{k}\sum_{i_j=1}^{n_j}\left[\frac{x_{ij}}{\mu}\ln\left(\frac{x_{ij}}{\mu_j}\right)+\frac{x_{ij}}{\mu}\ln\left(\frac{\mu_j}{\mu}\right)\right]+\sum_{j=1}^{k}\frac{n_j}{n}\sum_{i_j=1}^{n_j}\left[-\frac{1}{n_j}\ln\left(\frac{x_{ij}}{\mu_j}\right)\right]+\sum_{j=1}^{k}-\frac{n_j}{n}\ln\left(\frac{\mu_j}{\mu}\right)$$

Recognizing that $n_j / n = p_j$ $L_j = \sum_{i_j=1}^{n_j} -\frac{1}{n_j}\ln\left(\frac{x_{ij}}{\mu_j}\right)$ and that $L_B = \sum_{j=1}^{k} -\frac{n_j}{n}\ln\left(\frac{\mu_j}{\mu}\right)$ we can write

(21) $$I = \frac{1}{n}\sum_{j=1}^{k}\sum_{i_j=1}^{n_j}\left[\frac{x_{ij}}{\mu}\ln\left(\frac{x_{ij}}{\mu_j}\right)+\frac{x_{ij}}{\mu}\ln\left(\frac{\mu_j}{\mu}\right)\right]+\sum_{j=1}^{k} p_j L_j + L_B$$

where p_j is the population share of group *j*, L_j is the inequality within *j* and L_B is the inequality between the *k* subgroup means.

Using $\sum \frac{x_i}{\mu}\ln\frac{\mu_j}{\mu} = \sum \frac{\mu_j}{\mu}\ln\frac{\mu_j}{\mu}$ we get

(22) $$I = \frac{1}{n}\sum_{j=1}^{k}\sum_{i_j=1}^{n_j}\left[\frac{x_{ij}}{\mu}\ln\left(\frac{x_{ij}}{\mu_j}\right)+\frac{\mu_j}{\mu_j}\ln\frac{\mu_j}{\mu}\right]+\sum_{j=1}^{k} p_j L_j + L_B$$

With some algebra we get

(23) $$I = \sum_{j=1}^{k}\sum_{i_j=1}^{n_j}\left[\frac{n_j\mu_j}{n\mu}\left[\frac{1}{n_j}\frac{x_{ij}}{\mu_j}\ln\left(\frac{x_{ij}}{\mu_j}\right)+\frac{1}{n_j}\frac{\mu_j}{\mu_j}\ln\frac{\mu_j}{\mu}\right]\right]+\sum_{j=1}^{k} p_j L_j + L_B$$

(24) $$I=\sum_{j=1}^{k}\sum_{i_j=1}^{n_j}\left[\left[\frac{n_j\mu_j}{n\mu}\left[\frac{1}{n_j}\frac{x_{ij}}{\mu_j}\ln\left(\frac{x_{ij}}{\mu_j}\right)\right]\right]+\frac{n_j\mu_j}{n\mu}\ln\frac{\mu_k}{\mu}\right]+\sum_{j=1}^{k}p_jL_j+L_B$$

(25) $$I=\sum_{j=1}^{k}\sum_{i_j=1}^{n_j}\left[\frac{n_j\mu_j}{n\mu}\left[\frac{1}{n_j}\frac{x_{ij}}{\mu_j}\ln\left(\frac{x_{ij}}{\mu_j}\right)\right]\right]+\sum_{j=1}^{k}\frac{n_j\mu_j}{n\mu}\ln\left(\frac{\mu_k}{\mu}\right)+\sum_{j=1}^{k}p_lL_j+L_B$$

Noting that $q_j=\frac{n_j\mu_j}{n\mu}$ $T_j=\sum_{i_j=1}^{n_j}\frac{1}{n_j}\frac{x_{ij}}{\mu_j}\ln\left(\frac{x_{ij}}{\mu_j}\right)$ and $T_B=\sum_{j=1}^{k}\frac{n_j\mu_j}{n\mu}\ln\left(\frac{\mu_k}{\mu}\right)$ and invoking similar definitions as used in equation (20) we can write

(26) $$I=\sum_{j=1}^{k}\left(q_jT_j+p_jL_j\right)+\left(T_B+L_B\right)$$

Thus the decomposition of the proposed index requires that the measure of inequality within subgroup *j* can be written as $I_j=T_j+L_j$ where T_j and L_j are Theil's inequality indices. That is, we must be able to separate each within-group inequality estimate I^k into contributions T^k and L^k as each receives a different weighting for the decomposition of the index. It should be noted that this decomposition is somewhat messier than the standard GE decomposition due to the required separation of *T* and *L* and a more complex weighting system required to reconstruct the original measure. This messiness may be seen as a disadvantage of this technique.

Axiom (6) Diminishing transfers: I will satisfy the diminishing transfer principle if $T_{ji}(x)<T_{lk}(x)$ where $x_j-x_i=x_k-x_l$ and $x_i<x_k$. We can write

(27) $$T_{ji}-T_{lk}=\left(-\frac{1}{n}\frac{df(x)}{dx_i}+\frac{1}{n}\frac{df(x)}{dx_j}\right)-\left(-\frac{1}{n}\frac{df(x)}{dx_k}+\frac{1}{n}\frac{df(x)}{dx_j}\right)$$

which will be positive if $\frac{d^3f(x)}{dx^3}<0$ for all *x*. It is simple to verify that

(28) $$\frac{d^3f(x)}{dx^3}=\frac{-2}{x^3}-\frac{1}{\mu x^2}<0 \qquad x>0$$

and thus the measure satisfies the diminishing transfer axiom.

4. AN ALTERNATIVE FORM OF DECOMPOSITION

In this section we discuss an alternative method for analyzing an inequality metric. The proposed method is henceforth referred to as a 'dissection' or 'disaggregation' to

distinguish it from the standard method typical of GE measures where inequality is decomposed into within-group and between-group factors. As we shall see the proposed measure is well suited to this alternative disaggregation analysis.

Suppose we take distribution $\boldsymbol{x}$ and calculate an inequality metric of the form $I(x)=\sum_{i=1}^{n}(1/n)f(x_i;\mu)$ where the choice of $f(x;\mu)$ determines the particular index. Though the index I is a functional of $\boldsymbol{x}$ it may be of interest to ask what interpretation may be placed upon the individual component $(1/n)f(x_i;\mu)$ and what may be inferred from this component about the underlying income level x_i. As we are considering measures where the components are aggregated to determine the inequality metric, we suggest that under certain conditions it is reasonable to interpret $(1/n)f(x_i;\mu)$ as the contribution of income x_i to the index $I(x)$. To be able to place this interpretation upon $(1/n)f(x_i;\mu)$ however, we require some guidance as to what we believe an individual's 'contribution to inequality' should represent. In this paper we propose that the contribution an individual makes to inequality should be a non-negative reflection of the extremeness of that person's income level, such that persons with both low and high incomes make significant positive contributions which increase as incomes diverge further from the mean. Analogously persons with middling incomes should contribute less to inequality, and persons earning the exact mean income level should make a contribution of zero. In other words, if our inequality measure is the simply the sum (or average) of a scoring system which penalizes distributions for containing both low and high income levels; where the penalty for each income is directly observable, we suggest that each 'penalty' may represent the contribution of that income to the overall index.

This system implies two conditions for the index:

(I) The measure can be written in the form $I(x)=\sum_{i=1}^{n}(1/n)f(x;\mu)$ such that the individual components $f(x_i;\mu)$ are tractable.

(II) That $f(x)\geq 0$ for all x, $f(\mu)=0$, $f'(x)>0$ for $x<\mu$ and $f'(x)>0$ for $x>\mu$.

The Relative Mean Deviation index satisfies these conditions and provides a good example of the concept. The measure is [2]

$$RMD=\sum_{i=1}^{n}\frac{1}{n}\frac{|x_i-\mu|}{\mu} \tag{29}$$

[2] It should be noted that despite this attractive characteristic, this index makes a poor inequality measure as it is insensitive to transfers that do not cross the mean and thus fails the transfer axiom.

and is simply the average of the mean-standardized difference between each income and the average income. Here an individual earning the mean income level would have an individual component of zero and contribute nothing to the measured inequality, while the further an income moved away from the mean, either above or below, the greater that person's contribution. An advantage of using the Relative Mean Deviation or other measure which exhibits this property is that we may group together individuals in any arbitrary way into *k* exhaustive subgroups and observe the extent to which each subgroup, on average, contributes to the index. Thus we may write the index as

$$I = \frac{n_1}{n}\sum_{i_1=1}^{n_1} f(x_{i1};\mu) + \frac{n_2}{n}\sum_{i_2=1}^{n_2} f(x_{i2};\mu) \ldots + \frac{n_k}{n}\sum_{i_k=1}^{n_k} f(x_{ik};\mu) \tag{30}$$

and we define $I_j^c = \frac{1}{n_j}\sum_{i_j=1}^{n_j} f(x_{ij};\mu)$ as the average contribution of subgroup *j* to the index while $\sum_{i_j=1}^{n_j} f(x_{ij};\mu)$ is the total contribution for this group. These subgroup contributions give a reflection of the overall 'extremeness' of the group incomes, such that a group where every individual has a middling income will have an average of contribution of zero; and for any group, the average will always increase whenever any of the group incomes move away from the mean. We give these group averages the same interpretation as for the individual as the extent to which a particular subgroup contributes to the overall inequality measurement. Examples of such interpretations could be 'The lowest earning 10% of the income distribution drive 30% of measured inequality' or 'Persons from group A contribute twice as much to inequality, per person, than individuals from subgroup B'.

In the context of this group averaging process the motivation behind the non-negativity constraint for $f(x;\mu)$ in condition (II) becomes apparent. Measures which assign components that can be either positive or negative typically do relative to the mean of the distribution[3], where the sign of the components switches as incomes cross the sample average. This is problematic for our 'extremeness' interpretation as an offsetting effect will occur when some individuals within a subgroup lie on either side of the mean. Consider an example where we wish to interpret a subgroup component average for a group containing mostly low income earners, but where one individual from the group earns a spectacularly high income. We may ask 'should the presence of the high income earner increase or decrease the extent to which the subgroup, on average, contributes to the overall inequality estimate?'

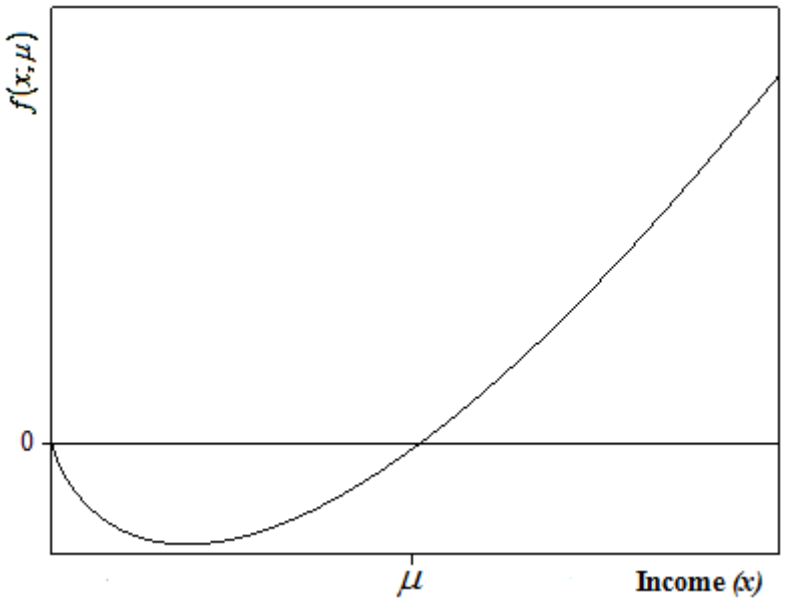
If we feel that the presence of this very high income reduces the extent to which the subgroup contributes to inequality we should use a measure that assigns both positive and negative components, as the inclusion of the high income earner will offset the components from lower income earners. This will ensure that the group average

[3] Such as GE indices given in equation (31)

component is reduced, and gives a rough representation of the relative average income of the subgroup. It is argued here however that the average *extremeness* of the incomes within a subgroup is more interesting and gives a better measure of the extent to which the subgroup contributes to overall inequality. As the presence of the solitary high income earner *increases* the extremeness of incomes within the group his or her presence should increase rather than decrease the group average contribution. Thus if we agree with the interpretation that a person's contribution to inequality is driven by the extremeness of the underlying income it is necessary to ensure that each component reflects the extremeness of the individual income in an absolute sense for the disaggregation of the inequality estimate to be meaningful.

This argument is well illustrated by examining plots of the evaluation functions of certain measures. Here we provide evaluation functions for Theil's measures and the proposed index. The evaluation functions for these measures are taken from equations (6)-(7).

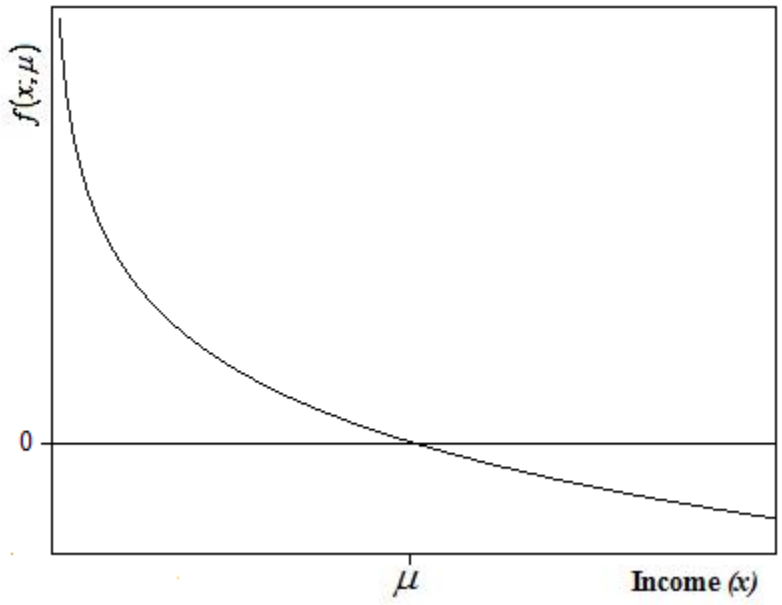
Figure 1. Evaluation function for Theil's T index



An examination of Figure 1 for Theil's T index[4] reveals the difficulty in interpreting a subgroup average of $f(x;\mu)$ across multiple x_{ij} as the extent to which the subgroup drives overall inequality. An average component of zero may occur with i. all $x_{ij}=0$, ii. all $x_{ij}=\mu$ or iii. some $x_{ij}<\mu$ and other $x_{ij}>\mu$. As it seems reasonable to conclude that these groups should contribute differently to the overall inequality estimate these subgroup averages may not be interpreted as a group contribution in the manner suggested.

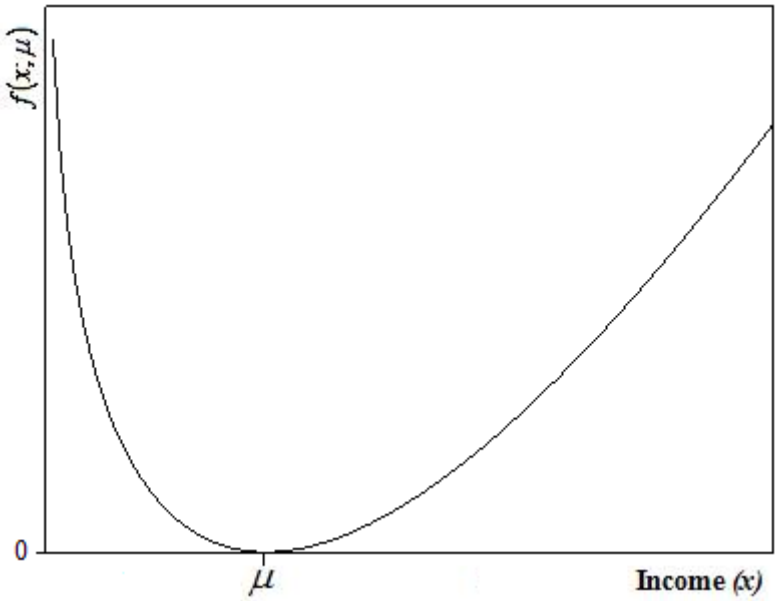
[4] We note that other formulations for Theil's T index exist, however this functional form seems to be the most widely presented. Similar arguments may be applied to other expressions for this measure.

Figure 2. Evaluation function for Theil's L index



A similar argument can be made concerning Theil's L measure. A subgroup where each $x_{ij} = \mu$ can have the same average component as a subgroup where some $x_{ij} < \mu$ and other $x_{ij} > \mu$. Thus in some instances a group average may increase as one income moves further away from the mean while in other cases this action will decrease the group average, depending on the incomes of other members of the subgroup.

Figure 3. Evaluation function for the new measure



The evaluation function for the new measure demonstrates the capability of this index for the proposed from of disaggregation. Each income level x_i gives a contribution to the index proportional to $f(x_i;\mu)$ such that extreme incomes contribute greatly and incomes that congregate around the mean contribute less. A group average of $f(x_i;\mu)$ across multiple x_i will always increase if any income moves further away from the mean and thus provides an acceptable measurement of the extent to which the subgroup drives the overall inequality metric.

Though it may be seen that many of the simpler ad hoc inequality indices such as the Relative Mean Deviation satisfy conditions (I) and (II) the property is unusual in more sophisticated measures.

Proposition 2: No existing axiomatically complete (GE) index satisfies condition (II).

Proof: Cowell (1981) and Foster (1983) show that the only axiomatically complete measures belong to the family of Generalized Entropy measures. This class of index is usually given as

$$I_{GE} = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\alpha^2-\alpha}\left[\left(\frac{x_i}{\mu}\right)^{\alpha}-1\right]\right] \tag{31}$$

where the parameter α dictates the sensitivity of the measure at higher and lower ends of the distribution. Formulations of Theil's L and T inequality measures may be derived as special cases of this class by taking the limits of equation (6) as $\alpha \to 0$ and $\alpha \to 1$ respectively. It is straightforward to see that the component

$$f(x_i;\mu) = \left[\frac{1}{\alpha^2-\alpha}\left[\left(\frac{x_i}{\mu}\right)^{\alpha}-1\right]\right] > 0 \text{ when } x_i > \mu \text{ and } \alpha > 1 \text{ or } x_i < \mu \text{ and } \alpha < 1 \tag{32}$$

and that the sign may be reversed by switching either but not both of the required inequalities. The measure will therefore assign both positive and negative components, making it unsuitable for the proposed analysis. In the special cases of Theil's T and L measures a similar argument may be used. The T index with evaluation function

$$f(x;\mu) = \frac{x}{\mu}\ln\left(\frac{x}{\mu}\right) \tag{33}$$

is shown in Figure 1 and is negative when $x_i < \mu$ and positive when $x_i > \mu$. The evaluation function for Theil's L measure is plotted in Figure 2 and is given as

(34) $$f(x;\mu) = -\ln\left(\frac{x}{\mu}\right)$$

This function is positive when $x_i < \mu$ and negative when $x_i > \mu$ and thus does not satisfy condition II.

Should α be set to two the GE equation yields a measure equal to half the square of the coefficient of Variation and may be written in a form consistent with conditions (1) and (2). It should be noted however that the index is not axiomatically complete as it fails the diminishing transfer principle (axiom 6). For a GE measure to satisfy this axiom it is required that $\alpha < 2$.

5. RECONCILING THE NEW FORM OF DISSAGREGATION WITH BETWEEN-GROUP AND WITHIN-GROUP INEQUALITY DECOMPOSITION

The proposed form of disaggregation is somewhat similar to the separation of inequality into within-group and between group factors typical of GE measures. The two techniques share a direct relationship, such that if we know the income and population shares of each group, the within and between-group inequalities and can separate these inequalities into T^k and L^k we can determine the contribution of each subgroup to the inequality metric as per equation (30). We have seen that

$$I = \sum_{j=1}^{k} I_j^c p_j = \sum_{j=1}^{k} q_j T_j + p_j L_j + (q_j - p_j)\ln\left(\frac{q_j}{p_j}\right)$$

It follows from the proof of axiom (5) that group contribution I_j^c may be determined as

(35) $$I_j^c = \frac{q_j T_j + p_j L_j + (q_j - p_j)\ln\left(\frac{q_j}{p_j}\right)}{p_j}$$

Furthermore it is straightforward to show that the estimates of within-group inequalities for the new measure are equal to the subgroup contributions in the special case where the between-group inequality is equal to zero. Suppose each $q_j = p_j$ i.e. there is no between-group inequality and all variation occurs within the subgroups. It is easy to see that (35) becomes

(36) $$I_j^c = T_j + L_j$$

which is simply the definition for the inequality within *j*. Thus the suggest form of decomposition is closely related to the standard method. The difference is that the standard method calculates within-group inequalities relative to each group mean, while the proposed method calculates group contributions relative to the overall sample mean.

6. AN EMPIRICAL ILLUSTRATION

In this section we provide an empirical demonstration of the new measure using U.S. census data taken from the IPUMS (Integrated Public Use Microdata Series http://usa.ipums.org/usa/) website. The 1-in-100 ACS (American Community Survey) dataset is used and we take the pre-tax wage and salary earnings of 1,496,561 individuals with non-zero incomes during 2007 to analyze U.S income inequality. We estimate the Theil L index to be equal to 0.581322 and Theil's T index at 0.46329 for the inequality within this dataset, with the proposed measure equal to the sum of these two estimates at 1.04462. By dissagregating the new index using equation (30) we can determine how much each racial group[5] drives the inequality measure. The results are in the table below.

Table 1. Racial contributions to inequality in the United States

Racial Group	Average Component	Population share	Ave comp × Pop Share
White	1.08505	0.75595	0.82024
Black/Negro	0.88216	0.11500	0.10145
Native	0.99309	0.00721	0.00716
Chinese	1.22825	0.01048	0.01287
Japanese	1.31612	0.00266	0.00350
Other Asian	1.08885	0.03377	0.03677
Other Race	0.76372	0.05904	0.04509
Two Races	1.10178	0.01480	0.01630
Three Races	1.12011	0.00110	0.00123
Total	**-**	**1**	**1.04462**

The first column of this table gives the average component of each subgroup calculated using the equation $I_j^c = \frac{1}{n_j}\sum_{i_j=1}^{n_j}\frac{x_{ij}-\mu}{\mu}\ln\left(\frac{x_{ij}}{\mu}\right)$. By weighting these averages with the population shares given in column two and summing them we can reconcile these group contributions with our overall inequality estimate. We can observe the extent to which each racial group drives this inequality statistic by the size of the average component or 'extremeness' of each group. Subgroups such as 'Chinese' and 'Japanese' have higher average components than the total population average and therefore contain relatively more extreme incomes than the population as a whole. Conversely persons of 'Black/Negro' or 'Native' racial groups earn more middling incomes and make lower contributions to inequality than other groups. The racial group 'White' makes up the majority of the population share and persons within this group earn slightly more extreme

[5] The racial labels in Table 1 are somewhat broad and more detail may be found on the IPUMS website. The racial group 'Native' for example contains persons of both Native Indian and Native Alaskan decent, while the group 'Other Asian' includes Pacific Islanders.

incomes ($I^{c}_{Whit} = 1.08505$) than the total population ($I = 1.04462$). It is worth emphasizing here that any subgroup in which every individual earned mean income level μ would have an average component of zero.

To distinguish these estimates from the within-group and between-group inequality estimates typical of GE measures we present these results for the inequality between and within our racial groups using our new measure. The results are in the table below.

Table 2. Inequalities within and between racial groups

Racial Group	Within Group	Between Component	Within Group (W)	Total
White	1.05495	0.00243	0.81782	-
Black/Negro	0.90634	0.00778	0.09367	-
Native	1.00940	0.00082	0.00633	-
Chinese	1.05852	0.00058	0.01229	-
Japanese	0.97861	0.00041	0.00309	-
Other Asian	0.99738	0.00076	0.03601	-
Other Race	0.74394	0.00739	0.03770	-
Two Races	1.15439	0.00071	0.01550	-
Three Races	1.17145	4.1E-05	0.00119	-
Total		**0.02093**	**1.02369**	**1.04462**

The first column gives the inequality estimate for the distribution of income within each racial group, while the second gives the group components to be aggregated to determine the inequality between the subgroups. The third column gives the within-group inequalities after they have been weighted by population and income shares as per equation (20), and the last column shows that total inequality may be reconciled using the within-group and between-group inequality estimates.

A cursory inspection of Table 2 reveals that some groups with high internal inequality such as 'Chinese', 'Two races' and 'Three Races' also had high average contributions in Table 1, while other groups with lower within-group inequality had lower group contributions (an exception is the subgroup 'Japanese'). To contrast these different methods of analyzing inequality we reemphasize that each within-group inequality is calculated relative to each group mean, while each racial group contribution from Table 1 is calculated with respect to the population mean rather than each specific group mean.

The similarity between these associated estimates for each group in the two tables may be explained by the low between-group inequality figure from Table 2. This estimate indicates there is little variation between the average incomes in each subgroup; hence the differences between the within-group inequalities calculated relative to each subgroup mean and the group contributions calculated relative to the single population mean are likely to be small. Should the average incomes of each subgroup become completely equalized across the groups, we would observe the between-group inequality estimate from Table 2 vanishing to zero and the within-subgroup inequalities converging to the group contributions from Table 1. Conversely, we would expect to see large differences

between each within-group inequality estimate and the corresponding subgroup contribution when the between-group inequality is high. Consider the example of a population containing an extravagantly wealthy subgroup with homogenous incomes. The inequality within this high earning subgroup is zero; however the average component of the group must be large as the underlying incomes are extreme relative to the population mean. Such a situation can only occur when there is a degree of between-group inequality and is somewhat illustrated by the case of the racial group 'Japanese'. The internal inequality of this subgroup is lower than most other groups, but due to an unusually high average income the group contribution is high.

7. CONCLUSION

This paper re-explores the link between inequality measurement and information theory. We show that the parallels between these concepts may be extended such that a new and attractive inequality index may be determined. The index exists in other fields of statistics and is referred to as the symmetrized Kullback-Liebler divergence, Symmetric Relative Entropy or J-Divergence. We show that the measure satisfies the axioms of measurement and possesses an intuitive disaggregation property which allows for the contribution of each income or subgroup to total inequality to be directly observable. The new measure is demonstrated with unit record data from the U.S. and we determine the average contribution to inequality of a range of racial subgroups. It was found that the incomes of U.S wage earners belonging to racial groups defined as 'Chinese', 'Japanese' or 'Three Races' are on average much more extreme than the entire population, while the opposite is true of persons belonging to 'Other race' and 'Black/Negro' racial groups

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