Scale characteristics in DEA: a unifying framework

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Agenda

- Scale elasticity and RTS
- DEA context: Existing difficulties and inconsistencies
- Theory and generalizations
- Application to EU agriculture

Assumption

We are concerned only with the technical (geometrical) characterization of efficient frontiers – no price information is assumed or used.

Scale elasticity and RTS

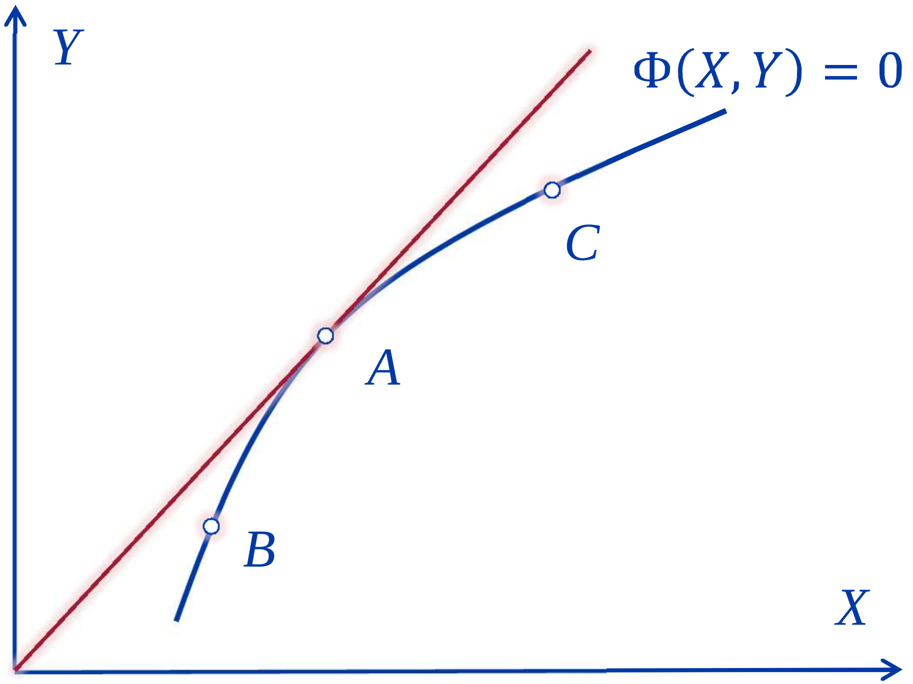
The standard RTS classification is based on scale elasticity ε.

In the smooth case,

$$\varepsilon > 1 \Leftrightarrow \text{IRS}$$

$$\varepsilon = 1 \Leftrightarrow \text{CRS}$$

$$\varepsilon < 1 \Leftrightarrow \text{DRS}$$

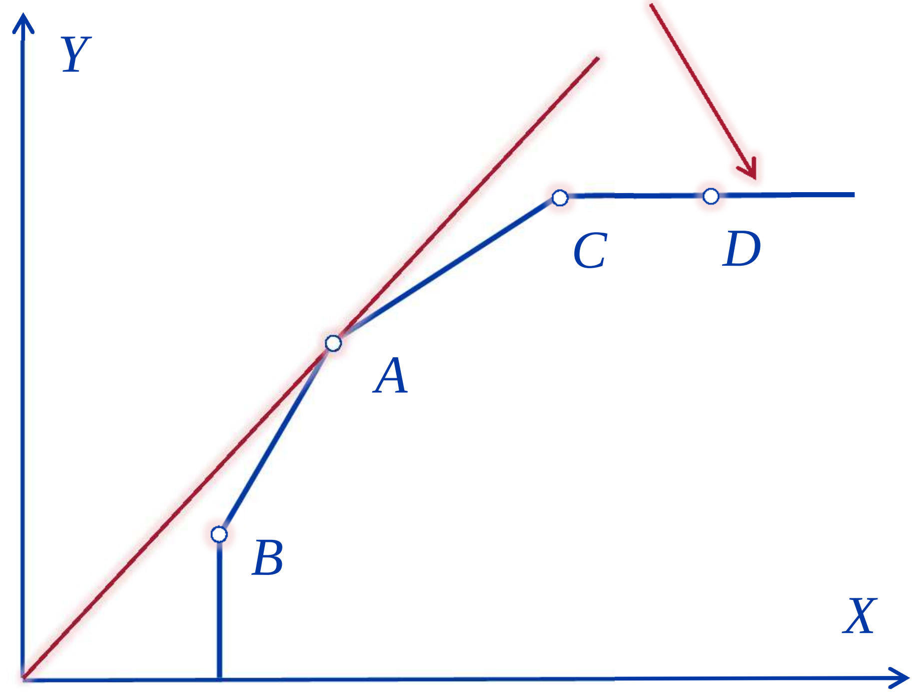


The easy smooth case

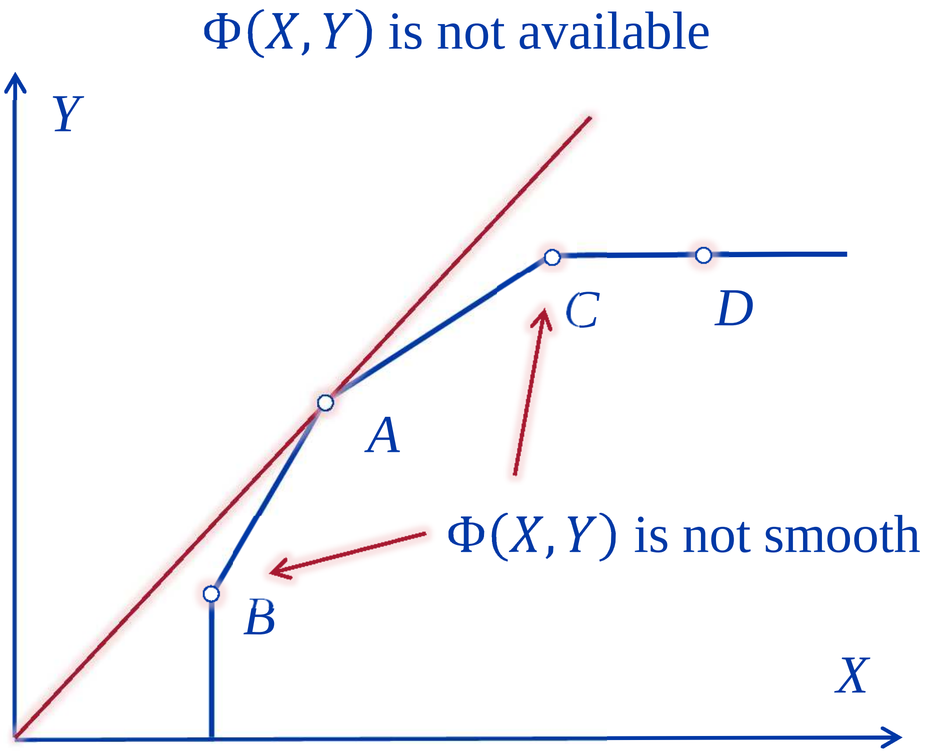
$$\varepsilon(X_0, Y_0) = -\frac{< X_0, \nabla_X \Phi(X_0, Y_0) >}{< Y_0, \nabla_Y \Phi(X_0, Y_0) >}$$

Difficulties in DEA

$\Phi(X,Y)$ is not available



Difficulties in DEA



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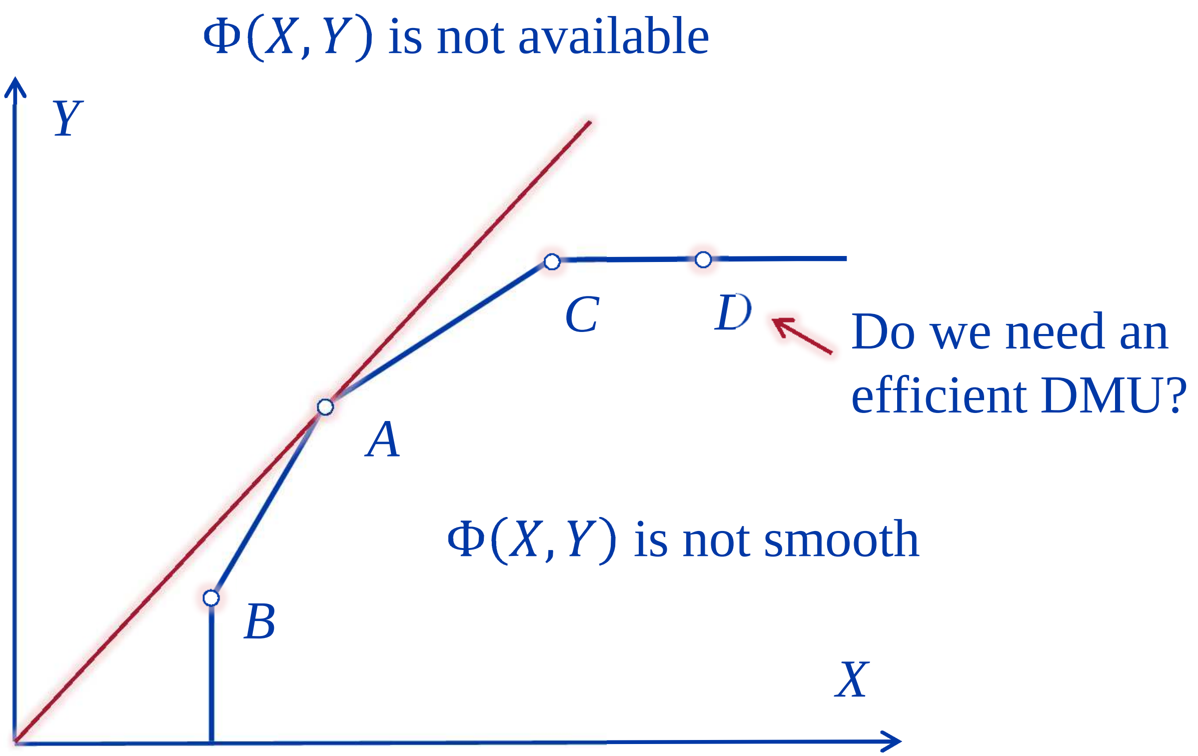

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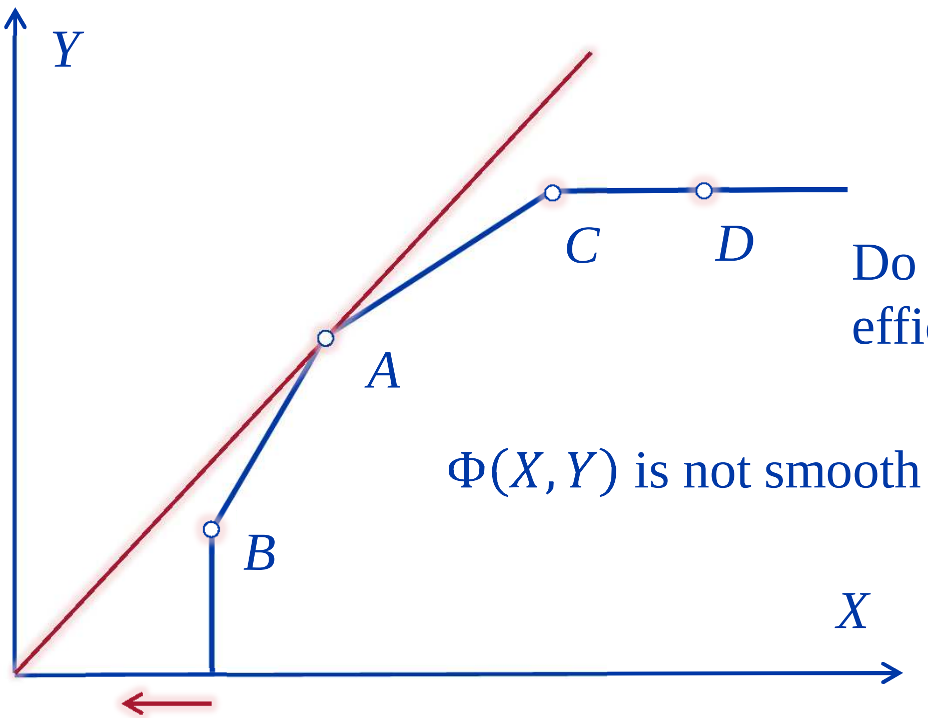

Do we need an efficient DMU?

Can't change X in some directions

Generalized setting

- Input set I, output set O
- Set $A \neq \emptyset$ – inputs and outputs that we want to change
- Set $B \neq \emptyset$ – outputs that respond to changes
- Set C – inputs and outputs kept constant

$$\mathrm{I} \cup \mathrm{O} = A \cup B \cup C$$

What is the response of outputs in set B with respect to marginal changes in the set A, provided the I/Os in C are kept constant?

Examples:

Scale elasticity: $A = I, B = \mathrm{O}$.

Short-run scale elasticity: $A \subset \mathrm{I}, B = \mathrm{O}$.

(Partial) elasticity measures

Let $(X_0, Y_0) = (X_0^A, X_0^C, Y_0^A, Y_0^B, Y_0^C)$

For the given A, B, C, define the output response function

$$\overline{\beta}(\alpha) = \max\left\{\beta \,\middle|\, (\alpha X_0^A, X_0^C, \alpha Y_0^A, \beta Y_0^B, Y_0^C) \in T \right\}$$

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Let $\bar{\beta}(1) = 1$. (The unit (X_0, Y_0) is efficient in the production of subset B.)

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Let $\bar{\beta}(1) = 1$. (The unit (X_0, Y_0) is efficient in the production of subset B.)

The elasticity of response of subvector Y_0^B with respect to marginal changes of the I/O bundle (X_0^A, Y_0^A) is defined as

$$\varepsilon_{A,B}(X_0, Y_0) = \bar{\beta}'(1)$$

provided the above derivative exists.

The easy smooth case

If $\Phi(X, Y)$ were known, then by the implicit function theorem,

$$\varepsilon_{A,B}(X_0, Y_0) = \overline{\beta}'(1) = -\frac{\left\langle X_0^A, \nabla_X^A \Phi(X_0, Y_0) \right\rangle + \left\langle Y_0^A, \nabla_Y^A \Phi(X_0, Y_0) \right\rangle}{\left\langle Y_0^B, \nabla_Y^B \Phi(X_0, Y_0) \right\rangle}$$

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In particular, for the scale elasticity,

$$\varepsilon_{\mathrm{I,O}}(X_0,Y_0)=-\frac{\langle X_0,\nabla_X\Phi(X_0,Y_0)\rangle}{\langle Y_0,\nabla_Y\Phi(X_0,Y_0)\rangle}$$

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If X and Y are scalars, we have

$$\varepsilon_{\mathrm{I,O}}(X_0, Y_0) = \frac{dY}{dX} / (Y/X)\Big|_{(X_0, Y_0)}$$

The difficult DEA case

We do not know

- the functional form $\Phi(X, Y)$ of the production frontier
- whether we can change (even marginally) the bundle (X_0^A, Y_0^A) by multiplier α
- whether $\bar{\beta}(\alpha)$ is differentiable at $\alpha = 1$ (probably not)

Response function as the optimal value of LP

To be specific, assume VRS. Then

$$\bar{\beta}(\alpha) = \max\left\{\beta \,\middle|\, (\alpha X_0^A, X_0^C, \alpha Y_0^A, \beta Y_0^B, Y_0^C) \in T_{VRS}\right\}$$

is simply:

$$
\begin{aligned}
\bar{\beta}(\alpha) = \max \quad & \beta \\
\text{subject to} \quad & \bar{X}^A \lambda \le \alpha X_0^A \\
& \bar{X}^C \lambda \le X_0^C \\
& -\bar{Y}^A \lambda \le -\alpha Y_0^A \\
& -\bar{Y}^B \lambda + \beta Y_0^B \le 0 \\
& -\bar{Y}^C \lambda \le -Y_0^C \\
& e\lambda = 1 \\
& \lambda \ge 0,\ \beta \text{ sign free}
\end{aligned}
$$

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\bar{\beta}(\alpha) = \max \quad & \beta \\
\text{subject to} \quad & \bar{X}^A \lambda \le \alpha X_0^A = X_0^A + (\alpha - 1) X_0^A \\
& \bar{X}^C \lambda \le X_0^C \\
& -\bar{Y}^A \lambda \le -\alpha Y_0^A = -Y_0^A - (\alpha - 1) Y_0^A \\
& -\bar{Y}^B \lambda + \beta Y_0^B \le 0 \\
& -\bar{Y}^C \lambda \le -Y_0^C \\
& e\lambda = 1 \\
& \lambda \ge 0,\ \beta \text{ sign free}
\end{aligned}
\qquad (1)
$$

Perturbation vector Δ

Sensitivity analysis comes to help

Let Γ be the domain of $\bar{\beta}(\alpha)$.

Theorem 1. (a) Γ is a closed interval.

(b) The optimal value (response) function $\bar{\beta}(\alpha)$ is continuous, concave and piecewise linear on its domain Γ.

Reference: Roos C, Terlaky T, Vial JP (2005) Interior point methods for linear optimization, Springer

Sensitivity analysis comes to help

Theorem 2. Let $\mathcal{D}^*$ be the set of optimal solutions to LP (1).

- If $\alpha = 1$ is not the right extreme point of Γ, then the right derivative $\beta'_+(1)$ exists and

$$\beta'_+(1) = \mathrm{Min}\langle\Delta, w\rangle, \text{ s.t. } w \in D^*$$

- If $\alpha = 1$ is not the left extreme point of Γ, then the left derivative $\beta'_-(1)$ exists and

$$\beta'_-(1) = \mathrm{Max}\langle\Delta, w\rangle, \text{ s.t. } w \in D^*$$

- If $\alpha = 1$ is the right (left) extreme point of Γ, the corresponding min (max) is unbounded.

Reference: Roos C, Terlaky T, Vial JP (2005) Interior point methods for linear optimization, Springer

The main DEA result

Theorem 3

(a) If $\alpha = 1$ is not the right extreme point of Γ, then the right elasticity $\varepsilon^{+}_{A,B}(X_0, Y_0) = \bar{\beta}'_{+}(1)$ exists, is finite and can be calculated as follows:

$$\begin{aligned}
\varepsilon^{+}_{A,B}(X_0, Y_0) = \min \quad & v^A X_0^A - \mu^A Y_0^A \\
\text{S.t.} \quad & v^A X_0^A + v^C X_0^C - \mu^A Y_0^A - \mu^C Y_0^C + \mu_0 = 1 \\
& v\bar{X} - \mu\bar{Y} + 1\mu_0 \geq 0 \\
& \mu^B Y_0^B = 1 \\
& v = (v^A, v^C),\ \mu = (\mu^A, \mu^B, \mu^C) \geq 0,\ \mu_0 \text{ sign free}
\end{aligned} \tag{2}$$

(b) If $\alpha = 1$ is not the left extreme point of Γ, then the left elasticity $\varepsilon^{-}_{A,B}(X_0, Y_0) = \bar{\beta}'_{-}(1)$ exists, is finite and can be calculated by program (2) in which Min is changed to Max

The main DEA result

Theorem 3 (continued)

(c) If $\alpha = 1$ is the right (left) extreme point of Γ, the corresponding min (max) is unbounded.

(d) If the "selective efficiency" condition $\overline{\beta}(1) = 1$ is not true, both of the above programs are infeasible.

A practical procedure

For each observed DMU *j* we solve the min and max programs (2), in one uninterrupted batch.

Note, there is no need to sort DMUs into efficient and inefficient ones. (This is irrelevant. The required condition $\overline{\beta}(1) = 1$ is tested automatically by the solver diagnostics.)

Three outcomes are possible.

Three outcomes

To be specific, consider the “Min” program (2) for the right elasticity $\varepsilon^{+}_{A,B}(X_0, Y_0)$.

1) If the Min program (2) has a finite optimal value, this is $\varepsilon^{+}_{A,B}(X_0, Y_0)$.

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1) If the Min program (2) has a finite optimal value, this is $\varepsilon^{+}_{A,B}(X_0, Y_0)$.

2) If program (2) is unbounded (optimal value is “$-\infty$”), then $\alpha = 1$ is the right extreme point of Γ and $\varepsilon^{+}_{A,B}(X_0, Y_0)$ is undefined. (The VRS technology does not have points to allow the required perturbation of the set A.

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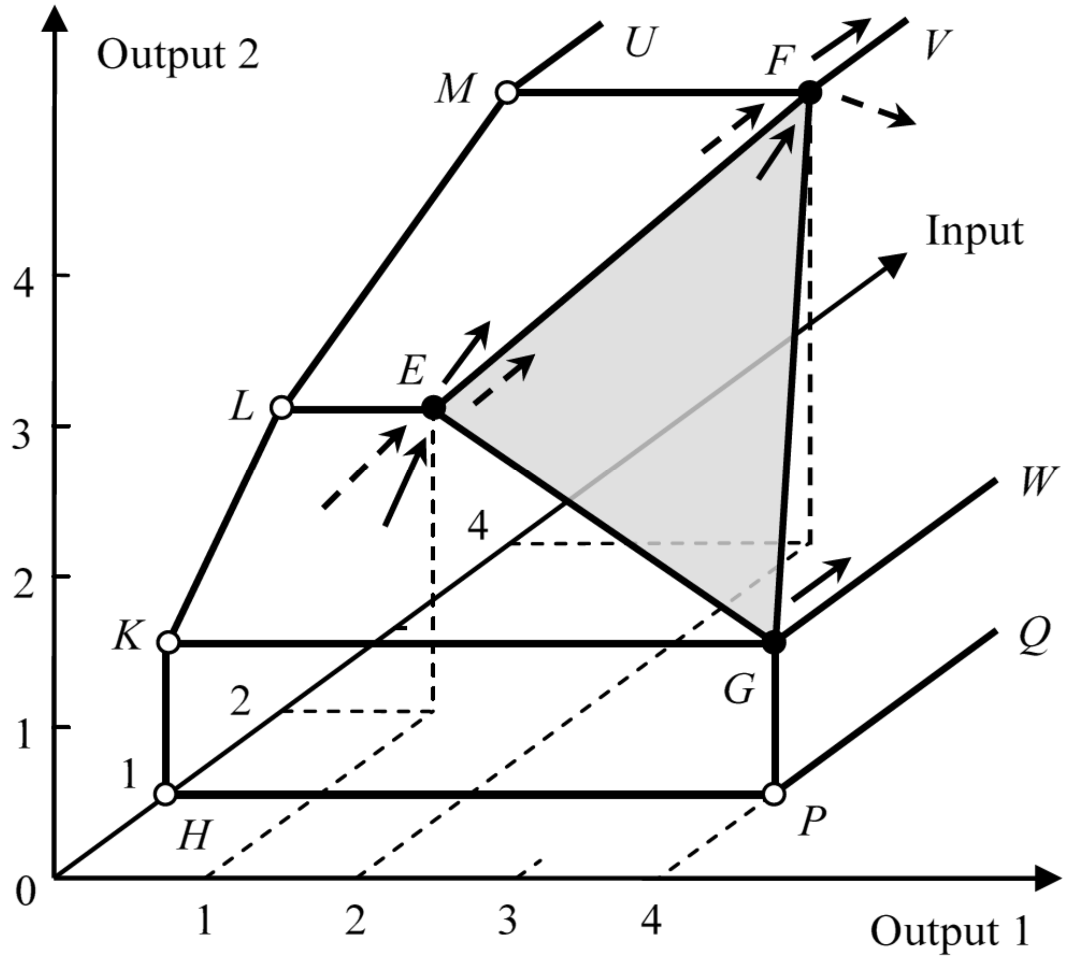
3) If program (2) is infeasible, then the unit (X_0, Y_0) does not satisfy condition $\overline{\beta}(1) = 1$. (This unit is not selectively efficient in the production of output bundle B.)

Example



Example

A = {Input}, B = {Output 2}, C = {Output 1}

$$\varepsilon^{+}_{A,B}(F) = \min \quad 4v_1$$

subject to

$$4v_1 - 2\mu_1 + \mu_0 = 1$$

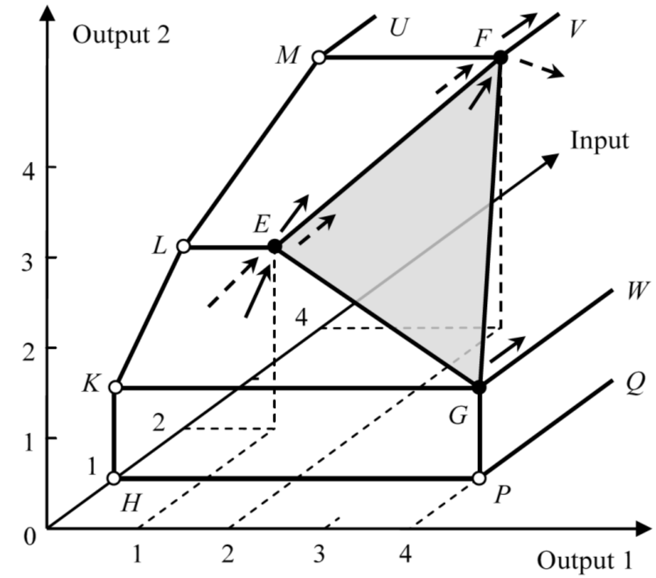
$$3\mu_2 = 1$$

$$2v_1 - 1\mu_1 - 2\mu_2 + \mu_0 \geq 0$$

$$4v_1 - 2\mu_1 - 3\mu_2 + \mu_0 \geq 0$$

$$1v_1 - 4\mu_1 - 1\mu_2 + \mu_0 \geq 0$$

$$v_1, \mu_1, \mu_2 \geq 0, \ \ \mu_0 \text{ sign free}$$



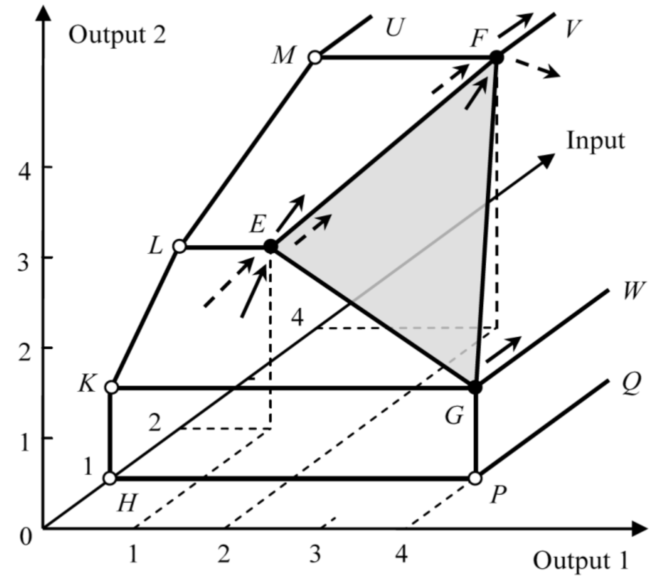
	Left	Right
E	1	0.5
F	0.7619	0
G	inf	0

Example

Let A = {Input, Output 1}, B = {Output 2}, $C = \varnothing$

$$\varepsilon^{+}_{A,B}(F) = \min \quad 4v_1 - 2\mu_1$$

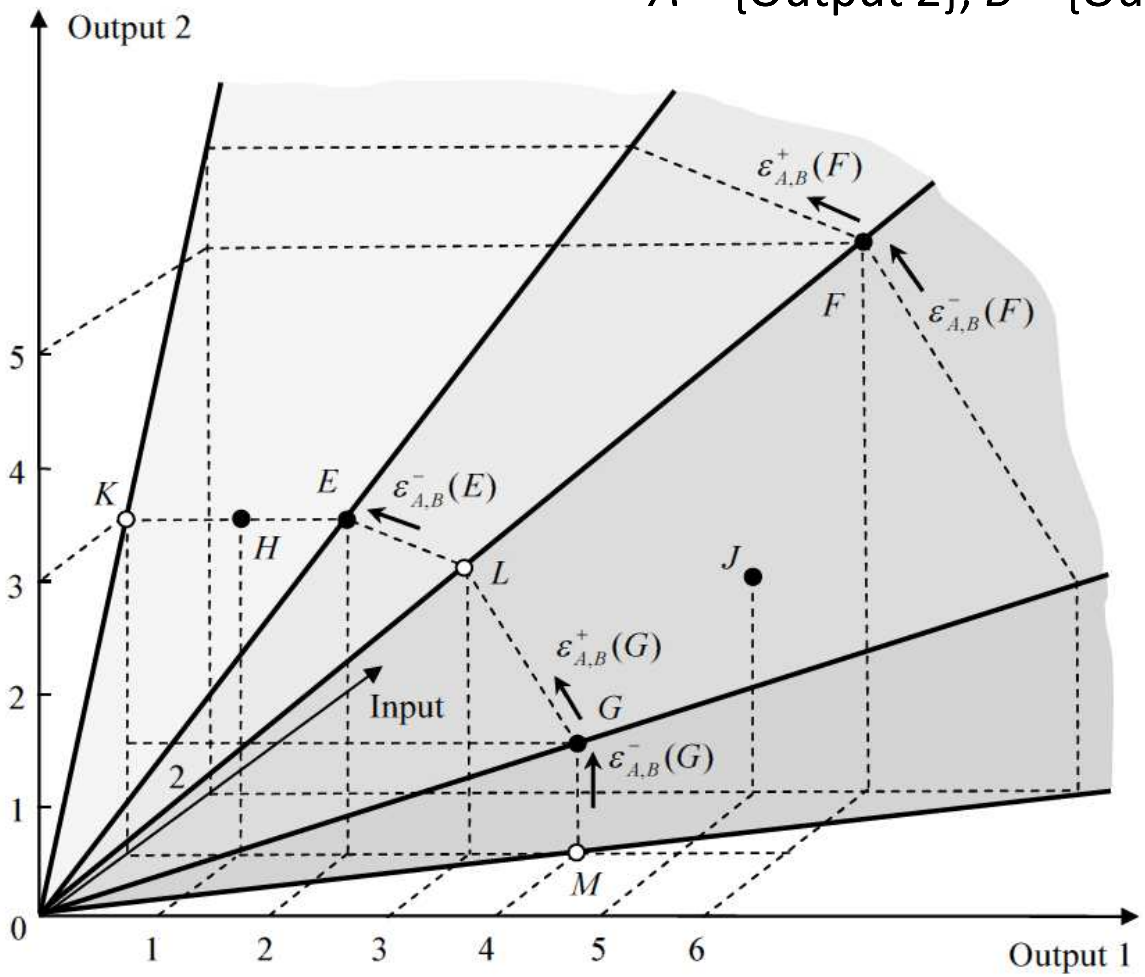
$$\begin{aligned}
\text{subject to} \quad & 4v_1 - 2\mu_1 + \mu_0 = 1 \\
& 3\mu_2 = 1 \\
& 2v_1 - 1\mu_1 - 2\mu_2 + \mu_0 \geq 0 \\
& 4v_1 - 2\mu_1 - 3\mu_2 + \mu_0 \geq 0 \\
& 1v_1 - 4\mu_1 - 1\mu_2 + \mu_0 \geq 0 \\
& v_1, \mu_1, \mu_2 \geq 0, \ \mu_0 \text{ sign free}
\end{aligned}$$



	Left	Right
E	1	0.5
F	0.6667	-0.6667
G	inf	inf

Example CRS

A = {Output 2}, B = {Output1}, C = {Input}



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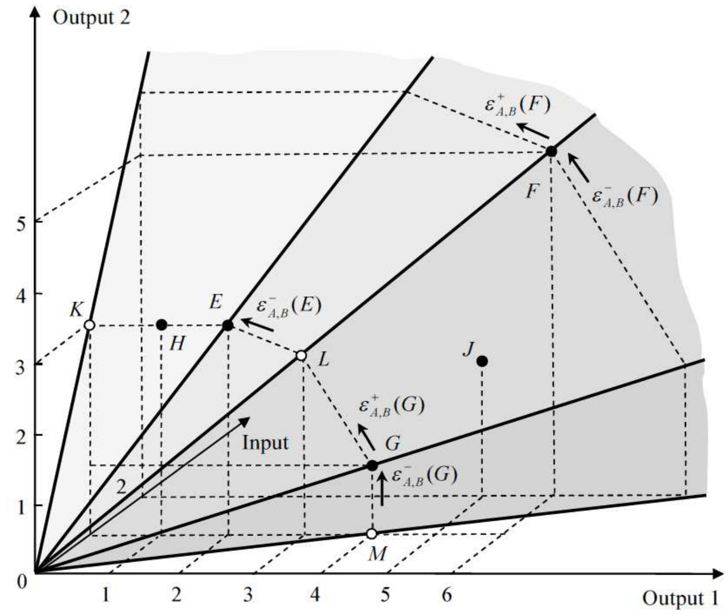

	$\overline{\beta}(1)$	Solver for Max	Solver for Min	Left elasticity	Right elasticity
E	1	-3	Unbounded	-3	Undefined
F	1	-0.55	-1.66	-0.55	-1.66
G	1	0	-0.16	0	-0.16
H	2	Infeasible	Infeasible	Undefined	Undefined
J	1.6	Infeasible	Infeasible	Undefined	Undefined

Summary of what we know

- We can define one-sided elasticities ε^- and ε^+ of response of any output set *B* with respect to marginal changes of the I/O set *A*, keeping I/O set *C* constant.
- If DMU (X_o, Y_o) is not radially efficient in the production of the output bundle *B*, neither ε^- nor ε^+ exists.
- If a marginal proportional increase of the I/O bundle *A* is infeasible in the technology, the right elasticity ε^+ does not exist. Similarly, if the decrease of *A* is infeasible, ε^- does not exist.
- All of this is diagnosed by solving two LPs for each DMU.

RTS classification for the given *A*, *B* and *C*

Assuming both right and left elasticities be positive, we define

$$[\varepsilon_+, \varepsilon_-] > 1 \Leftrightarrow \text{IRS}$$

$$1 \in [\varepsilon_+, \varepsilon_-] \Leftrightarrow \text{CRS}$$

$$[\varepsilon_+, \varepsilon_-] < 1 \Leftrightarrow \text{DRS}$$

Application to agriculture

- EU FADN database
- 100 – 3,000 farms in a region

Inputs:

Land, capital, costs, labour

Outputs:

Wheat, barley, oats, rye, potatoes, sugar beet,
other crops (price aggregated), livestock, farm net income

Elasticity measures and RTS

Long-run policy scenario

If we increase all inputs (land, assets, cost, labour) by 1%, how will the farm outputs respond?

Short-run policy scenario

If we increase only cost and labour by 1%, how will the farm outputs respond?

Family net income

If we increase the size of the farm by 1% (or only cost and labour by 1%) how will the farm net income respond?

Results

- There are different RTS types for the short and long-run policy scenarios in different EU regions.

  For example, in some EU regions (e.g. Austria, Ireland) farms predominantly exhibit IRS in the long-run scenario and DRS in the short run. (Over-farmed?)

- The elasticity of farm net income with respect to farm size (vector of farm inputs) is predominantly > 1. This implies IRS.

Literature

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Thank you!