Primaries, Strategic Voters and Heterogeneous Valences*

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Abstract. We propose a two-party model of policy promises and valence for office-seeking candidates under a two-stage electoral process with strategic voters. In each party, there are two candidates, where one of these candidates—called the party's advantaged candidate—has higher valence than the other. There are two equilibrium regimes. Which equilibrium arises depends on whether an advantaged candidate in one party can win both stages of the election with certainty. We provide the conditions for the existence of each regime and conduct comparative statics. In particular, we show that when an advantaged candidate's valence increases, the distance between the policy promises made by the two advantaged candidates decreases and when public opinion becomes more diverse, the advantaged candidates shift their policy promises toward their own parties' bliss points. Finally, we study the case where only one party holds a primary as well as the situation in which candidates strategically choose to enter a primary. Our model is robust under various extensions, and particularly useful for conducting comparative statics and providing testable predictions for electoral outcomes as public opinion changes.

Key Words: primary election, median voter, uncertainty, valence.
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1 Introduction

Across the world, primaries have become commonplace as a means of selecting a party's representative to compete in an upcoming general election. Besides the US, primary elections are also adopted in many European and Latin American countries (see Morton, 2006, Carey and Polga-Hecimovich, 2006). Informal primaries are also commonplace. For example, preselection—where party members vote for candidates to run for office at particular elections—is prevalent in Australia, Canada and New Zealand.

While there are many real-world examples of two-stage electoral processes, and an increasing body of empirical research that considers the effects of primaries on electoral prospects (see Carey and Polga-Hecimovich, 2006), theoretical analysis of party primaries remains limited (Hirano and Snyder, 2014).

This paper contributes to the theoretical analysis of primary elections by providing a framework for a two-stage electoral process in which primary voters consider policy promises, valence, and electability of candidates. In our model, there are two parties, each with two office-seeking primary candidates, and one of each party's candidates—called the party's *advantaged* candidate—has higher valence than the other. The four candidates select their policy promises simultaneously, following which both parties hold primary elections at the same time. The winning candidate from each primary election then competes in the general election. There is uncertainty regarding the realization of the median voter's bliss point in the general election. We assume that voters judge candidates based on two aspects: policy promises and valence. Further, voters in each party's primary choose strategically between their two candidates. In the model, we make the *no Etch a Sketch* assumption,[1] meaning that politicians are committed to a single policy promise for both the primary and general elections. Finally, we define a political equilibrium as a pure strategy Nash equilibrium.

In equilibrium, there are two possible regimes, which we term *regime 1* and *regime 2*. Without loss of generality, we assume that the valence of the left party's advantaged candidate is at least as high as that of the right party's advantaged candidate. In regime 1, the valence difference amongst candidates is sufficiently small such that there is no dominant candidate guaranteed to win both stages of the election. In this case, viable policy promises for each candidate is constrained. Under this regime, our results demonstrate that in equilibrium, the advantaged candidate in each party chooses the policy which maximizes the probability of winning the general election, subject to just beating the disadvantaged candidate in the primary election. In regime 2, the left party's advantaged candidate has much higher valence than

[1] This is a common assumption in the literature, which Hummel (2013) and Grofman et al. (2019) also adopt. Its origin lies in the 2012 Republican presidential primaries. Asked whether candidate, Mitt Romney, had moved too far right for the upcoming general election, his top aide, Eric Fehrnstrom, said that they would hit a reset button for the fall campaign, "almost like an Etch A Sketch." Two rival candidates then held up an Etch A Sketch toy as a visual aid to their audiences and criticized the remark. For further details, see: `http://en.wikipedia.org/wiki/Mitt_Romney_presidential_campaign,_2012`, viewed on June 25, 2020.

all other candidates such that there is some policy promise for the left party's advantaged candidate that guarantees her victories in both the primary and general elections.

Our model is robust under various extensions, and particularly useful for conducting comparative statics. Specifically, we provide testable predictions for how electoral outcomes are affected when public opinion shifts in one direction along the political spectrum. According to Harmon (2018), such shifts in public opinion could be caused by immigration, which is particularly relevant given the profound demographic changes experienced world-wide, especially this century (see Frey, 2015). Another set of comparative statics analyses several scenarios in which the variability of public opinion changes. First, we compare the equilibria in two societies, where the distribution of voters' bliss points in one society is clustered closer to the center than that in the other society. Next we consider the effect of a leftward shift in public opinion, in two scenarios that differ in terms of sets of voters changing their bliss points: under the first scenario, there are more voters in one society holding extreme left bliss points relative to the other; under the second scenario, the centrists' opinion shifts toward the left.

Within the general framework, we show that when the valence advantage increases for an advantaged candidate, the distance between policy promises chosen by the two advantaged candidates decreases. We also show that when the disadvantaged candidates' valences increase, this distance increases. Finally, we show that when public opinion becomes more diverse, the distance between policy promises made by the two advantaged candidates becomes greater. We then demonstrate our results by using computer simulation under the assumption that the median voter's bliss point is distributed according to a truncated normal distribution.

Additionally, several special cases of the model are analyzed. We consider the symmetric game where both parties' advantaged candidates have the same level of valence, while both parties' disadvantaged candidates also share the same—albeit lower—level of valence. We next study the case in which only one party holds a primary. A modified version of our model incorporating endogenous entry of candidates is also considered. Characterization of the equilibria for these special cases are provided.

In the theoretical literature, there have been several attempts to model a primary election. Our model extends Hummel (2013) by allowing valence levels to differ both between and within parties, and by allowing the distribution of public opinion to be asymmetric. This generalization enables us to analyze how relative changes in valence between candidates affect policy promises and electoral outcomes. It also allows us to consider the impact of shifts in public opinion on policy promises and election results. Further, the model can be used to predict how polarization of policy promises chosen by candidates could arise after public opinions become more diverse, or more specifically how policy promises change when there is an increase in the proportion of voters holding more extreme opinions.

The Literature Review

Owen and Grofman (2006) develop a two-stage electoral competition model with sincere voters and examine the candidates' equilibrium policy positions within a model of two-party competition in which candidates differ only in their policy promises.

Furthermore, Adams and Merrill (2008) propose a two-stage election model with two candidates in each party, each of whom attempts to maximize the joint winning probability in both the general and primary elections. In their model, candidates can differ in terms of valence and voters vote sincerely; and they prove the existence of a Nash equilibrium if each voter's utility is single-peaked. They show that when candidates in the same party have the same valence, they choose the same policy position in equilibrium. In addition, they show that as the other party's candidates' valence increases, candidates have an incentive to unilaterally shift their policy promises toward the general election median voter's bliss point. In our model voters are assumed to vote strategically instead of sincerely, and we show that this pattern does not hold; rather, when one advantaged candidate's valence increases, this pushes the other candidates' equilibrium policies away from the median of the overall median voter's bliss point, which we call the *center*, and toward their own party's median voter's bliss point.

The previous framework is extended in Grofman et al. (2019). In their analysis, voters are sincere, and candidates vary in valence. Their model differs from Adams and Merrill (2008) only in that the distribution of voter bliss points is deterministic. By assuming that candidates maximize the expected vote share in the general election, they show that a pure strategy Nash equilibrium always exists. In our model, because voters are strategic, when a policy chase arises between primary winners, a pure strategy Nash equilibrium could fail to exist. However, interestingly, primaries could play a role of stabilizing the game in the sense that a primary in each party constrains the advantaged candidates' choice of policy promises that could prevent a policy chase from arising. We use this intuition to provide the conditions for the existence of equilibrium.

In addition, Grofman et al. (2019) show that the winner of the general election needs not to be the candidate with the highest valence and points out that the choice of primary type (closed or open) may be important to determine the winner of the general election. Because in our model, the distribution of the median voter's bliss point is stochastic, who wins the general election is not deterministic. Further we consider a modified version of our model such that a candidate has a choice of entry to the elections by applying the concept of subgame-perfect equilibria. By studying the situation where only the left party holds a primary, we show that if the disadvantaged candidate of the left party is more concerned about a policy promise, she enters into the primary to influence the policy promise by the advantaged candidate of the left party even though the entry decreases the probability of winning the general election for her party. Hence, our analysis shows that the endogenous entry decision affects policy outcomes as well as their winning chances.

Our analysis is also close to the model presented by Hummel (2013), which examines a scenario that we term the *symmetric game* where both parties' advantaged candidates have the same level of valence, while both parties' disadvantaged candidates also share the same—albeit lower—level of valence. In addition, the distribution of the overall median voter's bliss point is symmetric. In our framework we can study the symmetric game as a special case. We show that an equilibrium always exists in the symmetric game and that if the difference between the advantaged and disadvantaged candidates' valences is small relative to the distance between the center and the parties' median voters' bliss points, then the equilibrium in Hummel (2013)—which is a regime 1 equilibrium—occurs. Otherwise, both advantaged candidates choose the center. The literature of political competition has extensively studied patterns of equilibrium policy outcomes in a two-party electoral model (see Owen and Grofman, 2006, Drouvelis et al., 2014a, Takayama et al., 2019). Two important patterns are *policy polarization* and *policy convergence*. When *policy convergence* arises, each party announces a policy located on the center. When *policy polarization* arises, each party chooses somewhere between the party's bliss point and the center. Our analysis about the symmetric game provides conditions under which policy polarization or convergence arises, while in our analysis within the general framework, under a regime 1 equilibrium, policy polarization arises.

In much of the literature, and in this paper, a candidate is modeled as a combination of a policy promise and a valence. There is a large body of empirical work that studies how these two factors affect electoral outcomes. Studies of the US, and evidence from recent cross-country research of Western Europe, suggest that changes to candidates' valence have important electoral consequences. In particular, they show that there is some relationship between valence, voters' choices, and the ideological distance between the candidates' policy promises (see Abney et al., 2013, Adams et al., 2011, Clark and Leiter, 2014, Green and Hobolt, 2008, Adams et al., 2012, Buttice and Stone, 2012). While many studies have also analyzed how the valence of party elites affects voters' electoral support for parties (Mondak, 1995, McCurley and Mondak, 1995, Stone and Simas, 2010), the relationship between valence, policy decisions, and electability remains uncertain.

The Organization of this Paper

The remainder of the paper is organized as follows. Section 2 describes the model and Section 3 provides the equilibrium existence theorem. Section 4 conducts the comparative statics, while Section 5 provides the equilibrium of the scenario in which there is only one primary. Section 6 discusses extensions including endogenous entry. Section 7 concludes.

2 The Model

In the model, there are two parties, which we refer to as party L and party R. The policy space $X = [-1, 1]$ is unidimensional. Party R and party L have core constituencies toward the right (closer to 1) and toward the left (closer to -1) in X, respectively.

We assume that each party has two candidates and the electoral process involves two elections: a primary election and a general election. First, both parties hold a primary election and voters in each party's primary choose strategically between their two candidates. Second, in the general election, the winning candidates from each primary election compete against each other.

The four candidates are assumed to be motivated solely by winning office. Each candidate is characterized by two aspects: strategy and characteristics. Each candidate chooses a *policy promise* and their characteristics are measured by a variable called *valence*. A candidate's valence variable is fixed exogenously, and benefits all voters independently of their political opinions if the candidate is elected. We assume that in each party, there is one advantaged candidate with *high valence*—denoted A—and one disadvantaged candidate with *low valence*—denoted D. This allows us to identify each candidate by their valence. In particular, party R has two candidates, R_A and R_D, with valences satisfying $R_A > R_D$, while party L also has two candidates, L_A and L_D, with valences satisfying $L_A > L_D$. Without loss of generality, we assume $L_A \geq R_A$.

There is a continuum of voters, each with a bliss point $b \in X$. We define the utility of a voter with bliss point $b \in X$, when the winning candidate has valence P_j for $P = L, R$ and $j = A, D$ and policy promise x, by:

$$v_b(x, P_j) = P_j - |x - b|. \tag{1}$$

We assume that the set of voters in the primary election of party k is a subset of the overall voting population, and has a measure denoted by μ_k for each $k = L, R$. Additionally, μ_L and μ_R are public information.

Let M_L and M_R be party L's and party R's median voter's bliss point with $M_R > M_L$. We call the median voter in the general election *the overall median voter*. We denote the bliss point of the overall median voter using M, and assume this is randomly distributed in $[M_L, M_R]$, with a cumulative distribution function F. We assume that F is continuously differentiable in its support $[M_L, M_R]$. Then, $F(x) = 0$ for $x \leq M_L$ and $F(x) = 1$ for $x \geq M_R$. Also, F is common knowledge. We denote the probability density function of F by f.

Assumption 1. *The hazard rate* $\frac{f(x)}{1-F(x)}$ *is increasing in* x *for all* $x \in [M_L, M_R]$ *and* $\frac{f(x)}{F(x)}$ *is decreasing in* x *for all* $x \in [M_L, M_R]$.

Assumption 1 is a standard assumption in the literature known as the *monotone hazard rate* (see, for example, Roemer, 1997, Saporiti, 2008, Hummel, 2013, Drouvelis et al., 2014b). This assumption guarantees the unique existence of maximizers in the expected payoffs and, while it is mainly for tractability, we also use the hazard rate to conduct comparative statics later in Section 5.

The sequence of the game is as follows.

Stage 0 Each candidate simultaneously and independently chooses their policy promise in each party. We assume that, if a politician wins the primary, she must commit to the same policy promise in the general election.

Stage 1 The primary election is held for each party.

Stage 2 The general election between the winners of each primary is held.

Suppose that party R's candidate—with policy r and valence R—and party L's candidate—with policy l and valence L—are the respective winners of their party's primary. Then, party R's candidate's probability of winning in the general election is,

$$\pi_R(r, R, l, L) = \Pr[v_M(r, R) \geq v_M(l, L)]. \tag{2}$$

When $|r - l| > |R - L|$, if $r < l$,

$$\pi_R(r, R, l, L) = \begin{cases} 0 & \text{if } \frac{R-L+l+r}{2} \leq M_L \\ F(\frac{R-L+l+r}{2}) & \text{if } \frac{R-L+l+r}{2} \in (M_L, M_R) \\ 1 & \text{if } \frac{R-L+l+r}{2} \geq M_R. \end{cases} \tag{3}$$

and if $r > l$,

$$\pi_R(r, R, l, L) = \begin{cases} 1 & \text{if } \frac{L-R+l+r}{2} \leq M_L \\ 1 - F(\frac{L-R+l+r}{2}) & \text{if } \frac{L-R+l+r}{2} \in (M_L, M_R) \\ 0 & \text{if } \frac{L-R+l+r}{2} \geq M_R. \end{cases} \tag{4}$$

Note that when $R = L$, (3) and (4) describe the outcome for all $l \neq r$. However, we also need to specify the probability of winning the general election for each candidate when $l = r$. This situation only arises in Corollary 1 when the game is symmetric (see Definition 2). When $L_A = R_A$ in the general election, we define a tiebreaking rule, $t : X \to [0, 1]$, such that for every $x \in X$,

$$t(x) = \pi_R(x, R_A, x, L_A). \tag{5}$$

For each $i, n = A, D$, define the expected payoff V_b of voter b, when candidate L_i is the winner in party L's primary and candidate R_n is the winner in party R's primary, by:

$$V_b(l_i, L_i, r_n, R_n) \quad = \quad \pi_R(r_n, R_n, l_i, L_i) v_b(r_n, R_n) + (1 - \pi_R(r_n, R_n, l_i, L_i)) v_b(l_i, L_i).$$

Let $w^k = (w^k_A, w^k_D)$ for each $k = L, R$, where each w^k_i denotes the probability that type $i = A, D$ wins the primary of party $k = L, R$. Then, for each $i = A, D$, the expected payoff of voter b in party L, when candidate L_i with policy l_i wins party L's primary is given by:

$$\mathrm{E}V_b(l_i, L_i, r_A, r_D) \quad = \quad w^R_A V_b(l_i, L_i, r_A, R_A) + (1 - w^R_A) V_b(l_i, L_i, r_D, R_D),$$

and the expected payoff of voter b in party R, when candidate R_i with policy r_i wins party R's primary, is given by:

$$\mathrm{E}V_b(r_i, R_i, l_A, l_D) \quad = \quad w^L_A V_b(l_A, L_A, r_i, R_i) + (1 - w^L_A) V_b(l_D, L_D, r_i, R_i).$$

In words, V_b is voter b's interim expected payoff when the winner of each primary election is known, while $\mathrm{E}V_b$ is voter b's ex ante expected payoff before the primaries.

The following definition expresses the notion of a pure strategy Nash equilibrium for the model.

Definition 1 (pure strategy Nash equilibrium)**.** *An equilibrium in a game of two-stage electoral competition is a collection of policy promises,* (l^*_A, l^*_D) *and* (r^*_A, r^*_D)*, and a collection of winning probabilities in the primary elections* $(w^{*k})_{k\in\{L,R\}} = (w^{*k}_A, w^{*k}_D)_{k\in\{L,R\}}$ *such that:*

I. $w^{*k}_A + w^{*k}_D = 1$ *for every* $k = L, R$*;*

II. for $i, j = A, D$*, in response to* (l^*_A, l^*_D, r^*_j)*, each* r^*_i *is a best response such that the following holds:*

(A) If $\mathrm{E}V_{M_R}(r_i, R_i, l^*_A, l^*_D) > \mathrm{E}V_{M_R}(r_j, R_j, l^*_A, l^*_D)$*, then* $w^{*R}_i = 1$*;*

(B) Whenever $w^{*L}_n > 0$ *for any* $n = A, D$*, there is no policy* r_i *such that candidate* R_i *can* profitably deviate *in the sense that*

$$\mathrm{E}V_{M_R}(r_i, R_i, l^*_A, l^*_D) > \mathrm{E}V_{M_R}(r^*_j, R_j, l^*_A, l^*_D),$$

and $\pi_R(r_i, R_i, l^*_n, L_n) > w^{*R}_i \cdot \pi_R(r^*_i, R_i, l^*_n, L_n)$*;*

III. for $i, j = A, D$*, in response to* (r^*_A, r^*_D, l^*_j)*, each* l^*_i *is a best response such that the following holds:*

(A) If $\mathrm{E}V_{M_L}(l^*_i, L_i, r^*_A, r^*_D) > \mathrm{E}V_{M_L}(l^*_j, L_j, r^*_A, r^*_D)$*, then* $w^{*L}_i = 1$*;*

(B) Whenever $w^{*R}_n > 0$ *for any* $n = A, D$*, there is no policy* l_i *such that candidate* L_i *can* profitably deviate *in the sense that*

$$\mathrm{E}V_{M_L}(l_i, L_i, r_A, r_D) > \mathrm{E}V_{M_L}(l^*_j, L_j, r^*_A, r^*_D),$$

and $(1 - \pi_R(r^*_n, R_n, l_i, L_i)) > w^{*L}_i \cdot (1 - \pi_R(r^*_n, R_n, l^*_i, L_i))$*.*

The equilibrium condition (A) states that a candidate who obtains the majority wins the primary with certainty. The equilibrium condition (B) requires that there are no other policy promises for any of the four candidates that would win them the primary election with certainty and yield a higher probability of winning the general election. In this definition of an equilibrium, we assume that the median voter theorem holds in primary elections. This is done to ensure analytical tractability and facilitate a simple analysis. Further, the validity of this assumption is reinforced by the results of an earlier version of this paper, Takayama (2014); when there is a reasonable concentration of primary voters around M_R or M_L compared to those around the tails of μ_R or μ_L there is a Condorcet winner, and the median voter theorem holds in primary elections.

The median voter theorem also holds in primary elections in Hummel (2013); here it is assumed that the number of primary voters—say m—is odd, and that $\frac{m+3}{2}$ voters have bliss points smaller than M_L in party L's primary, while $\frac{m+3}{2}$ voters have bliss points larger than M_R in party R's primary.[2]

3 Equilibrium Analysis

In this section, we present the main theorems. In doing so, it is useful to keep in mind that there are two possible regimes in equilibrium.

Regime 1. Advantaged candidates choose more centrist policy promises that are just close enough to their parties' median bliss points so as to guarantee their victory in their respective primaries. Candidate L_A does not win the general election with certainty.

Regime 2. There is a $l_A^* \in [M_L, M_R]$ that guarantees candidate L_A victories in both the primary and general elections.

3.1 The Existence Theorem

Theorem 1 provides the conditions under which regime 1 arises in equilibrium. In particular, a regime 1 equilibrium occurs when each advantaged candidate chooses the policy promise that maximizes their probability of winning the general election, subject to winning the primary election. In this way, the advantaged candidate wins each party's primary with certainty.

As we show in Propositions A-1 and A-2 (see Appendix A), while both parties' advantaged candidates win their respective primaries in regime 1, neither is guaranteed to win the general

[2]Although the focus is very different from ours, Bouton (2013) considers a runoff election and demonstrates that runoff systems with a threshold below 50 percent produce an Ortega effect such that the Condorcet loser of the election may win. Further, in contrast with Grofman et al. (2019), Bouton (2013) concludes that a sincere voting equilibrium does not always exist.

election in equilibrium. Since $w_A^{*L} = w_A^{*R} = 1$ in the regime 1 equilibrium, when examining the primary results, we only need to consider,

$$\begin{aligned} V_b(l_i, L_i, r_A, R_A) &= \mathrm{E}V_b(l_i, L_i, r_A, r_D); \\ V_b(l_A, L_A, r_i, R_i) &= \mathrm{E}V_b(r_i, R_i, l_A, l_D). \end{aligned}$$

Suppose r_n is the policy promise from party R's primary winner with valence R_n. Let $\alpha(r_n, R_n)$ be the set of policy promises that guarantees candidate L_A a victory in the general election, such that,

$$\alpha(r_n, R_n) = \left[r_n - (L_A - R_n), r_n + (L_A - R_n)\right].$$

Next, define correspondences β_L and β_R as follows: for each $r_A, l_A \in X$,

$$\begin{aligned} \beta_L(r_A, R_A) &= \{l : V_{M_L}(l, L_A, r_A, R_A) \geq \max_{l_D \in X} V_{M_L}(l_D, L_D, r_A, R_A)\}; \\ \beta_R(l_A, L_A) &= \{r : V_{M_R}(l_A, L_A, r, R_A) \geq \max_{r_D \in X} V_{M_R}(l_A, L_A, r_D, R_D)\}. \end{aligned}$$

Then, β_L and β_R are the sets of policy promises that guarantee candidates L_A and R_A victory in their respective primaries when the opponent party's primary winner is known to be the advantaged candidate. When β_L and β_R are nonempty, it is optimal for the advantaged candidates to select the policy promises in these sets.

For each $r_A, l_A \in X$, define $\hat{l}_D(r_A)$, $\hat{r}_D(l_A)$, $\hat{l}_A(r_A)$, and $\hat{r}_A(l_A)$ by:

$$\hat{l}_D(r_A) = \arg\max_{l_D \in X} V_{M_L}(l_D, L_D, r_A, R_A); \tag{6}$$

$$\hat{r}_D(l_A) = \arg\max_{r_D \in X} V_{M_R}(l_A, L_A, r_D, R_D); \tag{7}$$

$$\hat{l}_A(r_A) = \max\{l_A | V_{M_L}(l_A, L_A, r_A, R_A) \geq V_{M_L}(\bar{l}_D, L_D, r_A, R_A) \text{ for } \bar{l}_D \in \hat{l}_D(r_A)\}; \tag{8}$$

$$\hat{r}_A(l_A) = \min\{r_A | V_{M_R}(l_A, L_A, r_A, R_A) \geq V_{M_R}(l_A, L_A, \bar{r}_D, R_D) \text{ for } \bar{r}_D \in \hat{r}_D(l_A)\}. \tag{9}$$

Notice that $\hat{l}_D(r_A)$ and $\hat{r}_D(l_A)$ are correspondences and we can also show that $\hat{l}_D(r_A)$ and $\hat{r}_D(l_A)$ are nonempty. In Lemma A-1, we show that if candidate L_D's probability of winning the general election against candidate R_A is strictly positive when $l_D < r_A$, then $\hat{l}_D(r_A)$ is singleton. The same holds for $\hat{r}_D(l_A)$ when $r_D > l_A$ and candidate R_D's probability of winning the general election against candidate L_A is strictly positive. Otherwise, if $\pi_R(r_A, R_A, l_D, L_D) = 1$ for all $l_D \leq r_A$, then $V_{M_L}(l_D, L_D, r_A, R_A) = R_A - r_A + M_L$ for $l_D \leq r_A$, and $\hat{l}_D(r_A) \supseteq [M_L, r_A]$. This is because for every $l_D > r_A$ with $\pi_R(r_A, R_A, l_D, L_D) < 1$, we must have $R_A - r_A + M_L > L_D - l_D + M_L$ and thus, $V_{M_L}(l_D, L_D, r_A, R_A) < R_A - r_A + M_L$.

Figure 1 illustrates $\beta_R(l_A, L_A)$ and demonstrates how a regime 1 equilibrium works. For a fixed l_A, if each candidate from party R has a nonzero probability of winning the general election, then voter M_R's expected payoff from party R's candidates is bell-shaped. Then,

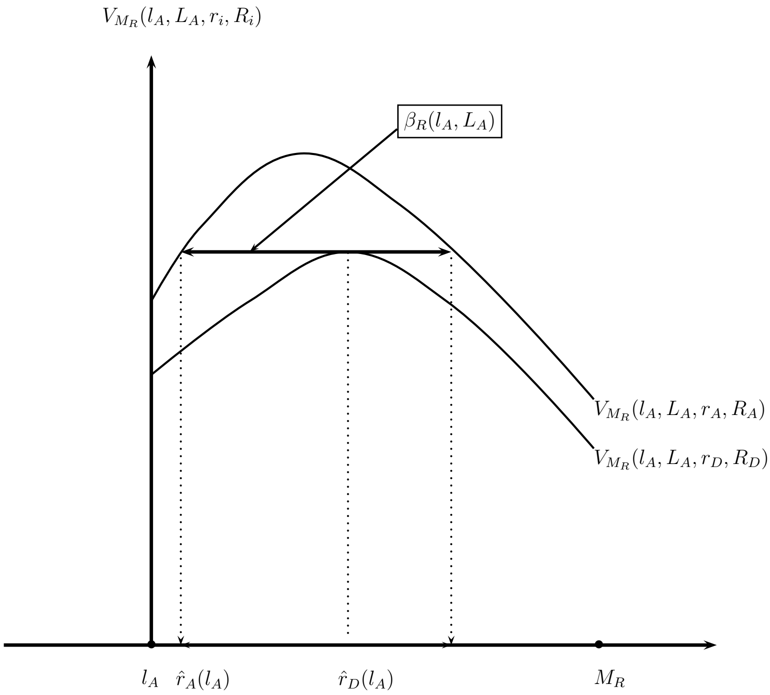


Figure 1: Illustration of $\beta_R(l_A, L_A)$

$\beta_R(l_A, L_A)$ is depicted as the interval of policy promises that yield a higher utility for candidate R_A than that achieved by the policy $\hat{r}_D(l_A)$.

The definition of a pure strategy Nash equilibrium uses an endogenous tiebreaking rule, the nature and motivation of which may not be immediately intuitive.[3] In a regime 1 equilibrium, the disadvantaged candidates choose the policy promises that maximize the expected payoff of their own party's median voters, denoted by $\hat{r}_D(l_A)$. Then, we can show that the set of policy promises that guarantee the advantaged candidate's victory in the primary, $\beta_R(l_A, L_A)$, is nonempty, as shown in Figure 1. Thus, the advantaged candidate chooses the policy promise $\hat{r}_A(l_A)$—within $\beta_R(l_A, L_A)$, the set of policies that guarantees victory in the primary—that maximizes their probability of winning the general election. Suppose the advantaged candidate chooses $\hat{r}_A(l_A)$, but their winning probability is smaller than one. Then, the advantaged candidate could obtain a certain victory in the primary by deviating from this policy promise to one that is slightly smaller or larger. Our endogenous tiebreaking rule allows an equilibrium in which the advantaged candidate can choose $\hat{r}_A(l_A)$ and still win the primary with certainty.[4]. The natural equilibrium concept has the agent with the highest valuation bidding the second

[3]The notion of an *endogenous sharing rule* with a Nash equilibrium is proposed in Simon and Zame (1990) (see, also, Jackson et al., 2002) and has been used in several studies to obtain the existence of a pure strategy Nash equilibrium. For example, Ausubel and Milgrom (2002) and Day and Milgrom (2008) use a similar definition of Nash equilibrium in the context of auction theory. See also Dell'Ariccia and Marquez (2006) in the banking literature and Prat and Rustichini (2003) in the common agency theory literature. Myerson (1993) also applies it in a political competition context.

[4]A similar issue arises in a first price auction when bidders' valuations are publicly known to all bidders

highest valuation and winning with certainty.

In other words, whenever a tie arises in the primary, there exists a slight deviation such that the median voter of each party strictly prefers the advantaged candidate over the disadvantaged candidate. For this slight deviation, we would imagine that the advantaged candidate wins the primary; the endogenous tiebreaking rule is consistent with this expectation. However, an endogenous tiebreaking rule in the primaries is not relevant for a regime 2 equilibrium since candidate L_A wins party L's primary with certainty, and an equilibrium strategy is unrestricted for party R's candidates.

Theorem 1 deals with the case where candidate L_A is not so strong. The conditions in Theorem 1 ensure that in equilibrium, the sets $\alpha(r_A^*, R_A)$ and $\beta_L(r_A^*, R_A)$ do not overlap, and $\beta_L(r_A^*, R_A)$ and $\beta_R(l_A^*, L_A)$ also do not overlap. This is because if $\alpha(r_A^*, R_A)$ and $\beta_L(r_A^*, R_A)$ overlap, candidate L_A can choose a policy promise in $\alpha(r_A^*, R_A) \cap \beta_L(r_A^*, R_A)$ that guarantees victory in both elections in equilibrium. However, this equilibrium is no longer in regime 1. Also, when $\beta_L(r_A^*, R_A)$ and $\beta_R(l_A^*, L_A)$ do not overlap, to maximize the probability of winning the general election, each candidate chooses the policy at the boundary of the β set that is closest to their opponent's policy. Thus, these conditions are sufficient for the existence of an equilibrium in regime 1.

Theorem 1. *Suppose $M_R - M_L > L_A - R_A \geq 0$. Suppose also that one of the following conditions holds.*

- *Suppose $L_A = R_A$, $L_A - R_D < M_R - z$ and $R_A - L_D < z - M_L$ for some $z \in (M_L, M_R)$.*
- *Suppose $L_A > R_A$ and that for some $z \in (M_L, M_R)$, the following conditions hold:*

$$\frac{\left(1 - F(\frac{M_R + z + L_A - R_D}{2})\right)^2}{\frac{1}{2} f(\frac{M_R + z + L_A - R_D}{2})} < L_A - R_A < \frac{\left(1 - F(\frac{M_R + M_L + L_A - R_D}{2})\right)^2}{\frac{1}{2} f(\frac{M_R + M_L + L_A - R_D}{2})} \tag{10}$$

$$\frac{\left(F(\frac{M_L + z + L_D - R_A}{2})\right)^2}{\frac{1}{2} f(\frac{M_L + z + L_D - R_A}{2})} < 2(L_A - R_A) < \frac{\left(F(\frac{M_L + M_R + L_D - R_A}{2})\right)^2}{\frac{1}{2} f(\frac{M_L + M_R + L_D - R_A}{2})}. \tag{11}$$

Then an equilibrium exists, and $(w_A^{*L}, w_D^{*L}) = (w_A^{*R}, w_D^{*R}) = (1, 0)$. Further, $\hat{l}_D(r_A^*)$ and $\hat{r}_D(l_A^*)$ are singletons, and the equilibrium satisfies $l_D^* \in \hat{l}_D(r_A^*)$, $r_D^* \in \hat{r}_D(l_A^*)$, $l_A^* = \hat{l}_A(r_A^*)$ and $r_A^* = \hat{r}_A(r_A^*)$. In addition, $l_A^* \leq z \leq r_A^*$ holds with at least one of the two inequalities being strict.

The first statement of Theorem 1 indicates that an equilibrium exists under a very mild condition when $L_A = R_A$. These two inequalities $L_A - R_D < M_R - z$ and $R_A - L_D < z - M_L$ for some $z \in (M_L, M_R)$ ensure that a policy chase between the advantaged candidates does not arise in equilibrium. The two inequalities in the second statement also ensure that a policy chase does not arise in equilibrium. First, it ensures that $L_A - R_A$ is not so large, and otherwise a regime 2 equilibrium may start to arise. Second, similarly to the first statement, R_D and L_D

are not so small and a policy chase does not arise because if one of them is small, then the β sets could overlap and a policy chase would possibly arise. In both statements, z takes a role of separating two regions of policy promises which candidates L_A and R_A could take in equilibrium in the sense that for policy $l_A \in [M_L, z]$, β_R is contained in $[z, M_R]$ and for policy $r_A \in [z, M_R]$, β_L is contained in $[M_L, z]$.[5]

In the literature on pure strategy Nash equilibria in political competition models, Aragones and Palfrey (2002) employ the standard Downsian model of two-candidate elections of two politicians; they show that when one candidate is more advantaged in terms of valence than the other, pure strategy Nash equilibria often do not exist.[6] Indeed, in our model, a pure strategy Nash equilibrium may fail to exist. Moreover, even if the existence of a pure strategy Nash equilibrium is guaranteed, one may fail to arise due to an unending "policy chase" in which the advantaged candidate continually pursues the disadvantaged candidate's policy promise over the entire policy space. However, if certain conditions are satisfied, the existence of the disadvantaged candidate in the primary may limit the extent of the chase by the advantaged candidate, and in this circumstance, an unending policy chase does not occur. If such conditions—which are specified in Theorem 1—hold, then a regime 1 equilibrium could exist.

Corollary 1 is a special case of Theorem 1 in the context of a symmetric game, as described in Definition 2 below. This corollary states that when the game is symmetric, an equilibrium always exists. An example of a symmetric game is Hummel (2013), which shows that in a symmetric game, if there is a solution to (6), (7), (8), and (9), then the game has an equilibrium. Corollary 1 shows that an equilibrium always exists in a symmetric game, regardless of whether the solution to the set of equations exists, and gives a condition under which the equilibrium that Hummel (2013) has shown arises and belongs to regime 1.

Formally, we define $\bar{m} = F^{-1}(\frac{1}{2})$ as the median of the distribution of the overall median voter's bliss point, while f is assumed to be symmetric with respect to $\bar{m} = 0$. We call this setting *symmetric* and formally define it as follows.

Definition 2. *Let $\bar{m} = 0$. Suppose that f is symmetric with respect to 0 with its support $[-a, a]$, and $R_A = L_A$ and $R_D = L_D$. We say that the game is* symmetric.

Corollary 1. *Suppose that the game is symmetric. Then an equilibrium always exists. In equilibrium, either (I) or (II) holds:*

(I) If $a \leq L_A - L_D$ and a tiebreaking rule $t(\bar{m}) = \frac{1}{2}$, then $(w_A^{*L}, w_D^{*L}) = (w_A^{*R}, w_D^{*R}) = (1, 0)$, $(l_A^*, r_A^*) = (\bar{m}, \bar{m})$ and $l_D^*, r_D^* \in X$.

[5] A similar idea is employed in Takayama et al. (2019) whose existence proof uses the idea of restricting the domains for each politician and applies the theorem of McLennan et al. (2011) to prove the existence of pure strategy Nash equilibria in political competition games.

[6] See also Saporiti (2008) and Takayama et al. (2019). The former uses the concept of the *better-reply security* proposed by Reny (1999) to consider the existence of pure strategy Nash equilibria while the latter uses the recently developed concept of discontinuous games by McLennan et al. (2011) to prove the existence of pure strategy Nash equilibria in games where the median voter's bliss point is stochastic.

(II) Otherwise, if $a > L_A - L_D$*, then* $(w_A^{*L}, w_D^{*L}) = (w_A^{*R}, w_D^{*R}) = (1, 0)$*, and* $(l_A^*, l_D^*, r_A^*, r_D^*)$ *solve* (6)*,* (7)*,* (8)*, and* (9)*. The equilibrium satisfies* $r_A^* = -l_A^*$*, and* $r_D^* = -l_D^*$ *where* r_A^* *and* r_D^* *respectively satisfy*

$$r_A^* + r_D^* - (L_A - L_D) = \frac{F(\frac{r_A^* - r_D^* - (L_A - L_D)}{2})}{\frac{1}{2} f(\frac{r_A^* - r_D^* - (L_A - L_D)}{2})} \tag{12}$$

$$r_A^* = (r_A^* + r_D^* - (L_A - L_D)) \cdot F(\frac{r_A^* - r_D^* - (L_A - L_D)}{2}). \tag{13}$$

The intuition behind Corollary 1 is as follows. As we formally state later, Proposition 4 in Hummel (2013) shows that when $(L_A - L_D)$ increases, the β sets expand. Consider the situation in which $(L_A - L_D)$ is greater than a such that $\beta_R(\bar{m}, L_A)$ and $\beta_L(\bar{m}, R_A)$ overlap (as shown in Lemma A-8). Notice that in a symmetric game, $\bar{m} \in \beta_R(\bar{m}, L_A)$ if and only if $\bar{m} \in \beta_L(\bar{m}, R_A)$. Thus, when $(L_A - L_D)$ is sufficiently large, $\bar{m} \in \beta_R(\bar{m}, L_A) \cap \beta_L(\bar{m}, R_A)$ holds. In this situation, as in Hotelling's model, both advantaged candidates choose $\bar{m} = 0$ in equilibrium. Conversely, when $(L_A - L_D)$ is small, both advantaged candidates are more constrained by competition in their primaries, and so the equilibrium in regime 1 that Hummel (2013) studies arises.

In reality, we observe both competitive and uncompetitive primaries. In our analysis, a regime 1 equilibrium describes a competitive primary, since it considers the case in which advantaged candidates are more constrained by primaries. However, we could well imagine that the presence of a very strong candidate could result in an uncompetitive primary. This is best described by a regime 2 equilibrium; since the most advantaged candidate has a policy promise that guarantees victories for her, she is less constrained by her party's primary, which may therefore be interpreted as relatively uncompetitive. Our theorem in this section provides sufficient conditions under which a regime 1 equilibrium exists. Although regime 2 equilibria are observed in reality, theoretically, the existence theorems of such equilibria may not be so interesting. Nonetheless, since they can provide useful insights into the discussion on the role of uncompetitive primaries, we keep the existence theorems in Appendix B.

4 Comparative Statics

In this section, we examine how changing model parameters—such as candidate valence and distribution of voters' bliss points—affects the equilibrium outcome. Our analyses yield testable hypotheses as propositions. First, we consider how a unilateral change of valence for some candidates affects the equilibrium. Second, we evaluate the impact of a decrease in the variance of the distribution of the overall median voter's bliss point; this can be interpreted as public opinion becoming more centrally concentrated. Third, we appraise the impact of a leftward shift in the distribution of the overall median voter's bliss point; this can be interpreted as public opinion shifting toward party L's ideology.

The extensive literature on candidate characteristics makes clear that, as a consequence of the decline in party identification (Garzia 2013, Kornberg 2005), individuals are increasingly basing their votes on candidates' personal attributes (Hayes 2009, Rahat and Sheafer 2007, Holian and Prysby 2014, Kinder et al. 1980). Alongside many other works, Abramowitz (1988) uses US congressional election data to estimate that candidates' characteristics were the main driver of election outcomes, while Buttice and Stone (2012) find that it was the candidates' valence differences that determined these electoral outcomes. A more concrete example is found in Holian and Prysby (2014), which analyzes the valence of Obama and Romney in the US 2012 elections. In particular, voters' perceptions with regards to empathy, integrity, and competence were higher for Obama, while their impressions of leadership were similar for both candidates. It was these former differences, according to the authors, that had a significant impact on the result and the observation that a significant majority of independent voters preferred Obama.

Assessing the interplay of policy promises and valence is a complicated task.[7] In this paper, we conduct two sets of exercises. In the first set, we consider the *symmetric game* by using a general form of distribution function for the median voter's bliss point. In the second set, we consider the case where the distribution function is uniform, triangular, and truncated normal. As the derivation of an equilibrium becomes very complicated in the last part of this section, we also conduct computer simulations to check the robustness of our results.

4.1 A Symmetric Case with a General Distribution Function

Changing Valence

We first consider how a unilateral change in valence affects the equilibrium. The literature provides examples of the impact of changes in the valence of opposing parties' candidates in symmetric games. Adams and Merrill (2008) examine the model in which each party has two candidates with identical valence. Then, they show that as one of the two opposing party's candidates' valences increase, the primary candidates of the other party locate their policy closer to the center to become more competitive in the general election. Intuitively, in the analysis of Adams and Merrill (2008), the candidates with relatively lower valence have to moderate their policy promises toward the center.

In contrast, Hummel (2013) considers the symmetric game and shows that as the valence difference between the advantaged and disadvantaged candidates increases, a candidate with higher valence can choose a more moderate policy promise in order to appeal to the general electorate while still winning the primary election. The following proposition is the formal statement of this result. Since the equilibrium in Hummel (2013) is equivalent to the one in

[7]Previously, in the literature, Adams et al. (2011) use a simple one-sided framework—in the context of an incumbent and challenger election—to illustrate the potential mechanism behind the connection between promises and valence.

our analysis of the symmetric game when $a > L_A - L_D$, and is also characterized by (12) and (13) of Corollary 1, the following result also holds in the symmetric game in our analysis.

Proposition (Hummel, 2013). *Consider the symmetric game studied in Theorem 1. Then when $R_A - R_D = L_A - L_D$ decreases, r_D^* decreases while r_A^* increases.*

In contrast, the next theorem assesses a scenario in which only candidate L_A's valence increases, while every other candidates' valence remains the same. This next theorem shows that when L_A increases, r_A^* and l_D^* move away from the center. The theorem also shows that r_D^* moves in the same direction as l_A^*. This is because candidate R_D chooses r_D^* to maximize the median voter M_R's expected payoff when voting for candidate R_D, given that candidate R_D will be facing candidate L_A, with a policy promise l_A^*, in the general election. More interestingly, the change in the direction of l_A^* depends on how strong—in terms of valence—candidate L_D is relative to candidate R_A. The theorem also states that there is an upper bound on candidate L_D's probability of winning the general election against candidate R_A, and if candidate L_D is strong enough relative to candidate R_A, then l_A^* moves toward M_L.

Theorem 2. *Suppose that the game is symmetric and $a > R_A - R_D = L_A - L_D$ holds.*[8] *Holding other exogenous variables constant, as L_A increases,*

- *r_A^* and $1 - \pi_R(r_A^*, R_A, l_A^*, L_A)$ increase.*
- *l_D^* decreases.*
- *Denote the probability that candidate L_D wins the general election against candidate R_A by F_D, and for given r_A^*, define $\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A}$ to be a function of F_D, which we denote by $\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A}(F_D)$. Let K be a solution in $(0, \frac{1}{2})$ for $\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A}(K) = 0$. Then K exists uniquely. Further, the following holds:*
  - *If $1 - \pi_R(r_A^*, R_A, l_D^*, L_D) < K$, l_A^* and r_D^* increase.*
  - *Otherwise, l_A^* and r_D^* decrease.*

Increasing Diversity of Bliss Points

Here, we examine the situation in which the center is the same in two societies, but the variance differs. We denote these two societies S and N. For simplicity, we consider the symmetric game in this subsection. Formally, we make the following assumption.

Assumption 2. *Let F_S and F_N be cumulative distribution functions with symmetric probability density functions f_S and f_N and a common support $[-a, a]$. Suppose that f_S and f_N satisfy the following: (1) $f_S(-a) < f_N(-a)$, (2) $f_S(0) > f_N(0)$, and (3) for all $x, x' \in [-a, 0]$ with $x < x'$, $\frac{f_S(x)}{f_S(x')} < \frac{f_N(x)}{f_N(x')} < 1$ and for all $x, x' \in [0, a]$ with $x < x'$, $\frac{f_S(x)}{f_S(x')} > \frac{f_N(x)}{f_N(x')} > 1$.*

[8] Recall from Definition 2 that f has support $[-a, a]$.

Assumption 2 implies the following properties (the proofs are found in Appendix A):

Property 1: For every $x \in [-a, 0]$, $\frac{F_S(x)}{f_S(x)} < \frac{F_N(x)}{f_N(x)}$. For every $x \in [0, a]$, $\frac{1-F_S(x)}{f_S(x)} < \frac{1-F_N(x)}{f_N(x)}$.

Property 2: For all $x \in [-a, 0]$, $F_S(x) < F_N(x)$ and for all $x \in [0, a]$, $F_S(x) > F_N(x)$.

We can compare the two societies in Assumption 2 according to the diversity of public opinion; while it is clustered closer to the center in society S, public opinion is more diverse in society N. In what follows, y^s denotes the equilibrium variable y in society $s = N, S$.

In equilibrium, the advantaged candidates R_A and L_A attempt to appeal to the voters of their own party. This is because, holding fixed the policy promises at the level of society S, the disadvantaged candidates' probability of winning the general election against the opponent party's nominee increases as the distribution shifts from F_S to F_N. This increases the attractiveness of the disadvantaged candidates to primary voters. Hence, to defend their prospects in the primaries, the advantaged candidates must shift their policy promises toward their own parties' bliss points. Then, the distance between the two policy promises increases. Theorem 3 describes this scenario.

Theorem 3. *Suppose that there are two societies N and S as defined in Assumption 2. Suppose also that both societies satisfy the conditions for the symmetric game, and a regime 1 equilibrium arises. Then $r_A^S < r_A^N$ and $l_A^N < l_A^S$.*

Hummel (2013) analyzes the scenario in which $F_N(x) = F_S(\frac{x}{\sigma})$ and shows that when σ increases, the advantaged candidates choose policy promises that are closer to their parties' bliss points. Theorem 3 is consistent with this result and shows that for a general F_N that satisfies Assumption 2, this tendency holds.

Shifting Median Voter's Bliss Point

We consider two different societies, N and S, with varying distributions of the overall median voter's bliss points. In society $s = N, S$, the cumulative distribution function is given by F_s, which has a well-defined probability density function f_s.

Suppose that for all $x, x' \in [-a, a]$, $\frac{f_S(x)}{f_S(x')} \leq \frac{f_N(x)}{f_N(x')}$ if $x < x'$. Let $\bar{m}_N$ satisfy $F_S(0) = F_N(\bar{m}_N) = \frac{1}{2}$. Under this assumption, the distribution of the overall median voter is more left skewed in society N when compared to society S. Further, the following three properties hold (the proofs are found in Appendix A):

Property 3: For every $x \in [-a, a]$, $\frac{f_N(x)}{1-F_N(x)} \geq \frac{f_S(x)}{1-F_S(x)}$.

Property 4: For every $x \in [-a, a]$, $\frac{f_N(x)}{F_N(x)} \leq \frac{f_S(x)}{F_S(x)}$.

Property 5: For every $x \in [-a, a]$, $F_S(x) \leq F_N(x)$ and hence $\bar{m}_S \geq \bar{m}_N$.

We now consider how the equilibrium policy promises move when the distribution in society S shifts to that in society N. While the formula that we obtain in the proof can be applied to any sized distribution shift, to obtain a simple intuition, we consider an infinitesimal change in the distribution function from F_S to F_N in the following two different contexts:

- Under the scenario of *extremists' change*, for $x \in [-a, -a+\bar{\epsilon}]$, $f_N(x) = (1+d(x))f_S(x)$, while for $x \in [a-\bar{\epsilon}, a]$, $f_N(x) = (1-d(x))f_S(x)$ where $d(x)$ is strictly smaller than $\bar{\epsilon}$ for every x.
- Under the scenario of *centrists' change*, for $x \in [-\bar{\epsilon}, 0]$, $f_N(x) = (1+d(x))f_S(x)$, while for $x \in [0, \bar{\epsilon}]$, $f_N(x) = (1-d(x))f_S(x)$ where $d(x)$ is strictly smaller than $\bar{\epsilon}$ for every x.

Under the scenario of centrists' change, the shift in public opinion implies that there is a large number of voters shifting their bliss points closer toward candidate L_A's policy promise from the center. As the shift is from the center, these voters' bliss points remain further away from candidate L_D's policy promise. The shift in public opinion toward party L's bliss point also benefits candidate L_A much more than candidate L_D in the primary election, because these new voters are attracted by both candidate L_A's policy promise and valence. Therefore, candidate L_A is less constrained by candidate L_D's primary challenge, and hence, is able to shift her policy toward the center to capture a greater share of votes in the general election.

Under the scenario of extremists' change, the shift in public opinion implies a large increase in the number of voters with bliss points between party L's bliss point and candidate L_D's policy promise. These new voters strictly prefer candidate L_A in the general election, forcing this candidate to shift her policy promise toward party L's bliss point to protect her chances of winning the primary.

Theorem 4. *Suppose that society S satisfies the conditions of the symmetric game. Suppose also that in both societies, a regime 1 equilibrium arises. Then, when the change between F_S and F_N is sufficiently small:*

- *Under the scenario of centrists' change, $l_A^N > l_A^S$.*
- *Under the scenario of extremists' change, $l_A^S > l_A^N$.*

4.2 A General Case with Specific Distribution Functions

First, we consider how a unilateral change in valence affects the equilibrium. Second, we analyze the effect of changes to the support of distribution of the median voter's bliss point and its moments in order to capture the potential effects that voters' characteristics can have on candidates' positions.

We start with the assumption that the median voter's bliss point is uniformly distributed, as in Kamada and Kojima (2014). In this setting, candidates L_D and R_D always choose the same policy promise regardless of valences and other candidates' policy promises. In order to explore higher interactions between the four candidates' valence and policy promises, we then study a setting that assumes a triangular distribution of the median voter's bliss point.

Using the triangular distribution assumption, we evaluate how changes in public opinion affect electoral outcomes. This is achieved by comparing two equilibria under two different distribution functions, F_N and F_S, such that F_N has heavier tails to reflect a society with more diverse opinions than F_S. Although we compare equilibria between societies S and N, the comparison can also be interpreted as how electoral outcomes change when public opinions change over time. These shifts have been documented in the literature. For example, Markussen (2008) shows that increases in economic growth cause a shift to the left of policy sentiments. Contrastingly, Harmon (2018) uses data from Denmark to show that an influx of immigrants could shift public opinions to the political right.

Finally, we use a numerical simulation under a truncated normal distribution to check the robustness of our theoretical results. The main results and intuition are preserved.

A Uniform Distribution Function

Proposition 1. *Suppose that F is uniformly distributed in* $[a, b]$. *Then, an equilibrium satisfies:*

$$\begin{aligned}
l_D^* &= a \\
r_D^* &= b \\
l_A^* &= \sqrt{2(L_A - L_D)(r_A^* - R_A - a) + (L_A^2 - L_D^2)} + a \\
r_A^* &= b - \sqrt{2(R_A - R_D)(b - L_A - l_A^*) + (R_A^2 - R_D^2)}
\end{aligned}$$

Using these equilibrium conditions we can analyze how policy promises of advantaged candidates change as a function of the variables of interest. The distance between policy promises of advantaged candidates can be thought as a degree of policy polarization because these two candidates win the primaries and their policy promises are proposed when they compete in the general election.

Theorem 5. *Suppose that F is uniformly distributed in* $[a, b]$. *Then, the following holds:*

$$\frac{\mathrm{d}(r_A^* - l_A^*)}{\mathrm{d}L_A} \leq 0 \quad \textit{and} \quad \frac{\mathrm{d}(r_A^* - l_A^*)}{\mathrm{d}L_D} \geq 0.$$

The intuition of this theorem is as follows. In an equilibrium of regime 1, when L_D increases, the advantaged candidate in party L becomes more constrained by her party's primary, thereby moving her policy promise toward her party's bliss point. The advantaged candidate in party R takes advantage of this opportunity by moving her policy promise toward the center, in order to increase her probability of winning the general election. Given that party R's advantaged candidate does not experience a change in the primary conditions, her movement toward the center is less aggressive than the shift of her left handed rival in the general election. On the other hand, when L_A increases, the advantaged candidate in party L faces less of a challenge in her primary, allowing her to move her policy promise toward the center and ultimately improve her chance of winning the general election. In the case of regime 2, the valence difference between advantaged and disadvantaged candidates is already high, meaning that marginal changes in the advantaged candidates' valence with holding everything else constant effect no change in their optimal promises.

These results differ from the results in Adams et al. (2011), where higher valences lead candidates to move their policy promises away from the center. In this regard, candidates experience dis-utility under the framework of Adams et al. (2011) when moving away from their own personal bliss points, even if they do so to win the general election. Then, having higher valence allows candidates to more honestly reflect their own bliss points in their policy promises without significantly losing electoral support, thereby minimizing this dis-utility. In addition, this "ideological constraint" is exogenous and fixed. However, in our setting, primaries act as a constraint for an advantaged candidate's promises; the valence of her primary rival determines the range of viable policy promises for the advantaged candidate (i.e., policy promises that guarantee a primary win). Hence, as the advantaged candidate's valence increases, the constraint on her policy promises imposed by the need to contest a primary relaxes.

The next theorem analyzes changes on the support. First, we assume that the support is expanded but the mean is preserved. Then, we assume that the support is expanded only on the right side.

Theorem 6. *Suppose that F is uniformly distributed in $[a, b]$. Suppose also that an equilibrium described in Theorem 1 exists.*

- *When $b - a$ increases while holding $\frac{a+b}{2}$, $\frac{\mathrm{d}(r_A^* - l_A^*)}{\mathrm{d}(b-a)} > 0$.*
- *When b increases, $\frac{\mathrm{d}(r_A^* - l_A^*)}{\mathrm{d}b} > 0$.*

In the first case, since the support is expanded while the mean is preserved, the median voter's bliss point in each party becomes more extreme. Hence, advantaged candidates move their policy promises accordingly to maintain viability in their primary elections. In the case when only b is increased, the mean moves to the right. Then, both candidates move their policy promises accordingly to the right, but the policy promise of the candidate in party R moves with a higher magnitude as her party's median voter's bliss point moves.

A Triangular Distribution Function

Consider a triangular distribution with a probability density function f on support $[a, b]$ defined as:

$$f(x) = \begin{cases} d - \frac{d}{a}x & \text{if } x \in [a, 0] \\ d - \frac{d}{b}x & \text{if } x \in [0, b] \end{cases}$$

where $a < 0$ and $b, d > 0$. Then, the cumulative distribution function F is:

$$F(x) = \begin{cases} -\frac{d}{2a}(x-a)^2 & \text{if } x \in [a, 0] \\ 1 - \frac{d}{2b}(b-x)^2 & \text{if } x \in [0, b] \end{cases}. \tag{14}$$

Thus,

$$\frac{F(x)}{\frac{1}{2}f(x)} = \begin{cases} x - a & \text{if } x < 0 \\ b - x & \text{if } x > 0. \end{cases}$$

Theorem 7. *Suppose that F is given by* (14). *Suppose also that an equilibrium described in Theorem 1 exists. Then,*

- *With respect to the change in L_A, we have* $\frac{\mathrm{d}(r_A^* - l_A^*)}{\mathrm{d}L_A} < 0$.
- *With respect to the change in L_D, we have* $\frac{\mathrm{d}(r_A^* - l_A^*)}{\mathrm{d}L_D} > 0$.

Theorem 7 implies that the same result in Theorem 6 still holds under the assumption of a triangular density function.

Now, we consider two different societies, N and S, with different distributions of the overall median voter's bliss point; public opinion is clustered closer to the center in society S, while it is more diverse in society N.

Theorem 8. *Suppose that the cumulative distribution functions F_N and F_S are given by* (14) *with support $[a_N, b_N]$ and $[a_S, b_S]$, respectively. Suppose also that in both societies, a regime 1 equilibrium exists. Then, when $a_N < a_S$, $r_A^N - l_A^N > r_A^S - l_A^S$.*

Theorem 8 shows that when public opinion becomes more diverse, the distance between the advantaged candidates' policy promises in party R and party L increases.

Now, suppose that in society S, the cumulative distribution function F_S is given by (14). Take $\bar{\epsilon}$ to be arbitrarily small. Then, consider the following scenario of *extremists' change*: for $x \in [-a, -a+\bar{\epsilon}]$, $f_N(x) = (1+d_a(x))f_S(x)$, while for $x \in [a-\bar{\epsilon}, a]$, $f_N(x) = (1-d_b(x))f_S(x)$ where $d_a(x)$ and $d_b(x)$ are strictly smaller than $\bar{\epsilon}$ for every x. It follows that under the scenario of extremists' change, for some small $\delta > 0$, we have $F_N(x) = F_S(x) + \delta$ for every $x \in [a + \bar{\epsilon}, b - \bar{\epsilon}]$. Then, when the difference $\bar{\epsilon}$ is small, we can compute how the policy promises change in the following form.

Proposition 2. *Suppose that in both societies, a regime 1 equilibrium arises. When $\bar{\epsilon}$ is sufficiently small, the following holds.*

Let $\pi = F\left(\frac{r_A^S + l_A^S + L_A - R_A}{2}\right)$, $\pi_L = F\left(\frac{r_A^S + l_D^S + L_D - R_A}{2}\right)$, *and* $\pi_R = F\left(\frac{l_A^S + r_D^S + L_A - R_D}{2}\right)$. *Let* $X = \frac{3(L_A - R_A) + 2(R_D - L_A) + 2b}{3(1-\pi) - 2(1-\pi_R)} - \frac{3(L_A - R_A) + 2(R_A - L_D) + 2a}{3\pi}$ *and* $Y = \frac{3\pi - 2\pi_L}{3\pi} - \frac{3(1-\pi)}{3(1-\pi)-2(1-\pi_R)}$.

Then, under the scenario of extremists' change*,*

$$(r_A^N - r_A^S) \cdot Y = \delta \cdot X;$$
$$(l_A^N - l_A^S) \cdot Y = \delta \cdot \left(\frac{3(L_A - R_A) + 2(R_D - L_A) + 2b}{3(1-\pi) - 2(1-\pi_R)} - \frac{2\pi_L}{3\pi} \cdot X \right).$$

Notice that π is the probability that candidate L_A wins against candidate R_A, π_L is the probability that candidate L_D wins against candidate R_A, and π_R is the probability that candidate L_A wins against candidate R_D.

Corollary 2. *Let* $b - a > (R_A - L_D) - (R_D - L_A)$ *and* $\frac{L_A - R_A}{2} > R_A - R_D - b$. *Suppose* $\pi > \pi_L$ *and* $\frac{1}{2} < \pi < \pi_R$. *Then,* $r_A^N < r_A^S$, *and* $l_A^N < l_A^S$.

Hence, under the scenario of extremists' change, the shift in public opinion implies a large increase in the number of voters with bliss points between party L's bliss point and candidate L_D's policy promise. The existence of these new voters forces candidate L_A to shift her policy promise toward the party's bliss point to protect her chances of winning the primary election. Following this, candidate R_A also moves her policy promise toward the left. Despite the difference in our settings, these results are in line with Callander (2005), who states that when candidates compete in several heterogeneous issues, an equilibrium requires that they each choose non-centrist policy promises and do not converge to the median voter's bliss point (in our case, the center).

A Truncated Normal Distribution Function

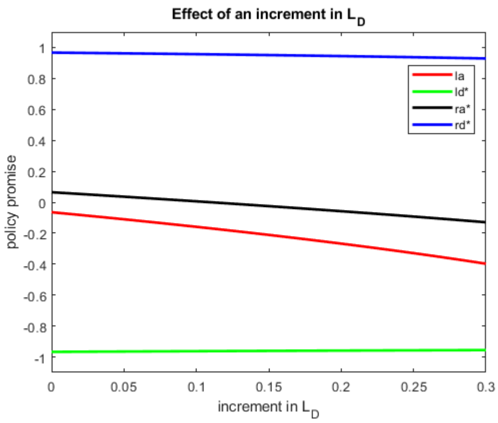
This section conducts comparative statics by performing numerical simulations under the assumption of a normally distributed median voter's bliss point. First, we define a truncated standard normal distribution with support $[-1, 1]$ and calibrate this such that the advantaged and disadvantaged candidates in each party have valences of 0.5 and 0, respectively. By Theorem 1, this initial setting constitutes a symmetric situation in regime 1. Then, we start increasing the left party's candidates' valence separately in 0.01 units—holding all other valences constant—until L_A reaches 1.0 and L_D reaches 0.3.

The results, starting from regime 1, are presented in figure 2. These differ from the results in the case of a uniform distribution in the following sense. When party L's advantaged candidate's valence, L_A, initially increases, her policy promise shifts toward party R's bliss point. However, at some point, further increases in L_A induce her policy promises to shift back towards the political center.

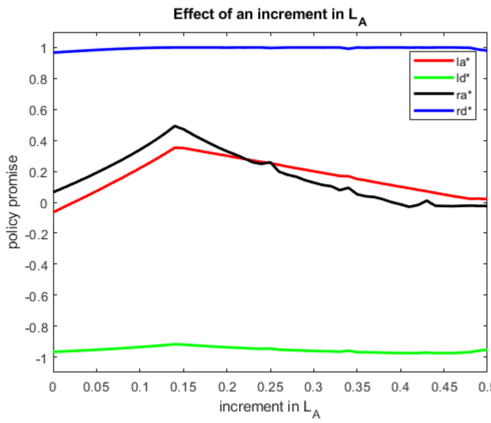
This can be potentially explained by recognising that in the normal distribution—unlike a uniform distribution—popular opinion (in the form of the probability density of the median voter's bliss point) is concentrated around the political center (the mean of the distribution of the median voter's bliss point). This implies that when the advantaged candidate's policy promise is sufficiently to the right, any incremental rightward shift in her policy promise yields a negligible increase in her probability of winning the general election. Furthermore, when party L's advantaged candidate has sufficiently higher valence than party R's advantaged candidate, she incurs no electoral penalty by choosing a policy promise close to the political center. This is because her valence advantage makes her the preferred general election candidate, even for voters with bliss points closer to her general election rival's. In other words, when popular opinion is sufficiently centrally concentrated and party L's advantaged candidate has sufficiently greater valence than party R's advantaged candidate, the left advantaged candidate obtains insignificant benefits from shifting her policy promise to the right. Moreover, her valence advantage allows her to capture a greater share of votes when moving closer to the political center.

It is also worth noting that once L_A has increased, then as we start increasing L_D, the advantaged candidate in the left party is more reluctant to move her policy promise away from the center and her position remains relatively flat. This can be somewhat explained by the fact that—under this distribution—the advantaged candidate is less constrained by the primary and, hence, rather than moving her policy promise toward the extreme, she is able to choose one closer to the center where the density is concentrated.

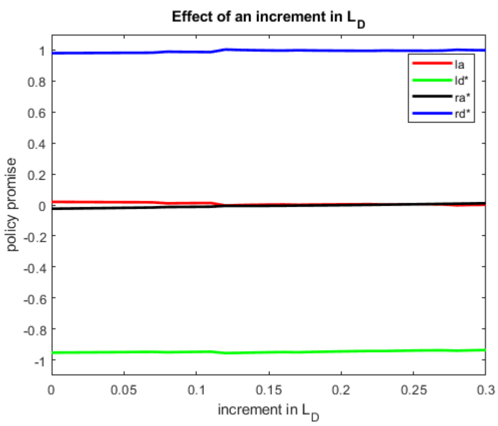
In Theorem 5—which studies the equilibrium in the setting of a uniform distribution—we show that the disadvantaged candidates set their promises at extreme support values. In the current case, we have greater interaction between candidates for policy choices but the effects from such interactions are quite small.



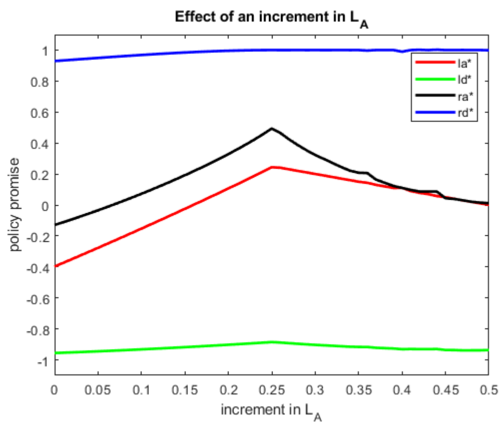
(a) $L_A = R_A = 0.5$, $L_D = R_D = 0$



(b) $L_A = R_A = 0.5$, $L_D = R_D = 0$



(c) $L_A = 1$, $R_A = 0.5$, $L_D = R_D = 0$

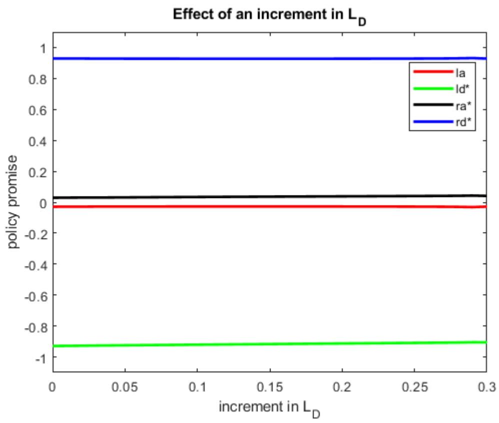


(d) $L_A = R_A = 0.5$, $L_D = 0.3$, $R_D = 0$

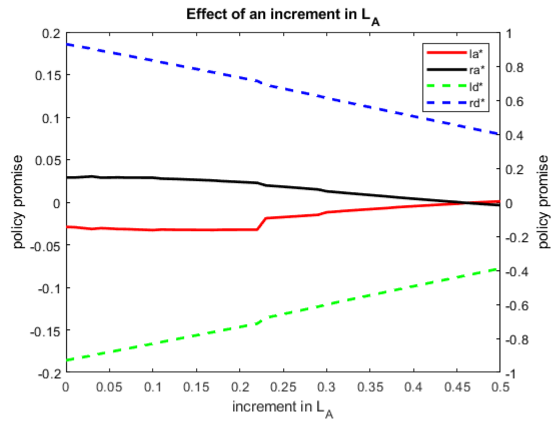
Figure 2: Increasing the valence starting from regime 1. Starting parameters indicated above.

The next simulations (figure 3) follow the same procedure, but instead we start from a symmetric situation where $L_A - L_D > b$. In line with Theorem 1, advantaged candidates set policies around the mean and they remain relatively flat given any change. When L_D increases, policies remain unaffected. On the other hand, when L_A increases, party R's advantaged candidate moves her policy promise closer to the center and her rival's policy promise in the general election follows the same path. While the disadvantaged candidates also follow this path, it is at a higher magnitude given their lower valence.

In contrast to the situation in regime 1, it is clear that there is greater interaction between the disadvantaged candidates relative to the advantaged candidates. The primary reason for this is that, when advantaged candidates have considerably higher valence than the disadvantaged, even a marginal displacement in the former's policy promises pressures the disadvantaged candidates to respond.



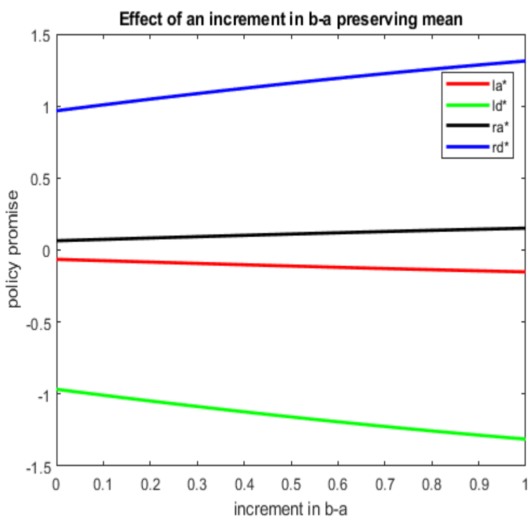
(a) $L_A = R_A = 1.1$, $L_D = R_D = 0$.



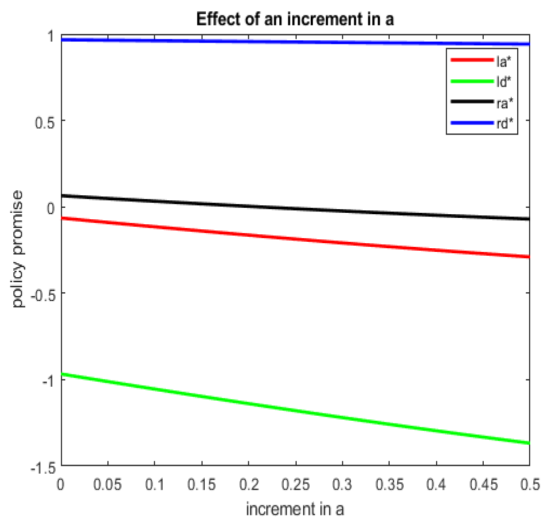
(b) $L_A = R_A = 1.1$, $L_D = R_D = 0$.

Figure 3: Increasing valence starting from large differences between advantaged and disadvantaged candidates. Starting parameters indicated above. Dash lines correspond to the secondary axis.

In relation to the support parameters, we analyze the effect of increasing $b - a$ while preserving the center. We also analyze the effect of increasing a. We present these results starting only under regime 1 given that—by Theorem 1—advantaged candidates set policy promises around the center when $(L_A - L_D)$ is sufficiently large. The results are the same as those in Theorem 6 and 7, which study the cases of a uniform distribution and a triangular distribution, respectively.



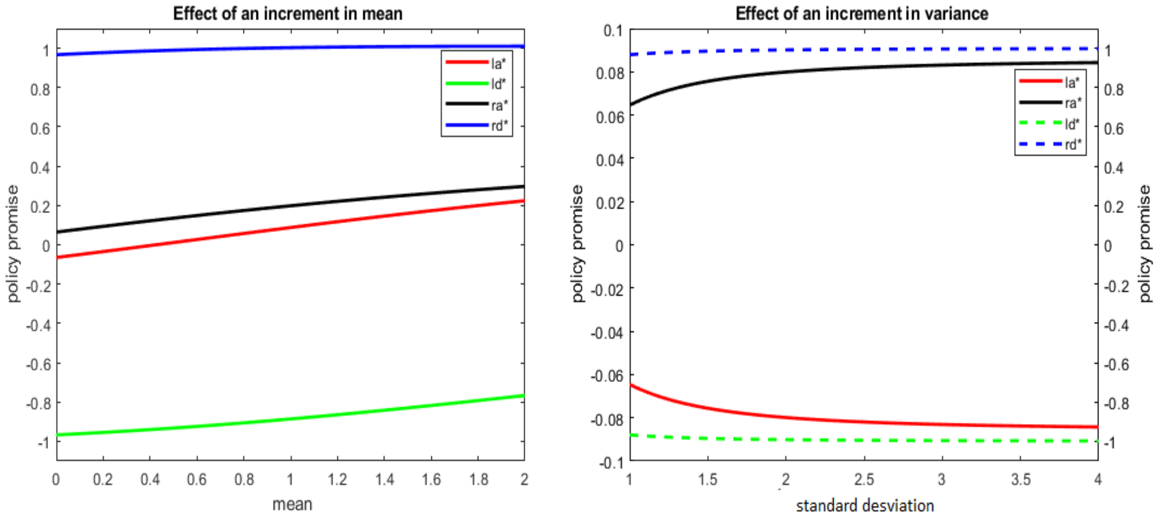
(a) $b - a$ from 1 to 2.



(b) a from -1 to -1.5.

Figure 4: Changing the support of normal distribution truncated in $(-1, 1)$. Starting parameters $L_A = R_A = 0.5$, $L_D = R_D = 0$ (regime 1).

In addition, we present the results when analysing the effect of an increase in the mean (the political center) and variance under regime 1. In figure 5, we see that as the mean increases and the median voter's bliss point changes, candidates move their policy promises accordingly. On the other hand, as the variance increases, the higher dispersion in public opinions impels candidates to move further away from the center, which is consistent with Theorem 7 (the case of a triangular distribution function) and Callander (2005).



(a) μ from 0 to 2.

(b) σ from 1 to 4.

Figure 5: Changing mean and variance. Starting parameters $L_A = R_A = 0.5$, $L_D = R_D = 0$ (regime 1). Dash lines correspond to the secondary axis.

5 The One-Primary Scenario

The content of this section is based on an earlier version of this paper, Takayama (2014). The following two propositions characterize the equilibrium in the case where only one party holds a primary. In this section, we assume that an incumbent in office is committed to a predetermined policy. Therefore, a policy chase between the two advantaged candidates does not occur unlike in the general scenario.

In Proposition 3, we consider the scenario where only party R holds a primary and candidate L_A is the incumbent in the office, committed to a policy promise $\bar{l}_A$. In Proposition 4, we consider the scenario where candidate R_A is the incumbent in the office, committed to a policy promise $\bar{r}_A$. In these propositions, we assume $\bar{r}_A > M_L$ and $\bar{l}_A < M_R$. It is highly unlikely that any incumbent would be committed to a policy promise that reflects the opponent's party ideology. Because $L_A \geq R_A$ by assumption, cases in Propositions 3 and 4 are not exactly symmetric.

Proposition 3. *Let $\bar{l}_A < M_R$. The following holds in an equilibrium of the scenario where only party R holds a primary.*

(a) Suppose that $L_A - R_A \geq M_R - \bar{l}_A$. Then any (r_D^*, r_A^*) and any (w_D^{*R}, w_A^{*R}) such that $w_D^{*R} + w_A^{*R} = 1$ is an equilibrium.

(b) Suppose that $L_A - R_A < M_R - \bar{l}_A$ and $M_R - \bar{l}_A > L_A - R_D$. If $\bar{l}_A + L_A - R_A \notin \beta_R(\bar{l}_A, L_A)$, then $\hat{r}_D(\bar{l}_A)$ is singleton and in the unique equilibrium, $r_D^* \in \hat{r}_D(\bar{l}_A)$, $r_A^* = \hat{r}_A(\bar{l}_A)$ and $(w_D^{*R}, w_A^{*R}) = (0, 1)$.

Proposition 4. *Let $\bar{r}_A > M_L$. An equilibrium always exists in the scenario where only party L holds a primary. Further, the following holds in this scenario.*

(a) Suppose that $\min\{L_A - R_A, R_A - L_D\} \geq \bar{r}_A - M_L$. Then any (l_D^*, l_A^*) and $(w_D^{*L}, w_A^{*L}) = (0, 1)$ is an equilibrium.

(b) Suppose that $R_A - L_D < \bar{r}_A - M_L$. Then $\hat{l}_D(\bar{r}_A)$ is singleton, and $l_D^* \in \hat{l}_D(\bar{r})$ and $(w_D^{*L}, w_A^{*L}) = (0, 1)$ constitute an equilibrium. Further, in equilibrium,

(1) If $\alpha(\bar{r}_A, R_A) \cap \beta_L(\bar{r}_A, R_A) \neq \emptyset$, then $l_A^* \in \alpha(\bar{r}_A, R_A) \cap \beta_L(\bar{r}_A, R_A)$,

(2) If $\alpha(\bar{r}_A, R_A) \cap \beta_L(\bar{r}_A, R_A) = \emptyset$, then $l_A^* = \hat{l}_A(\bar{r}_A)$.

(c) Suppose that $R_A - L_D \geq \bar{r}_A - M_L > L_A - R_A$. Then in equilibrium, $l_A^* \in [\bar{r}_A - L_A + R_A, \bar{r}_A]$, $l_D^* \in X$, and $(w_D^{*L}, w_A^{*L}) = (0, 1)$.

In both propositions, case (a) is the situation where candidate L_A is strong in terms of valence relative to both candidates R_A and the disadvantaged candidate. Case (b) is the situation where candidate L_A's valence advantage over candidate R_D is not sufficiently high to guarantee certain victory in the general election against candidate R_D. Case (c) of Proposition 4 is the situation where the disadvantaged candidate cannot win the general election with a strictly positive probability, while candidates L_A and R_A do not have significant difference in valence levels.

In Propositions 3 and 4, case (a) belongs to regime 1, and case (b) in Proposition 3 and case (b-2) in Proposition 4 belong to regime 2. Cases (b-1) and (c) in Proposition 4 are special cases of regime 1. In both cases, $\alpha_L(R_A, \bar{r}_A) \cap \beta_L(\bar{r}_A, R_A)$ is nonempty; however, a policy chase does not occur when candidate L_A chooses inside $\alpha_L(R_A, \bar{r}_A)$ because $\bar{r}_A$ is fixed.

6 Extensions Including Endogenous Entry

In this section, we discuss the robustness of our results under various extensions and evaluate a modified version of our model including the endogenous entry of a candidate.

Hummel (2013) explains that in the symmetric game, a regime 1 equilibrium is robust under the extensions such that there are three candidates in the primaries, candidates can take different policy promises in two stages, parties hold open primaries, and politicians also care about the policy outcomes. We do not repeat the same argument about a regime 1 equilibrium because the arguments in Hummel (2013) also apply to our results in the general setting. However, a couple of points require noting.

First, a regime 2 equilibrium is also robust under these extensions, because in this regime, candidate L_A is so strong in terms of valence and her status does not change even under these extensions. Hummel (2013) assumes that if a candidate loses the primary election, then the candidate obtains a utility of 0, if a candidate wins party k's primary, then she obtains a utility of $u_k \geq 0$ for each $k = L, R$, and the winner of the general election receives an additional utility of $u \geq 0$. Here we do not impose any particular utility and simply assume that each candidate attempts to maximize the probability of winning office. If we make a similar assumption to that of Hummel (2013), a regime 2 equilibrium still arises.[9]

Second, in a regime 1 equilibrium, when candidates also care about the policy outcomes, because the disadvantaged candidates cannot win the primaries with a strictly positive probability, they still maximize the expected payoffs of their own parties' median voter. In the Takayama et al. (2019) model, there are two candidates, who we can consider as our advantaged candidates. In a similar way as for our current proof for Theorem 1, we can show that when the β sets do not overlap, there is an equilibrium. However, depending on the positions of the advantaged candidates' bliss points, advantaged candidates may take policy promises inside of the β sets instead of the boundaries of the sets.

Finally, we consider a modified version of our model such that a candidate has a choice of entry to the elections. We can model this situation by using the concept of *subgame-perfect equilibria*.[10]

Here, we consider a scenario where candidate R_A is committed to a policy promise $\bar{r}_A$ and there is no primary in party R. We evaluate the endogenous decision of entry by candidate L_D. Suppose that candidate L_D has a bliss point $\bar{l}_D$ and the following utility: for $P_j = L_A, R_A$,

$$u_{\bar{l}_D}(x, P_j) = P_j - \kappa|x - \bar{l}_b|. \tag{15}$$

Compared with a voter's preference defined in (1), (15) has an additional element κ for the disutility caused by the policy difference with $\bar{l}_D$. One can imagine that as a politician, candidate L_D values valence more, or policy promises more. This is captured by κ. To consider a general case, we do not impose any assumption on κ, but obviously when $\kappa = 1$, candidate L_D's utility coincides with a voter's utility.

Proposition 5 points out several interesting features of the revised game with endogenous entry. First, if κ is large and the difference in valences between the two advantaged candidates is less of a concern, candidate L_D enters and influences the policy outcome. However, if κ is small and candidate L_D is less concerned about the policy outcome, she values more the difference in valences between the two advantaged candidates and does not enter. In other words,

[9] For specific statements, see Theorems B-1, B-2, and B-3 in Appendix B.

[10] By using Propositions 3 and 4 together with Theorems B-1, B-2, and B-3, and Corollary 1, we can consider various scenarios where an advantaged or disadvantaged candidate decides to enter the electoral race or not or the opponent party does not or does hold a primary.

if candidate L_D places less importance on policy outcomes, and the advantaged candidate in the opponent party has a sufficiently low valence level relative to the advantaged candidate in her own party, she does not enter, because her entry will constrain the policy promise of the advantaged candidate in her own party, potentially damaging the probability of her party winning the general election. In other words, Proposition 5 implies that the entry decision of candidate L_D affects not only the probability of winning the general election for candidate L_A but also her policy promise.

Proposition 5. *Suppose $L_A - R_A > 0$ holds, and candidate R_A is committed to policy $\bar{r}_A$, and candidate L_D's bliss point $\bar{l}_D$ satisfies $\bar{l}_D \le M_L$. Suppose that if candidate L_D does not enter, candidate L_A chooses $l_A^0 = \bar{r}_A - d$ for some d, and that if candidate L_D enters, candidate L_A chooses $l_A^1 = \hat{l}_A(\bar{r}_A)$ as in a regime 1 equilibrium. Then the followings hold:*

- *It is held that $d \le L_A - R_A$.*
- *If $F(\frac{\bar{r}_A + l_A^1 + L_A - R_A}{2})(\bar{r}_A - l_A^1) \le d$, then candidate L_D does not enter.*
- *If $F(\frac{\bar{r}_A + l_A^1 + L_A - R_A}{2})(\bar{r}_A - l_A^1) > d$, let $\bar{\kappa} = \frac{(1-F(\frac{\bar{r}_A + l_A^1 + L_A - R_A}{2}))(L_A - R_A)}{F(\frac{\bar{r}_A + l_A^1 + L_A - R_A}{2})(\bar{r}_A - l_A^1) - d}$. If $\kappa < \bar{\kappa}$, candidate L_D does not enter. Otherwise, if $\kappa > \bar{\kappa}$, candidate L_D enters.*

Finally, $\pi_R(\bar{r}_A, R_A, l_A^1, L_A) > \pi_R(\bar{r}_A, R_A, l_A^0, L_A) = 0$ and $\bar{r}_A - l_A^1 > \bar{r}_A - l_A^0$.

Second, as $l_A^0 = \bar{r}_A - d$ is candidate L_A's choice, candidate L_A can influence candidate L_D's decision of entry. By choosing d such that $F(\frac{\bar{r}_A + l_A^1 + L_A - R_A}{2})(\bar{r}_A - l_A^1) - d$ is greater than 0 but sufficiently small, candidate L_A can make the situation of $\kappa < \bar{\kappa}$ happen. In a sense, by choosing a d that is sufficiently close to $F(\frac{\bar{r}_A + l_A^1 + L_A - R_A}{2})(\bar{r}_A - l_A^1)$ and compensating for the disutility coming from the difference of policy outcomes for not entering to candidate L_D, candidate L_A can induce the situation where candidate L_D does not enter.

We note that Hummel (2013) also considers the endogenous entry decision into primaries in the symmetric game (namely, $L_A = R_A$). By introducing the case of $L_A > R_A$ (which allows us to study the case of $\bar{\kappa} > 0$), Proposition 5 adds another interesting insight into the decisions of candidates about whether or not to enter the race.

7 Discussions

In this paper, we proposed a simple model of a two-stage electoral process with primaries followed by a general election. By allowing voters to value both policy promises and valence, we showed that there are possibly two equilibrium regimes, and provided the conditions under which each regime arises. Our model is simple and tractable. At the same time, it is sufficiently general to cover special scenarios such as a symmetric game or the case of a one-party primary, and may also lend itself to further extensions such as uncertainty over candidates' valence. Our model is also rich as we provide the sufficient conditions for each equilibrium regime, and can obtain the equilibrium in Hummel (2013) as a special case of our equilibrium analysis.

Primaries have become popular across the world. Our results from comparative statics can be tested in a real example. In the United States there are two main parties, the Republicans and the Democrats. Over a period of 6 months, primaries are held in different states to choose each party's presidential nominee. As the president is usually uncontested as his party's nominee, it is common that competitive presidential primaries are held only in one party in election cycles. Similarly for congressional and gubernatorial primaries, competitive primaries are held only in one party unless the incumbent is not running. Additionally, in each step of the election process, there are typically only two candidates with reasonable chance of winning. Therefore, our model setting is applicable to this context. Moreover, Hummel (2013)'s framework doesn't consider asymmetric electoral settings in which only one party holds a primary. This fact is particularly relevant as equilibrium outcomes are highly dependent on the circumstances that candidates face during the primaries. Hence, if no primaries are held in one party, this will impact on the candidates' policy promises and the final outcome.

There are also other countries where it is not uncommon for only one party to hold primaries. One sided primaries have been typically realised in Italy, Germany and Japan. In the case of Latin America, even though primaries are typically held by all parties, the existence of a multi-party primary system would complicate the analysis. However, our model would be a good starting point for analyzing a multi-party primary system and at the same time, there are also cases in Latin American countries where one and two-side competitive primaries have been held for local and national elections. In Chile, for example, since primaries were implemented in 2017, for presidential election two-side primaries have been held in each occasion, whereas in local elections also we find one-side primaries.

Because our framework is simple but general, there are many ways to extend the simple framework studied here. In the real world, primaries sequentially aggregate a number of different individual choices as well as party leaders and decide over time which candidate to endorse. Next, we could attempt to relax the assumption about the same policy across the two stages of the elections by allowing politicians to revise a policy promise within a certain constraint in the general election. As there is valence in the model, and valence also captures the honesty of candidates, it may be interesting to see how the equilibrium policy promises vary, if candidates can choose a different policy in the general election within a certain distance, which is a function of valence, from their policy promise in the primary.

There is a view that having primaries results in more extreme polarization because candidates need to appeal to partisan primary voters, whose preferences are typically more extreme than those of voters in the general election, and so candidates tend to take more extreme positions. Polarization of policy promises is a significant contemporary concern (Benoit and Laver, 2006, McCarty et al., 2006, 2013, Woon, 2018, Hill and Tausanovitch, 2018). However, the empirical results are mixed. Overall, while primaries may cause polarization in limited circumstances (Bullock and Clinton, 2011), they could also serve to reduce polarization because the general elections will continue to exert some moderating pressure on candidates (Hirano et al., 2010, Hall, 2015). Our model, particularly analyses on a regime 1 equilibrium, can be used to answer this question.

There is yet another stream of the literature that considers how the decision to hold primaries affects policy outcomes. Serra (2011) proposes a model of party elites in which those who have a different bliss point from the party's median voter decide whether or not to hold a primary election. Snyder and Ting (2011) consider a model with primaries in which the candidate can differ in quality, but they do not include endogenous candidate policy promises in their model. Woon (2018) uses behavioral game theory and experiments to examine how primaries affect policy promises. Our model can also be used to examine how having a primary affects policy outcomes.

In this paper, we assumed that valence and the policy promise are independent and that voters' utility is a function of these two variables. Also, in our paper, we assumed that the valence values are exogenously given, and are public information. It may also be interesting to relax these assumptions and analyze how the equilibrium changes.

Adams and Merrill (2009) have developed a model of policy-seeking parties in a parliamentary democracy, and in their model, party elites are uncertain about voters' evaluations of the parties' valence. Finally, Adams et al. (2013) extend their model to situations where voters hold coalitions of parties collectively responsible for their valence-related performances. It would then be interesting to specify uncertainty about valence and consider aspects of the collective responsibility for this, as in Adams et al. (2013).

Examining how different institutional arrangements affect the equilibrium outcomes in our model would also be interesting. One example is crossover voting. Chen and Yang (2002) and Cho and Kang (2014) consider an open primary election with two competing candidates and the possibility of crossover voting. In our current model, crossover voting is eliminated by assumption and our setting is then more akin to closed primaries.

Another interesting question is whether the open primary system leads to more moderate outcomes than our current model. Galderisi and Ezra (2001) state that in general, the direct primary where voters directly vote for a primary candidate was thought to be a method of controlling the corrupting influences of the urban political machine. Ware (2002) reviews the political history of the US, and asserts that the direct primary was the result of an attempt by politicians to subject previously informal procedures to formal rules, which started in the late 1880s. Gerber and Morton (1998) empirically examine how different primary election laws affect the types of candidates elected in nonpresidential American elections. It would be useful to evaluate whether these theoretical predictions match the empirical observations in Gerber and Morton (1998). In this sense, Casas (2018) proposes a model and shows that open primaries result in candidates that are expected to be ideologically more extreme than those in closed primaries. Our model is stable enough to lend itself to these various extensions.

Appendix A: Proofs

Proofs of Theorem 1

Derivation of Equations (3) *and* (4). Let $\sigma(l,L,r,R)$ satisfy

$$v_{\sigma(l,L,r,R)}(l,L)=v_{\sigma(l,L,r,R)}(r,R).$$

Then, $\sigma(l,L,r,R)$ does not exist, if $|r-l|<|R-L|$. If $|r-l|>|R-L|$, let

$$\sigma(l,L,r,R)=\begin{cases}\frac{(L-R)+(l+r)}{2} & \text{if } l<r\\ \frac{(R-L)+(l+r)}{2} & \text{if } r<l,\end{cases} \tag{A-1}$$

and the probability that candidate R wins the general election is

$$\pi_R(r,R,l,L)=\begin{cases}\Pr(M>\sigma(l,L,r,R))=1-F(\sigma(l,L,r,R)) & \text{if } l<r\\ \Pr(M<\sigma(l,L,r,R))=F(\sigma(l,L,r,R)) & \text{if } r<l.\end{cases}$$

$\square$

Lemma A-1.

1. *Suppose that a winner in party L's primary has a policy promise $\bar{l}$ and a valence $\bar{L}$. Then, for each $i=A,D$, $V_{M_R}(\bar{l},\bar{L},x,R_i)$ is a continuous function of x on $[\bar{l},M_R]$, and when $\pi_R(x,R_i,\bar{l},\bar{L})>0$,*

   - *$V_{M_R}(\bar{l},\bar{L},x,R_i)$ is*
     - *single-peaked if $\frac{\partial}{\partial x}\left(V_{M_R}(\bar{l},\bar{L},x,R_j)\right)\big|_{x=M_R}\leq 0$;*
     - *increasing otherwise.*
   - *When $V_{M_R}(\bar{l},\bar{L},x,R_i)$ is singled-peaked, the maximal value is*

     $$\bar{l}+\bar{L}-M_R+\frac{\left(1-F(\frac{x^*+\bar{l}+\bar{L}-R_i}{2})\right)^2}{\frac{1}{2}f(\frac{x^*+\bar{l}+\bar{L}-R_i}{2})},$$

     where x^* satisfies the first-order condition

     $$R_i+x^*-\bar{L}-\bar{l}=\frac{(1-F(\frac{x^*+\bar{l}+\bar{L}-R_i}{2}))}{\frac{1}{2}f(\frac{x^*+\bar{l}+\bar{L}-R_i}{2})}.$$

   - *The maximizer of $V_{M_R}(\bar{l},\bar{L},x,R_i)$ exists uniquely on $[\bar{l},M_R]$.*

2. *Suppose that a winner in party R's primary has a policy promise $\bar{r}$ and a valence $\bar{R}$. For every $i=A,D$, $V_{M_L}(x,L_i,\bar{r},\bar{R})$ is a continuous function of x on $[M_L,\bar{r}]$, and when $\pi_R(\bar{r},\bar{R},x,L_i)<1$,*

- *$V_{M_L}(x, L_i, \bar{r}, \bar{R})$ is*
  - *single-peaked if $\frac{\partial}{\partial x}\left(V_{M_L}(x, L_i, \bar{r}, \bar{R})\right)\big|_{x=M_L} \geq 0$;*
  - *decreasing otherwise.*
- *When $V_{M_L}(x, L_i, \bar{r}, \bar{R})$ is singled-peaked, the maximal value is*

$$\bar{R} - \bar{r} + \frac{\left(F(\frac{\bar{r}+x^*+L_i-\bar{R}}{2})\right)^2}{\frac{1}{2}f(\frac{\bar{r}+x^*+L_i-\bar{R}}{2})} + M_L,$$

where x^* satisfies the first-order condition:

$$L_i + \bar{r} - \bar{R} - x^* = \frac{F(\frac{\bar{r}+x^*+L_i-\bar{R}}{2})}{\frac{1}{2}f(\frac{\bar{r}+x^*+L_i-\bar{R}}{2})}.$$

- *The maximizer of $V_{M_L}(x, L_i, \bar{r}, \bar{R})$ always exists uniquely on $[M_L, \bar{r}]$.*

Proof of Lemma A-1. By symmetry, we only prove the first case. Taking the first derivatives, then

$$\frac{\partial}{\partial x}\left(V_{M_R}(\bar{l}, \bar{L}, x, R_i)\right) = \pi'_R(x, R_i, \bar{l}, \bar{L})(R_i + x - (\bar{l} + \bar{L})) + \pi_R(x, R_i, \bar{l}, \bar{L}),$$

where $\pi'_R(x, R_i, \bar{l}, \bar{L})$ denotes the first derivative of $\pi_R(x, R_i, \bar{l}, \bar{L})$ with respect to x.

When $x = \bar{l}$, $R_i + x - (\bar{l} + \bar{L}) \leq 0$. Further, $\pi'_R(x, R_i, \bar{l}, \bar{L}) < 0$ when $x > \bar{l}$. Thus $\frac{\partial}{\partial x}\left(V_{M_R}(\bar{l}, \bar{L}, x, R_i)\right) > 0$. If

$$\frac{\partial}{\partial x}\left(V_{M_R}(\bar{l}, \bar{L}, x, R_i)\right)\Big|_{x=M_R} \leq 0,$$

then there is an $\bar{x}$ to satisfy

$$\pi'_R(\bar{x}, R_i, \bar{l}, \bar{L})(R_i + \bar{x} - (\bar{l} + \bar{L})) + \pi_R(\bar{x}, R_i, \bar{l}, \bar{L}) = 0.$$

Then, note that when $\bar{l} > x$ and $\pi_R(x, R_i, \bar{l}, \bar{L}) > 0$,

$$-\frac{\pi'_R(\bar{x}, R_i, \bar{l}, \bar{L})}{\pi_R(\bar{x}, R_i, \bar{l}, \bar{L})} = \frac{\frac{1}{2}f(\frac{\bar{x}+\bar{l}+\bar{L}-R_i}{2})}{1 - F(\frac{\bar{x}+\bar{l}+\bar{L}-R_i}{2})}. \tag{A-2}$$

Also,

$$-\frac{\pi'_R(\bar{x}, R_i, \bar{l}, \bar{L})}{\pi_R(\bar{x}, R_i, \bar{l}, \bar{L})} = \frac{1}{R_i + \bar{x} - (\bar{l} + \bar{L})}. \tag{A-3}$$

By Assumption 1, for every $x > \bar{x}$, we have

$$\frac{\frac{1}{2}f(\frac{\bar{x}+\bar{l}+\bar{L}-R_i}{2})}{1 - F(\frac{\bar{x}+\bar{l}+\bar{L}-R_i}{2})} > \frac{1}{R_i + x - (\bar{l} + \bar{L})}$$

and for every $x < \bar{x}$, we have

$$\frac{\frac{1}{2}f(\frac{\bar{x}+\bar{l}+\bar{L}-R_i}{2})}{1-F(\frac{\bar{x}+\bar{l}+\bar{L}-R_i}{2})} < \frac{1}{R_i + x - (\bar{l}+\bar{L})}.$$

When there is no such $\bar{x}$, the expected payoff is increasing. Thus we obtain the first and second statements.

By (A-3), we obtain the first-order condition, and the maximal value is obtained by

$$\begin{aligned}
&V_{M_R}(\bar{l}, \bar{L}, x^*, R_i) + M_R \\
&= F(\frac{x^* + \bar{l} + \bar{L} - R_i}{2})\left(\bar{L} + \bar{l}\right) + (1 - F(\frac{x^* + \bar{l} + \bar{L} - R_i}{2}))\left(R_i + x^*\right) \\
&= F(\frac{x^* + \bar{l} + \bar{L} - R_i}{2})\left(\bar{L} + \bar{l}\right) \\
&\qquad\qquad + (1 - F(\frac{x^* + \bar{l} + \bar{L} - R_i}{2}))\left(\bar{l} + \bar{L} + \frac{(1-F(\frac{x^*+\bar{l}+\bar{L}-R_i}{2}))}{\frac{1}{2}f(\frac{x^*+\bar{l}+\bar{L}-R_i}{2})}\right).
\end{aligned}$$

To prove the last item, notice that when $\pi_R(x, R_i, \bar{l}, \bar{L}) > 0$,

$$x - \bar{l} > \max\{R_i - \bar{L}, \bar{L} - R_i\}.$$

Then, we obtain $v_{M_R}(x, R_i) > v_{M_R}(\bar{l}, \bar{L})$. Hence, if there is some $r \geq \bar{l}$ with $\pi_R(r, R_i, \bar{l}, \bar{L}) = 0$, we have

$$V_{M_R}(\bar{l}, \bar{L}, r, R_i) = \bar{L} + \bar{l} - M_R < \max V_{M_R}(\bar{l}, \bar{L}, x, R_i).$$

Thus, from the first item of this lemma, the maximizer is either x^* that satisfies the first-order condition, or M_R. This completes the proof. □

Lemma A-2.

1. *Suppose that a winner in party L's primary has a policy promise $\bar{l}$ and a valence $\bar{L}$. Then, for each $i = A, D$, $V_{M_R}(\bar{l}, \bar{L}, x, R_i)$ is a continuous function of x on $[\bar{l}, M_R]$, and when $\pi_R(x, R_i, \bar{l}, \bar{L}) > 0$, $V_{M_R}(\bar{l}, \bar{L}, x, R_i)$ is increasing for $x \leq \bar{l}$.*

2. *Suppose that a winner in party R's primary has a policy promise $\bar{r}$ and a valence $\bar{R}$. For every $i = A, D$, $V_{M_L}(x, L_i, \bar{r}, \bar{R})$ is a continuous function of x on $[M_L, \bar{r}]$, and when $\pi_R(\bar{r}, \bar{R}, x, L_i) < 1$, $V_{M_L}(x, L_i, \bar{r}, \bar{R})$ is decreasing for $x \geq \bar{r}$.*

Proof of Lemma A-2. By symmetry, we only prove the second statement. Suppose that for some $x \geq \bar{r}$, $\pi_R(\bar{r}, \bar{R}, x, L_i) < 1$, and note that the first derivative of $V_{M_L}(x, L_i, \bar{r}, \bar{R})$ with respect to x when $x > \bar{r}$ is

$$f(\frac{\bar{R} - L_i + x + \bar{r}}{2})((L_i - x) - (\bar{R} - \bar{r})) - F(\frac{\bar{R} - L_i + x + \bar{r}}{2}).$$

When $\pi_R(\bar{r}, \bar{R}, x, L_i) < 1$ and $\bar{r} < x$, $\bar{r} < x - (L_i - \bar{R})$. Thus, the first derivative of $V_{M_L}(x, L_i, \bar{r}, \bar{R})$ is negative. This completes the proof. □

Lemma A-3.

1. *Suppose that a winner in party L's primary has a policy promise $\bar{l}$ and a valence $\bar{L}$. Then, for each $i = A, D$, the maximizer of $V_{M_R}(\bar{l}, \bar{L}, x, R_i)$ for $x > \bar{l}$ maximizes $V_{M_R}(\bar{l}, \bar{L}, x, R_i)$ for $x \in X$.*

2. *Suppose that a winner in party R's primary has a policy promise $\bar{r}$ and a valence $\bar{R}$. For every $i = A, D$, the maximizer of $V_{M_L}(x, L_i, \bar{r}, \bar{R})$ maximizes $V_{M_L}(x, L_i, \bar{r}, \bar{R})$ for $x \in X$.*

Proof of Lemma A-3. By Lemmas A-1 and A-2, the maximizer of $V_{M_L}(x, L_i, \bar{r}, \bar{R})$ for $x < \bar{r}$ maximizes $V_{M_L}(x, L_i, \bar{r}, \bar{R})$ for $x \in X$ with $\pi_R(\bar{r}, \bar{R}, x, L_i) < 1$. When $\pi_R(\bar{r}, \bar{R}, x, L_i) = 1$, $L_i < \bar{R}$ and $x \in [\bar{r} - \bar{R} + L_i, \bar{r} + \bar{R} - L_i]$. Thus, $L_i - x \leq \bar{R} - \bar{r}$. Thus,

$$V_{M_L}(x, L_i, \bar{r}, \bar{R}) = \bar{R} - \bar{r} + M_L.$$

If there is $x < \bar{r}$ with $\pi_R(\bar{r}, \bar{R}, x, L_i) < 1$, the maximizer of $V_{M_L}(x, L_i, \bar{r}, \bar{R})$ for $x < \bar{r}$ maximizes $V_{M_L}(x, L_i, \bar{r}, \bar{R})$ for $x \in X$. If there is no such x, for every $x < \bar{r}$,

$$V_{M_L}(x, L_i, \bar{r}, \bar{R}) = \bar{R} - \bar{r} + M_L.$$

Thus, even in this case, every $x < \bar{r}$ is a maximizer of $V_{M_L}(x, L_i, \bar{r}, \bar{R})$. This completes the proof. $\square$

To prove Theorems 1 and 1, we begin by stating the propositions that help characterize an equilibrium. Our strategy is that we first provide the properties that a set of policy promises must satisfy as an equilibrium in regime 1. Then we give the conditions under which such a set of equilibrium policy promises exists.

For every $l_A \in X$, we define the set of candidate R_A's strategies that the median voter M_R strictly prefers to any policy promise that can be made by candidate R_D, by $\beta_R^\circ(l_A, L_A) \subseteq \beta_R(l_A, L_A)$ as follows:

$$\beta_R^\circ(l_A, L_A) \equiv \{r : V_{M_R}(l_A, L_A, r, R_A) > \max_{r_D \in X} V_{M_R}(l_A, L_A, r_D, R_D)\}.$$

Suppose that $(l_A^*, l_D^*, r_A^*, r_D^*)$ and (w^{*L}, w^{*R}) constitute an equilibrium.

Proposition A-1. *Suppose that the conditions for Theorem 1 hold. If $r_A^* > M_L$, then $w_A^{*L} = 1$ in equilibrium.*

Proof of Proposition A-1. Suppose $L_A - R_A < M_R - M_L$ and suppose $r_A^* > M_L$. We will show that there is always an l_A such that

$$\mathrm{E}V_{M_L}(l_A, L_A, r_A^*, r_D^*) > \mathrm{E}V_{M_L}(l_D^*, L_D, r_A^*, r_D^*) \quad \text{(A-4)}$$

$$1 - \pi_R(r_A^*, R_A, l_A, L_A) > 0. \quad \text{(A-5)}$$

Then by the definition of equilibrium, we obtain $w_A^{*L} = 1$.

Because $r_A^* > M_L$ and $L_A - R_A < M_R - M_L$, there is always an $l_A \in [M_L, r_A^*]$ such that $1 - \pi_R(r_A^*, R_A, l_A, L_A) > 0$. Notice that when $1 - \pi_R(r_A^*, R_A, l_A, L_A) > 0$, we have $1 - \pi_R(r_D^*, R_D, l_A, L_A) > 0$ (for any r_D^*) and

$$\mathrm{E}V_{M_L}(l_A, L_A, r_A^*, r_D^*) > \mathrm{E}V_{M_L}(l_A, L_D, r_A^*, r_D^*),$$

because

$$\begin{aligned} V_{M_L}(l_A, L_D, r_A^*, R_A) &> V_{M_L}(l_A, L_D, r_A^*, R_A); \\ V_{M_L}(l_A, L_D, r_D^*, R_D) &> V_{M_L}(l_A, L_D, r_D^*, R_D). \end{aligned}$$

Then we have $\max_{l_A} \mathrm{E}V_{M_L}(l_A, L_A, r_A^*, r_D^*) > \max_{l_D} \mathrm{E}V_{M_L}(l_D, L_D, r_A^*, r_D^*)$. By Lemma A-3, the maximizer of $\mathrm{E}V_{M_L}(l_A, L_A, r_A^*, r_D^*)$, denoted by l_A^{**}, belongs to the interval $[M_L, r_A^*]$, and because $r_A^* > M_L$ and $L_A - R_A < M_R - M_L$, $1 - \pi_R(r_A^*, R_A, l_A^{**}, L_A) > 0$. Therefore, l_A^{**} always satisfies (A-4) and (A-5). This completes the proof. $\square$

Proposition A-2. *Suppose that the conditions for Theorem 1 hold. If candidate L_A wins party L's primary and $\beta_R^\circ(l_A^*, L_A)$ is nonempty, $w_A^{*R} = 1$ in equilibrium.*

Proof of Proposition A-2. Suppose $L_A - R_A < M_R - M_L$. Suppose that candidate L_A wins party L's primary, and $w_A^{*L} = 1$. Suppose that $\beta_R^\circ(l_A^*, L_A)$ is nonempty. Now, on the contrary, suppose $w_A^{*R} \neq 1$. Then by the definition of equilibrium,

$$V_{M_R}(l_A^*, L_A, r_A^*, R_A) \leq V_{M_R}(l_A^*, L_A, r_D^*, R_D). \tag{A-6}$$

Case 1. (A-6) *holds with strict inequality.* Then, $w_D^{*R} = 1$ and $w_A^{*R} = 0$ by the definition of equilibrium. Then immediately we obtain a contradiction, and because $\beta_R^\circ(l_A^*, L_A)$ is nonempty, there is an $r_A \in \beta_R^\circ(l_A^*, L_A)$ such that

$$\begin{aligned} V_{M_R}(l_A^*, L_A, r_A, R_A) &> V_{M_R}(l_A^*, L_A, r_D^*, R_D); \\ \pi_R(r_A, R_A, l_A^*, L_A) &> w_A^{*R} \cdot \pi_R(r_A^*, R_A, l_A^*, L_A) = 0. \end{aligned}$$

Case 2. (A-6) *holds with equality.* Then because $\beta_R^\circ(l_A^*, L_A)$ is nonempty and $w_A^{*R} < 1$, by Lemma A-1, we can take $r_A = r_A^* + \epsilon$ for ϵ sufficiently small such that

$$\begin{aligned} V_{M_R}(l_A^*, L_A, r_A, R_A) &> V_{M_R}(l_A^*, L_A, r_D^*, R_D); \\ \pi_R(r_A, R_A, l_A^*, L_A) &> w_A^{*R} \cdot \pi_R(r_A^*, R_A, l_A^*, L_A). \end{aligned}$$

By the definition of equilibrium, this contradicts the assumption that w_A^{*R} and r_A^* constitute an equilibrium. $\square$

Lemma A-4. *For every $l_A, r_A \in X$,*

- *If $\pi_R(r, R_A, l_A, L_A) > 0$ for some $r \in X$, $\beta_R^\circ(l_A, L_A)$ is nonempty.*
- *If $\pi_R(r_A, R_A, l, L_A) < 1$ for some $l \in X$, $\beta_L^\circ(r_A, R_A)$ is nonempty.*
- *$\beta_L(r_A, R_A)$ and $\beta_R(l_A, L_A)$ are nonempty.*

Proof of Lemma A-4. As the proof is symmetric, we only prove the claims for $\beta_R(l_A, L_A)$. We show that

$$\max_r V_{M_R}(l_A, L_A, r, R_A) \geq \max_r V_{M_R}(l_A, L_A, r, R_D) \tag{A-7}$$

and if there is an r with $\pi_R(r, R_A, l_A, L_A) > 0$, $\beta_R^\circ(l_A, L_A)$ is nonempty.

Fix l_A and r in X. If $\pi_R(r, R_A, l_A, L_A) > 0$, we have $v_{M_R}(r, R_A) > v_{M_R}(l_A, L_A)$. Because $v_{M_R}(r, R_A) > v_{M_R}(r, R_D)$ and $\pi_R(r, R_A, l_A, L_A) > 0$ implies $\pi_R(r, R_A, l_A, L_A) > \pi_R(r, R_D, l_A, L_A)$,

$$V_{M_R}(l_A, L_A, r, R_A) > V_{M_R}(l_A, L_A, r, R_D). \tag{A-8}$$

Thus, indeed $\beta_R^\circ(l_A, L_A)$ is nonempty, which gives us the first statement. Alternatively, if $\pi_R(r, R_A, l_A, L_A) = 0$, then

$$V_{M_R}(l_A, L_A, r, R_A) = V_{M_R}(l_A, L_A, r, R_D) = L_A - M_R + l_A. \tag{A-9}$$

Because r and l_A are arbitrary and either (A-8) or (A-9) holds, we conclude that for every $r \in X$, (A-7) holds. □

For every $l_A, r_A \in X$, recall that we have defined $\hat{l}_D(r_A)$ and $\hat{r}_D(l_A)$ by

$$\hat{l}_D(r_A) = \arg\max_{l_D \in X} V_{M_L}(l_D, L_D, r_A, R_A);$$
$$\hat{r}_D(l_A) = \arg\max_{r_D \in X} V_{M_R}(l_A, L_A, r_D, R_D),$$

and $\hat{l}_A(r_A)$ and $\hat{r}_A(l_A)$ by

$$\hat{l}_A(r_A) = \max\{l_A | V_{M_L}(l_A, L_A, r_A, R_A) \geq V_{M_L}(\bar{l}_D, L_D, r_A, R_A) \text{ for } \bar{l}_D \in \hat{l}_D(r_A)\};$$
$$\hat{r}_A(l_A) = \min\{r_A | V_{M_R}(l_A, L_A, r_A, R_A) \geq V_{M_R}(l_A, L_A, \bar{r}_D, R_D) \text{ for } \bar{r}_D \in \hat{r}_D(l_A)\}.$$

Lemma A-5. *Suppose $M_R - M_L > L_A - R_A$.*

- *If $L_A = R_A$ and $L_A - R_D < M_R - z$ for some $z \in (M_L, M_R)$, then for every $x \leq z$, $x + L_A - R_A$ does not belong to $\beta_R(x, L_A)$.*
- *If $L_A > R_A$ and there is $\bar{x}$ such that*

$$\frac{\frac{1}{2} f(\frac{M_R+\bar{x}+L_A-R_D}{2})}{\left(1 - F(\frac{M_R+\bar{x}+L_A-R_D}{2})\right)^2} = \frac{1}{L_A - R_A}, \tag{A-10}$$

then for every $x \leq \bar{x}$, $x + L_A - R_A$ does not belong to $\beta_R(x, L_A)$.

Proof of Lemma A-5. Fix $l_A = x$. Let $\hat{x}_R \in \hat{r}_D(x)$. Then $x + L_A - R_A$ does not belong to $\beta_R(x, L_A)$ if and only if

$$V_{M_R}(x, L_A, \hat{x}_R, R_D) > \lim_{y^+ \to x+L_A-R_A} V_{M_R}(x, L_A, y, R_A). \tag{A-11}$$

By the first-order condition in Lemma A-1,

$$R_D + \hat{x}_R = L_A + x + \frac{(1 - F(\frac{\hat{x}_R+x+L_A-R_D}{2}))}{\frac{1}{2}f(\frac{\hat{x}_R+x+L_A-R_D}{2})}. \tag{A-12}$$

The left-hand side (LHS) of (A-11) is

$$\begin{aligned}&(1 - F(\frac{\hat{x}_R + x + L_A - R_D}{2}))(R_D + \hat{x}_R) + F(\frac{\hat{x}_R + x + L_A - R_D}{2})(L_A + x) - M_R \\ =&L_A + x + \frac{\left(1 - F(\frac{\hat{x}_R+x+L_A-R_D}{2})\right)^2}{\frac{1}{2}f(\frac{\hat{x}_R+x+L_A-R_D}{2})} - M_R.\end{aligned}$$

The right-hand side (RHS) of (A-11) is

$$\begin{aligned}&(1 - F(x + L_A - R_A))(L_A - M_R + x + L_A - R_A) + F(x + L_A - R_A)(L_A - M_R + x) \\ &= L_A - M_R + x + (1 - F(x + L_A - R_A))(L_A - R_A).\end{aligned}$$

Thus, (A-11) holds when

$$\frac{\frac{1}{2}f(\frac{\hat{x}_R+x+L_A-R_D}{2})(1 - F(x + L_A - R_A))}{\left(1 - F(\frac{\hat{x}_R+x+L_A-R_D}{2})\right)^2} \leq \frac{1}{(L_A - R_A)}.$$

Notice that when $L_A = R_A$, (A-11) holds for every $x \leq z$, because $L_A - R_D < M_R - z$ implies $\hat{x}_R + x + L_A - R_D \leq M_R + z + L_A - R_D < 2M_R$. Thus, suppose $L_A > R_A$. Then, if

$$\max \frac{\frac{1}{2}f(\frac{\hat{x}_R+x+L_A-R_D}{2})(1 - F(x + L_A - R_A))}{\left(1 - F(\frac{\hat{x}_R+x+L_A-R_D}{2})\right)^2} \leq \frac{1}{(L_A - R_A)},$$

then $x + L_A - R_A$ does not belong to $\beta_R(x, L_A)$. If there is $\bar{x}$ such that

$$\frac{\frac{1}{2}f(\frac{M_R+\bar{x}+L_A-R_D}{2})}{\left(1 - F(\frac{M_R+\bar{x}+L_A-R_D}{2})\right)^2} = \frac{1}{(L_A - R_A)},$$

for every $x \leq \bar{x}$, (A-11) holds and $x + L_A - R_A$ does not belong to $\beta_R(x, L_A)$, because by Assumption 1,

$$\max \frac{\frac{1}{2}f(\frac{\hat{x}_R+x+L_A-R_D}{2})(1 - F(x + L_A - R_A))}{\left(1 - F(\frac{\hat{x}_R+x+L_A-R_D}{2})\right)^2} \leq \frac{\frac{1}{2}f(\frac{M_R+\bar{x}+L_A-R_D}{2})}{\left(1 - F(\frac{M_R+\bar{x}+L_A-R_D}{2})\right)^2} = \frac{1}{(L_A - R_A)},$$

□

Lemma A-6. *Suppose* $\frac{M_R-M_L}{2} > L_A - R_A$.

- *If $L_A = R_A$ and $R_A - L_D < z - M_L$ for some $z \in (M_L, M_R)$, then for every $y \geq z$, $y - L_A + R_A$ does not belong to $\beta_L(y, R_A)$.*

- *If $L_A > R_A$ and there is a $\bar{y}$ to satisfy*

$$\frac{\frac{1}{2}f(\frac{\bar{y}+M_L+L_D-R_A}{2})}{\big(F(\frac{\bar{y}+M_L+L_D-R_A}{2})\big)^2} = \frac{1}{2(L_A - R_A)}, \tag{A-13}$$

then for every $y \geq \bar{y}$, $y - L_A + R_A$ does not belong to $\beta_L(y, R_A)$.

Proof of Lemma A-6. Fix $r_A = y$. Let $\hat{y}_L \in \hat{l}_D(y)$. Then $y - L_A + R_A$ does not belong to $\beta_L(y, R_A)$ if and only if

$$V_{M_L}(\hat{y}_L, L_D, y, R_A) > \lim_{x^- \to y - L_A + R_A} V_{M_L}(x, R_A, y, L_A), \tag{A-14}$$

To compute these expected payoffs in (A-14), denote p the winning probability of candidate R_A when candidate L_A takes $y - L_A + R_A$. Then, $p = 0$ if $L_A > R_A$, and $p = F(y)$ if $L_A = R_A$ by (4). Then (A-14) can be rewritten as

$$\begin{aligned}&(1 - F(\tfrac{y+\hat{y}_L+L_D-R_A}{2}))(R_A - y) + F(\tfrac{y+\hat{y}_L+L_D-R_A}{2})(L_D - \hat{y}_L) + M_L \\ &> (1-p)(L_A - y + L_A - R_A + M_L) + p(L_A - y + M_L).\end{aligned} \tag{A-15}$$

By Lemma A-1, the LHS of (A-15) is

$$R_A - y + \frac{\big(F(\frac{y+\hat{y}_L+L_D-R_A}{2})\big)^2}{\frac{1}{2}f(\frac{y+\hat{y}_L+L_D-R_A}{2})} + M_L.$$

Thus, (A-15) is rewritten as

$$R_A - y + \frac{\big(F(\frac{y+\hat{y}_L+L_D-R_A}{2})\big)^2}{\frac{1}{2}f(\frac{y+\hat{y}_L+L_D-R_A}{2})} + M_L > L_A - y + M_L + (1-p)(L_A - R_A). \tag{A-16}$$

If $L_A = R_A$, (A-16) holds for every $y \geq z$, because $R_A - L_D < z - M_L$ implies $y + \hat{y}_L + L_D - R_A > z + M_L + L_D - R_A > M_L$. Now, suppose that $L_A > R_A$, and then (A-16) holds if and only if

$$\frac{\frac{1}{2}f(\frac{y+\hat{y}_L+L_D-R_A}{2})}{\big(F(\frac{y+\hat{y}_L+L_D-R_A}{2})\big)^2} < \frac{1}{2(L_A - R_A)}.$$

Thus, if

$$\max \frac{\frac{1}{2}f(\frac{y+\hat{y}_L+L_D-R_A}{2})}{\big(F(\frac{y+\hat{y}_L+L_D-R_A}{2})\big)^2} < \frac{1}{2(L_A - R_A)},$$

then $y + L_A - R_A$ does not belong to $\beta_L(y, R_A)$. Thus, if there is a $\bar{y}$ to satisfy

$$\frac{\frac{1}{2}f(\frac{\bar{y}+M_L+L_D-R_A}{2})}{\big(F(\frac{\bar{y}+M_L+L_D-R_A}{2})\big)^2} = \frac{1}{2(L_A - R_A)},$$

then for every $y \geq \bar{y}$, $y + L_A - R_A$ does not belong to $\beta_L(y, R_A)$ because by Assumption 1,

$$\max \frac{\frac{1}{2} f(\frac{y+\hat{y}_L+L_D-R_A}{2})}{\left(F(\frac{y+\hat{y}_L+L_D-R_A}{2})\right)^2} < \frac{\frac{1}{2} f(\frac{\bar{y}+M_L+L_D-R_A}{2})}{\left(F(\frac{\bar{y}+M_L+L_D-R_A}{2})\right)^2} = \frac{1}{2(L_A - R_A)}.$$

$\square$

Lemma A-7. *Suppose $M_R - M_L > L_A - R_A \geq 0$, and there are $\bar{x}$ and $\bar{y}$ that satisfy*

$$\{\max \beta_L(r_A, R_A) : r_A \in [\bar{y}, M_R]\} \cap \{\min \beta_R(l_A, L_A) : l_A \in [M_L, \bar{x}]\} = \varnothing. \tag{A-17}$$

Then an equilibrium exists, and $(w_A^{*L}, w_D^{*L}) = (w_A^{*R}, w_D^{*R}) = (1, 0)$. Further, $\hat{l}_D(r_A^*)$ and $\hat{r}_D(l_A^*)$ are singletons, and the equilibrium satisfies $l_D^* \in \hat{l}_D(r_A^*)$, $r_D^* \in \hat{r}_D(l_A^*)$, $l_A^* = \hat{l}_A(r_A^*)$ and $r_A^* = \hat{r}_A(r_A^*)$.

Proof of Lemma A-7.

Step 1. We show that there is a fixed point $(l_A^*, l_D^*, r_A^*, r_D^*)$ such that $l_A^* = \hat{l}_A(r_A^*), l_D^* \in \hat{l}_D(r_A^*), r_A^* = \hat{r}_A(l_A^*), r_D^* \in \hat{r}_D(l_A^*)$.

Now, $[M_L, \bar{x}]^2 \times [\bar{y}, M_R]^2$ is a compact and convex subset of a metric space. By the maximum theorem, $BR_D^L(r_A) = \hat{l}_D(r_A)$ and $BR_D^R(l_A) = \hat{r}_D(l_A)$ are nonempty, upper semicontinuous, and compact-valued. Notice that for every $r_A \in [\bar{y}, M_R]$ and $l_A \in [M_L, \bar{x}]$, there are r_D and l_D that satisfy $\pi_R(r_D, R_D, l_A, L_A) > 0$ and $\pi_R(r_A, R_A, l_D, L_D) < 1$. Then Lemma A-1 is applicable, and the maximizers of the expected payoffs uniquely exist for every $r_A \in [\bar{y}, M_R]$ and $l_A \in [M_L, \bar{x}]$. Therefore, $BR_D^L(r_A) = \hat{l}_D(r_A)$ and $BR_D^R(l_A) = \hat{r}_D(l_A)$ are convex-valued for every $r_A \in [\bar{y}, M_R]$ and $l_A \in [M_L, \bar{x}]$.

By (A-17), $\beta_R(l_A, L_A) \cap \beta_L(r_A, R_A) = \varnothing$ when $r_A \geq \bar{y}$ and $l_A \leq \bar{x}$, for every $r \in \beta_R(l_A, L_A)$, $\pi_R(r, R_A, l_A, L_A) > 0$ and for every $l \in \beta_L(r_A, R_A)$, $\pi_R(r_A, R_A, l, L_A) < 1$. By Lemma A-4, $\beta_R^\circ(l_A, L_A)$ and $\beta_L^\circ(r_A, R_A)$ are nonempty. Then also applying Lemma A-1, $\beta_R(l_A, L_A)$ and $\beta_L(r_A, R_A)$ are nonempty and compact intervals. Therefore, $BR_A^L(r_D)$ and $BR_A^R(l_D)$ are nonempty values for every $r_D \geq \bar{y}$ and $l_D \leq \bar{x}$, and indeed singletons. Because the expected payoffs are continuous with respect to each candidate's policy promise, $BR_A^L(r_D)$ and $BR_A^R(l_D)$ are also continuous and convex because they are singletons.

Thus, by Kakutani's fixed point theorem, there is a fixed point $(l_A^*, l_D^*, r_A^*, r_D^*)$ in $[M_L, \bar{x}]^2 \times [\bar{y}, M_R]^2$ such that

$$BR(l_A^*, l_D^*, r_A^*, r_D^*) = (BR_A^L(l_D^*), BR_D^L(r_A^*), BR_A^R(r_D^*), BR_D^R(l_A^*)) = (l_A^*, l_D^*, r_A^*, r_D^*).$$

Finally, note that $\{\hat{l}_A(r_A) : r_A \in [\bar{y}, M_R]\} = \{\max \beta_L(r_A, R_A) : r_A \in [\bar{y}, M_R]\}$ and $\{\hat{r}_A(l_A) : l_A \in [M_L, \bar{x}]\} = \{\min \beta_R(l_A, L_A) : l_A \in [M_L, \bar{x}]\}$. Thus we obtain $l_A^* = \hat{l}_A(r_A^*), l_D^* \in \hat{l}_D(r_A^*), r_A^* = \hat{r}_A(l_A^*), r_D^* \in \hat{r}_D(l_A^*)$.

Step 2. We show that the fixed point whose existence is shown in Step 1 is an equilibrium. Notice that when $r_A^* = \hat{r}_A(l_A^*)$, $r_A^* > M_L$. Thus by Proposition A-1, $w_A^{*L} = 1$. Further, when

$l_A^* + L_A - R_A \notin \beta_R(l_A^*, L_A)$ by Lemma A-5, $\pi_R(r_A, R_A, l_A^*, L_A) > 0$ for every $r_A \in \beta_R(l_A^*, L_A)$ and thus $\beta_R^\circ(l_A^*, L_A)$ is nonempty by the first statement of Lemma A-4. Thus by Proposition A-2, $w_A^{*R} = 1$.

Party R. First, we will show that there is no incentive for candidate R_A to deviate from $\hat{r}_A(l_A^*)$. Notice that $\hat{r}_A(l_A^*)$ maximizes the probability of winning the general election among policy promises in $\beta_R(l_A^*, L_A)$. Because candidate R_A's probability of winning the general election is a strictly decreasing continuous function of r_A, and otherwise candidate R_A could improve her winning chances by moving further to the right while still winning the primary election, that is, there is an r_A such that

$$\begin{aligned} V_{M_R}(l_A^*, L_A, r_A, R_A) &> V_{M_R}(l_A^*, L_A, r_A^*, R_A); \text{ and} && \text{(A-18)} \\ \pi_R(r_A, R_A, l_A^*, L_A) &> \pi_R(r_A^*, R_A, l_A^*, L_A). && \text{(A-19)} \end{aligned}$$

This contradicts an equilibrium condition. Thus, candidate R_A does not have other strategies to deviate from $\hat{r}_A(l_A^*)$. Meanwhile, if $r_D^* \notin \hat{r}_D(l_A^*)$, then there is an r_A for which (A-18) and (A-19) simultaneously hold. This is a contradiction with the definition of equilibrium.

Party L. The proof for party L's candidates can be done in the same fashion as the first part. Intuitively, this is because candidate L_A's probability of winning the general election is a strictly decreasing continuous function of l_A, and otherwise candidate L_A could improve her winning chances by moving further to the left while still winning the primary election, which contradicts an equilibrium condition. Then candidate L_A can win the primary with probability one by adopting a policy promise in the interior of $\beta_L(r_A^*, R_A)$, and if she adopts a position outside this interval, candidate L_D could defeat her by adopting a policy promise in $\hat{l}_D(r_A^*)$. In addition, her probability of winning the general election is a strictly decreasing continuous function of l_A, so in any equilibrium we must have $l_A^* = \hat{l}_A(r_A^*)$. □

Proof of Theorem 1. Suppose $L_A = R_A$. By Lemmas A-5 and A-6, for some $z \in (M_L, M_R)$, the following holds:

$$\{\hat{l}_A(r_A) : r_A \in [z, M_R]\} \cap \{\hat{r}_A(l_A) : l_A \in [M_L, z]\} = \varnothing.$$

Then by Lemma A-7, we obtain the result when the first condition holds.

Suppose $L_A > R_A$. It suffices to show that $\bar{x}$ and $\bar{y}$ with $\bar{x} \leq \bar{y}$ which satisfy (A-10) and (A-13) exist by Lemma A-7. By the intermediate value theorem, the existence of $\bar{x}$ and $\bar{y}$ in $[M_L, M_R]$ is direct due to the assumption of monotone hazard rates (Assumption 1). It remains to show $\bar{x} \leq \bar{y}$. By (10) and (11), there is a $z \in (M_L, M_R)$ such that $\bar{y} > z > \bar{x}$ holds, because

Assumption 1 implies:

$$\frac{\big(1-F(\frac{M_R+z+L_A-R_D}{2})\big)^2}{\frac{1}{2}f(\frac{M_R+z+L_A-R_D}{2})} < \frac{\big(1-F(\frac{M_R+\bar{x}+L_A-R_D}{2})\big)^2}{\frac{1}{2}f(\frac{M_R+\bar{x}+L_A-R_D}{2})} = L_A - R_A$$
$$< \frac{\big(1-F(\frac{M_R+M_L+L_A-R_D}{2})\big)^2}{\frac{1}{2}f(\frac{M_R+M_L+L_A-R_D}{2})}; \text{ and}$$
$$\frac{\big(F(\frac{M_L+z+L_D-R_A}{2})\big)^2}{\frac{1}{2}f(\frac{M_L+z+L_D-R_A}{2})} < 2(L_A - R_A)$$
$$= \frac{\big(F(\frac{M_L+\bar{y}+L_D-R_A}{2})\big)^2}{\frac{1}{2}f(\frac{M_L+\bar{y}+L_D-R_A}{2})} < \frac{\big(F(\frac{M_L+M_R+L_D-R_A}{2})\big)^2}{\frac{1}{2}f(\frac{M_L+M_R+L_D-R_A}{2})}.$$

Finally, $l_A^* \leq z \leq r_A^*$ holds with one of the inequalities being strict, because l_A^* and r_A^* must belong to the domain of the fixed point, which is contained in $[M_L, z]$ and $[z, M_R]$. □

Proposition A-3. *Suppose $L_A - R_A < M_R - M_L$ and the conditions for Theorem 1 hold. Then an equilibrium satisfies*

$$L_D + r_A^* - R_A - l_D^* = \frac{F(\frac{r_A^*+l_D^*+L_D-R_A}{2})}{\frac{1}{2}f(\frac{r_A^*+l_D^*+L_D-R_A}{2})} \tag{A-20}$$

$$F(\frac{l_A^* + r_A^* + L_A - R_A}{2})(L_A - l_A^* - R_A + r_A^*)$$
$$= (L_D + r_A^* - R_A - l_D^*) \cdot F(\frac{l_D^* + r_A^* + L_D - R_A}{2}) \tag{A-21}$$

$$R_D + r_D^* - L_A - l_A^* = \frac{(1-F(\frac{r_D^*+l_A^*+L_A-R_D}{2}))}{\frac{1}{2}f(\frac{r_D^*+l_A^*+L_A-R_D}{2})} \tag{A-22}$$

$$(1 - F(\frac{l_A^* + r_A^* + L_A - R_A}{2}))(R_A + r_A^* - L_A - l_A^*)$$
$$= (R_D + r_D^* - L_A - l_A^*) \cdot (1 - F(\frac{l_A^* + r_D^* + L_A - R_D}{2})). \tag{A-23}$$

Proof of Proposition A-3. By the first-order condition in Lemma A-1, we obtain (A-20) and (A-22). Similarly, by Lemma A-1, we obtain the results. □

Lemma A-8. *In a symmetric game,*

- *when regime 1 arises, $l_A^* = -r_A^*$, and $l_D^* = -r_D^*$ where r_A^* and r_D^* respectively solve* (12) *and* (13).
- $\bar{m} \notin \beta_R(\bar{m}, L_A) \cap \beta_L(\bar{m}, R_A)$ *if and only if $a > L_A - L_D$.*

Proof of Lemma A-8. The first item can be shown by the symmetry of the equilibrium and the four equations in Theorem 1. Now, we illustrate the second item. Let $r_A^* = 0 (= \bar{m})$. Then by (7) of Theorem 1, we obtain

$$r_D^* - (L_A - L_D) = \frac{1 - F(\frac{r_D^*+L_A-L_D}{2})}{\frac{1}{2}f(\frac{r_D^*+L_A-L_D}{2})}.$$

When $r_A^* = l_A^* = 0$, we obtain

$$V_{M_R}(l_A^*, L_A, r_A^*, R_A) = L_A - M_R$$

and by Proposition A-3,

$$V_{M_R}(l_A^*, L_A, r_D^*, R_D) = L_A - M_R + (r_D^* - (L_A - L_D))(1 - F(\frac{r_D^* + L_A - L_D}{2})).$$

Then, $V_{M_R}(l_A^*, L_A, r_A^*, R_A) < V_{M_R}(l_A^*, L_A, r_D^*, R_D)$, which implies $\bar{m} \notin \beta_R(\bar{m}, L_A)$, if and only if $a > L_A - L_D$. $\square$

Proof of Corollary 1. First, as in case (I), when $(l_A^*, r_A^*) = (\bar{m}, \bar{m})$, because a tiebreaking rule $t(\bar{m})$ specifies $\pi_R(\bar{m}, R_A, \bar{m}, L_A) = \frac{1}{2}$, by Lemma A-1, $\beta_R^\circ(\bar{m}, L_A)$ and $\beta_L^\circ(\bar{m}, R_A)$ are nonempty. By Propositions A-1 and A-2, $(w_A^{*L}, w_D^{*L}) = (w_A^{*R}, w_D^{*R}) = (1, 0)$. Case (II) is a special case of Theorem 1. When the set of equilibrium policy promises is as in case (II), again by Lemma A-1 and Propositions A-1 and A-2, we obtain $(w_A^{*L}, w_D^{*L}) = (w_A^{*R}, w_D^{*R}) = (1, 0)$.

Now, we show that if $\bar{m} \in \beta_R(\bar{m}, L_A) \cap \beta_L(\bar{m}, R_A)$ and $t(\bar{m}) = \frac{1}{2}$, case (I) arises, and otherwise, case (II) arises. Then the rest follows by Lemma A-8. Case (II) can be proved in the same way with Theorem 1. Thus, we focus on the proof of case (I). First, we prove that $r_A^* = l_A^* = \bar{m}$ is an equilibrium. Suppose that $r_A^* = l_A^* = \bar{m}$ and then $F(\frac{r_A^* + l_A^*}{2}) = 1 - F(\frac{r_A^* + l_A^*}{2}) = t(\bar{m}) = \frac{1}{2}$. Given $l_A^* = \bar{m}$, for every $r < l_A^*$, candidate R_A's probability of winning the general election is increasing and for every $r > l_A^*$, it is decreasing. Symmetrically, given $r_A^* = \bar{m}$, this is true for candidate L_A. Thus, $r_A^* = l_A^* = \bar{m}$ is an equilibrium.

Second, consider the disadvantaged candidates. Because choosing $\bar{m}$ gives a primary win with certainty for both advantaged candidates, there is no strategy for each disadvantaged candidate to improve their chance of winning in the primary. Thus any strategy in X can be an equilibrium strategy for each disadvantaged candidate. $\square$

Proofs of Results in Section 4

Proofs of Properties 1 and 2. Property 1 is verified as follows. For all $x, x' \in [-a, 0]$ with $x < x'$, we have:

$$\int_{-\infty}^{x} \frac{f_S(t)}{f_S(x)} \mathrm{d}t = \frac{F_S(x)}{f_S(x)} < \frac{F_N(x)}{f_N(x)} = \int_{-\infty}^{x} \frac{f_N(t)}{f_N(x)} \mathrm{d}t.$$

By symmetry, $\frac{1-F_S(-x)}{f_S(-x)} < \frac{1-F_N(-x)}{f_N(-x)}$ for all $x \in [-a, 0]$, and thus we can rewrite it as follows. For all $x \in [0, a]$,

$$\frac{1 - F_S(x)}{f_S(x)} < \frac{1 - F_N(x)}{f_N(x)}.$$

Property 2 is obtained by comparing the regions below $f_S(x)$ or $f_N(x)$ with the line $x = 0$. By Assumption 2 and because $F_S(0) = F_N(0)$, at one point $x' \in [-a, 0]$, and we have $f_S(x') = f_N(x')$. For $x \in [-a, x']$, $f_S(x) < f_N(x)$, which gives us $F_S(x) < F_N(x)$. Since

$F_S(0) = F_N(0)$, we also have $F_S(x) < F_N(x)$ for every $x \in [x', 0]$. By symmetry, for all $x \in [-a, 0]$,

$$1 - F_S(-x) = F_S(x) < 1 - F_N(-x) = F_N(x),$$

which we can rewrite as $F_S(x) > F_N(x)$ for all $x \in [0, a]$. □

Proofs of Properties 3, 4 and 5. Property 3 can be verified by:

$$\frac{1 - F_N(x)}{f_N(x)} = \int_x^\infty \frac{f_N(t)}{f_N(x)} \mathrm{d}t \le \int_x^\infty \frac{f_S(t)}{f_S(x)} \mathrm{d}t = \frac{1 - F_S(x)}{f_S(x)},$$

and similarly, property 4 can be verified by:

$$\frac{F_N(x)}{f_N(x)} = \int_0^x \frac{f_N(t)}{f_N(x)} \mathrm{d}t \ge \int_0^x \frac{f_S(t)}{f_S(x)} \mathrm{d}t = \frac{F_S(x)}{f_S(x)}.$$

Property 5 can be obtained by either property 3 or 4 because

$$F_S(x) = \exp\left(-\int_x^\infty \frac{f_S(t)}{F_S(t)} \mathrm{d}t\right) \le \exp\left(-\int_x^\infty \frac{f_N(t)}{F_N(t)} \mathrm{d}t\right) = F_N(x). \qquad \text{(A-24)}$$

Property 5 is often called *first-order stochastic dominance* when the inequality is strict, in which case we say that F_S first-order stochastically dominates F_N. This implies that the overall median voter tends to be closer to $-a$ with a higher probability in society N than in society S. Properties 1 and 2 imply the same. □

Proofs of the rest of the results in Section 4 involve complicated calculations by taking the first derivatives of four equations A-20 to A-23. Thus we keep them in the separate appendix.

Proofs of Propositions 3, 4 and Proposition 5

Proof of Proposition 3. The first case (a) is proved in a similar manner with Theorem B-1. Now, we consider case (b). Because $M_R - \bar{l}_A > L_A - R_D$, for some r_D, $\pi_R(r_D, R_D, \bar{l}_A, L_A) > 0$. By Lemma A-1, $\hat{r}_D(\bar{l}_A)$ is singleton. Further, $\beta_R^\circ(\bar{l}_A, L_A)$ is nonempty. Thus, $(w_D^{*R}, w_A^{*R}) = (0, 1)$ by Proposition A-2. The rest follows from the proof for party R in Theorem 1. □

Proof of Proposition 4. The proof of case (a) is the same with case (a) in Proposition 3. First, $(w_D^{*L}, w_A^{*L}) = (0, 1)$ is obtained by Proposition A-1 because $\bar{r}_A > M_L$. In case (b), $l_D^* \in \hat{l}_D(\bar{r}_A)$ is proved in the same way as for the proof for party L in Theorem 1. Thus it suffices to show $l_A^* \in \alpha(\bar{r}_A, R_A) \cap \beta_L(\bar{r}_A, R_A)$ in case (b-1), and otherwise, in case (b-2), $l_A^* = \hat{l}_A(\bar{r}_A)$. In case (b-1), candidate L_A can win the primary with probability one and still win the general election with a probability arbitrarily close to one by moving slightly to the right of the rightmost such policy, and in equilibrium she must take a position in $\alpha(\bar{r}_A) \cap \beta_L(\bar{r}_A, R_A)$. No matter what position candidate L_D adopts, neither candidate L_D nor candidate R_A have any way to improve. The proof of case (c)'s $l_A^* \in [\bar{r}_A - L_A + R_A, \bar{r}_A + L_A - R_A]$ is immediate because in this region, candidate L_A can secure a victory in the general election. Because $\bar{r}_A - M_L \leq L_A - R_D$, candidate L_D cannot win the general election against candidate R_A and thus the equilibrium policy promise is unrestricted for candidate L_D.

Finally, if $L_A - R_A \geq R_A - L_D$, then cases (a) and (c) are the only ones possible. In contrast, if $L_A - R_A < R_A - L_D$, then all of cases (a), (b), and (c) cover all those possible. As a result, in either situation, there is always an equilibrium. This completes the proof. $\square$

Proof of Proposition 5. If candidate L_D does not run for the primary, candidate L_A chooses a policy promise in $\alpha_L(\bar{r}_A, R_A)$ as described in Proposition 4. Suppose candidate L_A chooses $l_A^0 = \bar{r}_A - d \in \alpha_L(\bar{r}_A, R_A)$, and thus we have $d \leq L_A - R_A$. Then the payoff that candidate L_D receives is $L_A - \kappa(\bar{r}_A - d) + \kappa \bar{l}_D$.

If candidate L_D runs and candidate L_A chooses l_A^1, because $l_A^1 \geq M_L$ and $M_L \leq \bar{l}_D$, the expected payoff of candidate L_D is

$$(1 - F(\frac{\bar{r}_A + l_A^1 + L_A - R_A}{2}))(R_A - \kappa \bar{r}_A + \kappa \bar{l}_D) + F(\frac{\bar{r}_A + l_A^1 + L_A - R_A}{2})(L_A - \kappa l_A^1 + \kappa \bar{l}_D).$$

Thus, if $(1 - F(\frac{\bar{r}_A + l_A^1 + L_A - R_A}{2}))(L_A - R_A) < F(\frac{\bar{r}_A + l_A^1 + L_A - R_A}{2})\kappa(\bar{r}_A - l_A^1) - \kappa d$, candidate L_D enters, and otherwise, candidate L_D does not enter. If $F(\frac{\bar{r}_A + l_A^1 + L_A - R_A}{2})(\bar{r}_A - l_A^1) \leq d$, because $L_A \geq 0$ and $\pi_R(\bar{r}_A, R_A, l_A^1, L_A) > 0$ in a regime 1 equilibrium, $\bar{r}_A > l_A^1 + L_A - R_A$. Then, $(1 - F(\frac{\bar{r}_A + l_A^1 + L_A - R_A}{2}))(L_A - R_A) > 0$. Thus, candidate L_D does not enter.

Suppose $F(\frac{\bar{r}_A + l_A^1 + L_A - R_A}{2})(\bar{r}_A - l_A^1) > d$. Let $\bar{\kappa} = \frac{(1 - F(\frac{\bar{r}_A + l_A^1 + L_A - R_A}{2}))(L_A - R_A)}{F(\frac{\bar{r}_A + l_A^1 + L_A - R_A}{2})(\bar{r}_A - l_A^1) - d}$. If $\kappa < \bar{\kappa}$, candidate L_D does not enter. Otherwise, if $\kappa > \bar{\kappa}$, candidate L_D enters.

Finally, to complete the proof, we note that $\pi_R(\bar{r}_A, R_A, l_A^1, L_A) > 0$ in a regime 1 equilibrium and $\pi_R(\bar{r}_A, R_A, l_A^0, L_A) = 0$ because candidate L_A chooses a policy promise l_A^0 in $\alpha_L(\bar{r}_A, R_A)$ as described in Proposition 4. Then $\bar{r}_A - l_A^1 > \bar{r}_A - l_A^0$ because $l_A^0 \in \alpha_L(\bar{r}_A, R_A)$ and $l_A^1 \notin \alpha_L(\bar{r}_A, R_A)$. $\square$

References

Abney, Ronni, James Adams, Michael Clark, Malcolm Easton, Lawrence Ezrow, Spyros Kosmidis, and Anja Neundorf, "When does valence matter? Heightened valence effects for governing parties during election campaigns," *Party Politics*, 2013, *19* (1), 61—-82.

Abramowitz, Alan I, "Explaining Senate election outcomes," *American Political science review*, 1988, *82* (2), 385–403.

Adams, James and Samuel III Merrill, "Candidate and Party Strategies in Two-Stage Elections Beginning with a Primary," *American Journal of Political Science*, 2008, *52* (2), 344–359.

__ **and** __ , "Policy-Seeking Parties in a Parliamentary Democracy with Proportional Representation: A Valence-Uncertainty Model," *British Journal of Political Science*, 2009, *39* (3), 539–558.

__ **, Jane Green, and Caitlin Milazzo**, "Has the British Public Depolarized Along With Political Elites? An American Perspective on British Public Opinion," *Comparative Political Studies*, 2012, *45* (4), 507—-530.

__ **, Lawrence Ezrow, Samuel III Merrill, and Zeynep Somer-Topcu**, "Does Collective Responsibility for Performance Alter Party Strategies? Policy-Seeking Parties in Proportional Systems," *British Journal of Political Science*, 2013, *43* (10), 1–23.

__ **, Samuel Merrill III, Elizabeth N Simas, and Walter J Stone**, "When candidates value good character: A spatial model with applications to congressional elections," *The Journal of Politics*, 2011, *73* (1), 17–30.

Aragones, Enriqueta and Thomas R Palfrey, "Mixed equilibrium in a Downsian model with a favored candidate," *Journal of Economic Theory*, 2002, *103* (1), 131–161.

Ausubel, Lawrence M. and P. Milgrom, "Ascending Auctions with Package Bidding," *Frontiers of Theoretical Economics*, 2002, *1*, 1–42.

Benoit, Kenneth and Michael Laver, *Party policy in modern democracies*, Routledge, 2006.

Bouton, Laurent, "A theory of strategic voting in runoff elections," *The American Economic Review*, 2013, *103* (4), 1248–1288.

Bullock, Will and Joshua D. Clinton, "More a Molehill than a Mountain: The Effects of the Blanket Primary on Elected Officials' Behavior from California," *The Journal of Politics*, 2011, *73* (3), 915–930.

Buttice, Matthew K. and Walter J. Stone, "Candidates Matter: Policy and Quality Differences in Congressional Elections," *The Journal of Politics*, 2012, *74* (12), 870—-887.

Callander, Steven, "Electoral competition in heterogeneous districts," *Journal of Political Economy*, 2005, *113* (5), 1116–1145.

Carey, John M. and John Polga-Hecimovich, "Primary Elections and Candidate Strength in Latin America," *The Journal of Politics*, 2006, *68* (3), 530–543.

Casas, Agustin, "Ideological extremism and primaries," *Working Paper*, 2018.

Chen, Kong-Pin and Sheng-Zhang Yang, "Strategic Voting in Open Primaries," *Public Choice*, 2002, *112* (1/2), 1–30.

Cho, Seok-ju and Insun Kang, "Open Primaries and Crossover Voting," *Journal of Theoretical Politics*, 2014. DOI: 10.1177/0951629814531671.

Clark, Michael and Debra Leiter, "Does the Ideological Dispersion of Parties Mediate the Electoral Impact of Valence? A Cross-National Study of Party Support in Nine Western European Democracies," *Comparative Political Studies*, 2014, *47* (2), 171–202.

Day, Robert and P. Milgrom, "Core-selecting package auctions," *International Journal of Game Theory*, 2008, *36*, 393–407.

Dell'Ariccia, Giovanni and Robert Marquez, "Lending Booms and Lending Standards," *The Journal of Finance*, 2006, *61* (5), 2511–2546.

Drouvelis, Michalis, Alejandro Saporiti, and Nicolaas J. Vriend, "Political motivations and electoral competition: Equilibrium analysis and experimental evidence," *Games and Economic Behavior*, January 2014, *83*, 86–115.

_ , _ , and Nicolaas J Vriend, "Political motivations and electoral competition: Equilibrium analysis and experimental evidence," *Games and Economic Behavior*, 2014, *83*, 86–115.

Frey, William, *Diversity Explosion: How New Racial Demographics are Remaking America*, Washington: Brookings Institution Press, 2015.

Galderisi, Peter and Marni Ezra, "Congressional Primaries in Historical and Theoretical Context," in Peter Galderisi, Marni Ezra, and Michael Lyons, eds., *Congressional Primaries and the Politics of Representation*, Oxford: Rowman & Littlefield Publishers, INC, 2001.

Garzia, Diego, "Changing parties, changing partisans: The personalization of partisan attachments in Western Europe," *Political Psychology*, 2013, *34* (1), 67–89.

Gerber, Elisabeth and Rebecca Morton, "Primary Election Systems and Representation," *The Journal of Law, Economics, & Organization*, 1998, *14* (2), 304–324.

Green, Jane and Sara B. Hobolt, "Owning the issue agenda: Party strategies and vote choices in British elections," *Electoral Studies*, 2008, *27* (3), 460–476.

Grofman, Bernard, Orestis Troumpounis, and Dimitrios Xefteris, "Electoral Competition with Primaries and Quality Asymmetries," *Journal of Politics*, 2019, *81* (1), 260–273.

Hall, Andrew B., "What Happens When Extremists Win Primaries?," *American Political Science Review*, 2015, *109* (1), 18–42.

Harmon, Nikoraj A., "Immigration, Ethnic Diversity, and Political Outcomes: Evidence from Denmark," *Scandinavian Journal of Economics*, 2018, *120*, 1043–1074.

Hayes, Danny, "Has television personalized voting behavior?," *Political Behavior*, 2009, *31* (2), 231–260.

Hill, Seth J. and Chris Tausanovitch, "Southern realignment, party sorting, and the polarization of American primary electorates, 1958–2012," *Public Choice*, Jul 2018, *176* (1), 107–132.

Hirano, Shigeo and James M. Snyder Jr., "Primary Elections and the Quality of Elected Officials," 2014. Working Paper.

_ , _ , Stephen Daniel Ansolabehere, and John Mark Hansen, "Primary elections and partisan polarization in the U.S. Congress," *Quarterly Journal of Political Science*, 2010, *5* (2), 169–191.

Holian, David B and Charles Prysby, "Candidate character traits in the 2012 presidential election," *Presidential Studies Quarterly*, 2014, *44* (3), 484–505.

Hummel, Patrick, "Candidate strategies in primaries and general elections with candidates of heterogeneous quality," *Games and Economic Behavior*, 2013, *78*, 85–102.

Jackson, Matthew O., Leo K. Simon, Jeroen M. Swinkels, and William R. Zame, "Communication and Equilibrium in Discontinuous Games of Incomplete Information," *Econometrica*, 2002, *70* (5), 1711–1740.

Kamada, Yuichiro and Fuhito Kojima, "Voter preferences, polarization, and electoral policies," *American Economic Journal: Microeconomics*, 2014, *6* (4), 203–36.

Kinder, Donald R, Mark D Peters, Robert P Abelson, and Susan T Fiske, "Presidential prototypes," *Political behavior*, 1980, *2* (4), 315–337.

Kornberg, Allan, "Democratic Challenges, Democratic Choices: The Erosion of Political Support in Advanced Industrial Democracies," *Perspectives on Politics*, 2005, *3* (2), 397–398.

Markussen, Simen, "How the left prospers from prosperity," *European Journal of Political Economy*, 2008, *24* (2), 329–342.

McCarty, Nolan, Keith T. Poole, and Howard Rosenthal, *Polarized America: The Dance of Ideology and Unequal Riches*, Cambridge, MA: The MIT University Press, 2006.

_ , _ , and _ , *Political Bubbles: Financial Crises and the Failure of American Democracy*, Princeton, NJ: Princeton University Press, 2013.

McCurley, Carl and Jeffery J. Mondak, "Inspected by #1184063113: The influence of incumbents' competence and integrity in U.S. house elections," *American Journal of Political Science*, 1995, *39* (4), 864–885.

McLennan, Andrew, Paulo K. Monteiro, and Rabee Tourky, "Games with discontinuous payoffs: a strengthening of Reny's existence theorem," *Econometrica*, 2011, *79* (5), 1643–1664.

Mondak, Jeffery J., "Competence, integrity, and the electoral success of Congressional incumbents," *The Journal of Politics*, 1995, *57* (4), 1043–1069.

Morton, Rebecca, *Analyzing Elections: New Institutionalism in American Politics*, New York, NY: W. W. Norton & Company, 2006.

Myerson, Roger, "Incentives to Cultivate Favored Minorities Under Alternative Electoral Systems," *The American Political Science Review*, 1993, *87* (4), 856–869.

Owen, Guillermo and Bernard Grofman, "Two-Stage Electoral Competition in Two-Party Contest: Persistent Divergence of Party Positions," *Social Choice and Welfare*, 2006, *26* (3), 547–569.

Prat, Andrea and Aldo Rustichini, "Games Played through Agents," *Econometrica*, 2003, *71* (4), 989–1026.

Rahat, Gideon and Tamir Sheafer, "The personalization (s) of politics: Israel, 1949–2003," *Political communication*, 2007, *24* (1), 65–80.

Reny, Philip J, "On the existence of pure and mixed strategy Nash equilibria in discontinuous games," *Econometrica*, 1999, *67* (5), 1029–1056.

Roemer, John E., "Political-Economic Equilibrium When Parties Represent Constituents: The Unidimensional Case," *Social Choice and Welfare*, 1997, *14* (4), 479–502.

Saporiti, Alejandro, "Existence and uniqueness of Nash equilibrium in electoral competition games: The hybrid case," *Journal of Public Economic Theory*, 2008, *10* (5), 827–857.

Serra, Gilles, "Why Primaries? The Party's Tradeoff between Policy and Valence," *Journal of Theoretical Politics*, 2011, *23* (21), 21–55.

Simon, Leo K. and William R. Zame, "Discontinuous Games and Endogenous Sharing Rules," *Econometrica*, 1990, *58* (4), 861–872.

Snyder, James. M. Jr and M. M. Ting, "Electoral selection with parties and primaries," *American Journal of Political Science*, 2011, *55*, 782–796.

Stone, Walter J. and Elizabeth N. Simas, "Candidate valence and ideological placement in U. S. house elections," *American Journal of Political Science*, 2010, *54* (2), 371–388.

Takayama, Shino, "A Model of Two-stage Electoral Competition with Strategic Voters," 2014. School of Economics Discussion Paper Series No. 525, University of Queensland.

_ , Yuki Tamura, and Terence Yeo, "Nash Equilibrium and Party Polarization in an Electoral Competition Model," *Journal of Public Economic Theory*, 2019, *21* (2), 219–240.

Ware, Alan, *The American Direct Primary: Party Institutionalization and Transformation in the North*, UK: Cambridge University Press, 2002.

Woon, Jonathan, "Primaries and Candidate Polarization: Behavioral Theory and Experimental Evidence," *American Political Science Review*, 2018, *112* (4), 826–843.

Primaries, Strategic Voters and Heterogeneous Valences: Appendix B

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A regime 2 equilibrium

There are three possible cases that result in regime 2. In the first case, candidate L_A's valence advantage over the other candidates is sufficiently high that candidate L_A wins the primary and general elections with certainty regardless of the policy promises of all other candidates. Theorem B-1 provides the conditions under which this case arises. In Theorems B-2 and B-3, we consider the situations in which candidate L_A can win both the primary and general elections with certainty by choosing M_L (Theorem B-2) or M_R (Theorem B-3). Thus, these two theorems provide the conditions under which candidate L_A chooses M_L or M_R and regime 2 arises.

Obviously, regime 2 arises when candidate L_A is strong in terms of valence. Consider two cases: case 1, $M_R - M_L \leq L_A - R_A$, and case 2, $M_R - M_L > L_A - R_A$. In case 1, candidate L_A is so strong that candidate L_A wins the general election with certainty regardless of the policy promises of candidate R_A or R_D. Furthermore, for any primary voter in party R, the expected payoff resulting from either party's R candidate with any policy promise is simply the payoff of having candidate L_A elected, so the winning probabilities w_D^{*R} and w_A^{*R} are unrestricted. Theorem B-2 considers this situation. In case 2, both regimes 1 and 2 can arise. Theorem B-3 provides the conditions under which regime 1 arises, even in case 2.

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In summary, the three theorems yield the conditions under which, in equilibrium, regime 2 arises and candidate L_A chooses $\frac{M_R+M_L}{2}$ (Theorem B-1), M_L (Theorem B-2), and M_R (Theorem B-3).[1]

Theorem B-1. *Suppose that* $\frac{M_R-M_L}{2} < \min\{L_A - L_D, L_A - R_A\}$. *Then any strategy profile such that* $l_A^* = \frac{M_R+M_L}{2}$ *and* (l_D^*, r_A^*, r_D^*), *and any collection of primary winning probabilities* $(w_A^{*R}, w_D^{*R}) \in [0,1]^2$ *such that* $w_A^{*R} + w_D^{*R} = 1$, *and* $(w_A^{*L}, w_D^{*L}) = (1,0)$ *constitutes an equilibrium. Finally,* $\pi_R(l_A^*, L_A, r_n^*, R_n) = 0$ *for every* $n = A, D$.

Theorem B-2. *Suppose that* $L_A - R_A \geq M_R - M_L$. *Then any strategy profile such that* $l_A^* = M_L$ *and* (l_D^*, r_A^*, r_D^*), *and any collection of primary winning probabilities* $(w_A^{*R}, w_D^{*R}) \in [0,1]^2$ *such that* $w_A^{*R} + w_D^{*R} = 1$, *and* $(w_A^{*L}, w_D^{*L}) = (1,0)$ *constitutes an equilibrium. Finally,* $\pi_R(l_A^*, L_A, r_n^*, R_n) = 0$ *for every* $n = A, D$.

Theorem B-3. *Suppose* $M_R - M_L > L_A - R_A > 0$.

1. *If* $M_R - M_L \leq R_A - L_D$, *then any strategy profile such that* $l_A^* = M_R$ *and* (l_D^*, r_A^*, r_D^*) *such that* $r_A^*, r_D^* \in [M_R - (L_A - R_A), M_R]$, *and any collection of primary winning probabilities* $(w_A^{*R}, w_D^{*R}) \in [0,1]^2$ *such that* $w_A^{*R} + w_D^{*R} = 1$, *and* $(w_A^{*L}, w_D^{*L}) = (1,0)$ *constitute an equilibrium. Finally,* $\pi_R(l_A^*, L_A, r_n^*, R_n) = 0$ *for every* $n = A, D$.

2. *If* $M_R - M_L > R_A - L_D$ *and for each* $n = A, D$, *there is* $\bar{d}_n > 0$ *that satisfies*

$$\bar{d}_n = (L_A - R_n) - \frac{\left(F(M_R + \frac{L_D - R_n}{2})\right)^2}{\frac{1}{2} f(M_R + \frac{L_D - R_n}{2})}, \tag{B-1}$$

then any strategy profile such that $l_A^* = M_R$ *and* (l_D^*, r_A^*, r_D^*) *such that* $r_A^* \geq M_R - \bar{d}_A$, $r_D^* \geq M_R - \bar{d}_D$ *and any collection of primary winning probabilities* $(w_A^{*R}, w_D^{*R}) \in [0,1]^2$ *such that* $w_A^{*R} + w_D^{*R} = 1$, *and* $(w_A^{*L}, w_D^{*L}) = (1,0)$ *constitute an equilibrium. Finally,* $\pi_R(l_A^*, L_A, r_n^*, R_n) = 0$ *for every* $n = A, D$.

It is easy to see that when L_A increases, $\bar{d}_n > 0$ is likely to hold for each $n = A, D$. When L_D increases, by Assumption 1, $\bar{d}_n > 0$ is less likely to hold. Being more advantaged with a higher valence makes it easier for candidate L_A to win both elections by choosing M_R, whereas facing a stronger competitor L_D in the primary makes it more difficult to win office by choosing M_R.

[1] Hummel (2013) assumes that if a candidate loses in the primary election, then the candidate obtains a utility of 0, if a candidate wins party k's primary, then she obtains a utility of $u_k \geq 0$ for each $k = L, R$, and the winner of the general election receives an additional utility of $u \geq 0$. We do not impose any particular utility and simply assume that each candidate attempts to maximize the probability of winning office. If we make a similar assumption to that of Hummel (2013), a regime 2 equilibrium still arises under the conditions of Theorems B-1, B-2, and B-3, although candidates R_A and R_D would attempt to maximize the chance of winning party R's primary.

The proofs of Theorems B-1, B-2, and B-3

Proof of Theorem B-1. We claim that $l_A^* = \frac{M_R+M_L}{2}$ is a policy promise for candidate L_A that guarantees the candidate a victory in both party L's primary as well as in the general election, regardless of the policy promises of any of the other candidates.

First, we show that if candidate L_A wins party L's primary, then L_A wins the general election with certainty, regardless of the identity or policy promise of party R's primary winner.

Given $\frac{M_R-M_L}{2} < L_A - R_A$, for any $r_A \in X$ we must have $\pi_R\left(r_A, R_A, l_A^*, L_A\right) = 0$ given $v_b(l_A^*, L_A) > v_b(r_A, R_A)$ for all voters with bliss points $b \in [M_L, M_R]$. The same holds for any $r_D \in X$, i.e. $\pi_R(r_D, R_D, l_A^*, L_A) = 0$ given $R_D < R_A$. Hence, $\pi_R\left(r_i, R_i, l_A, L_A\right) = 0$ for $i = A, D$ and any $r_i \in X$.

Second, we show that candidate L_A wins party L's primary with certainty in equilibrium, and $w_A^{*L} = 1$. By assumption, $\frac{M_R-M_L}{2} < \min\left\{L_A - R_A, L_A - L_D\right\}$ and $R_A > R_D$, hence $v_{M_L}(l_A^*, L_A) > \max\left\{L_D, R_A, R_D\right\}$. As $\pi_R\left(r_i, R_i, l_A, L_A\right) = 0$ for any $i \in A, D$, therefore $\mathrm{E}V_{M_L}(l_A^*, L_A, r_A, r_D) = v_{M_L}(l_A^*, L_A)$ for any $r_A, r_D \in X$. Then because $V_{M_L}(l_D, L_D, r_i, R_i) \leq \max\left\{L_D, R_A, R_D\right\}$ for all $l_D \in X$, we obtain

$$\mathrm{E}V_{M_L}(l_A^*, L_A, r_A, r_D) > V_{M_L}(l_D, L_D, r_i, R_i),$$

for all $l_D \in X$, and $i \in \{A, D\}$.

Thus, $\mathrm{E}V_{M_L}(l_A^*, L_A, r_A, r_D) > \mathrm{E}V_{M_L}(l_D, L_D, r_A, r_D)$, and the median voter in party L, M_L, strictly prefers candidate L_A in the primary election regardless of candidate L_D's policy promise or the outcome of party R's primary. Then by the definition of equilibrium, candidate L_A is guaranteed victory in party L's primary in equilibrium. $\square$

Proof of Theorem B-2. Let $l_A^* = M_L$. Because $L_A - R_A \geq M_R - M_L$, $\pi_R(r_n^*, R_n, l_A^*, L_A) = 0$ for any $r_n^* \in X$ and thus $V_{M_R}(r_n^*, R_n, l_A^*, L_A) = L_A - M_R + M_L$. Further, obviously we also have $V_{M_L}(l_A^*, L_A, r_n^*, R_n) = L_A > V_{M_L}(l_D^*, L_D, r_n^*, R_n)$ for any $l_D^* \in X$ and each $n = A, D$. Therefore,

$$\mathrm{E}V_{M_L}(l_A^*, L_A, r_A^*, r_D^*) = L_A > \mathrm{E}V_{M_L}(l_D^*, L_D, r_A^*, r_D^*),$$

which completes the proof. $\square$

Proof of Theorem B-3. Suppose $M_R - M_L > L_A - R_A > 0$. Here, we consider the two cases: $M_R - M_L \leq R_A - L_D$ and $M_R - M_L > R_A - L_D$.

First, suppose $M_R - M_L \leq R_A - L_D$. We show that by choosing M_R, candidate L_A wins the primary and $w_A^{*L} = 1$. For all $r_A \in [M_L, M_R]$ and $l_D \in X$,

$$\begin{aligned} \pi_R(r_A, R_A, l_D, L_D) &= 1; \\ V_{M_L}(l_D, L_D, r_A, R_A) &= R_A - r_A + M_L. \end{aligned}$$

If $r_A, r_D \geq M_R - (L_A - R_A)$, for each $n = A, D$,

$$V_{M_L}(M_R, L_A, r_n, R_n) = L_A - M_R + M_L.$$

Thus, $V_{M_L}(l_D, L_D, r_A, R_A) \leq V_{M_L}(M_R, L_A, r_A, R_A)$ for $r_A \geq M_R - (L_A - R_A)$. Next, notice that $M_R - M_L \leq R_A - L_D \leq L_A - L_D$. When $r_D \geq M_R - (L_A - R_A)$, regardless of $\pi_R(r_D, R_D, l_D, L_D)$, because $L_A - M_R \geq L_D - M_L$ and $R_D - r_D < L_A - M_R$,

$$\begin{aligned}
&V_{M_L}(l_D, L_D, r_D, R_D) = \\
&\pi_R(r_D, R_D, l_D, L_D)(L_D - l_D) + (1 - \pi_R(r_D, R_D, l_D, L_D))(R_D - r_D) + M_L \\
&\leq V_{M_L}(M_R, L_A, r_D, R_D).
\end{aligned}$$

Therefore, $V_{M_L}(l_D, L_D, r_n, R_n) \leq V_{M_L}(M_R, L_A, r_n, R_n)$ for each $n = A, D$. Then

$$\mathrm{E}V_{M_L}(l_D, L_D, r_A, r_D) < \mathrm{E}V_{M_L}(M_R, L_A, r_A, r_D).$$

Thus, we obtain $w_A^{*L} = 1$ in this case. Further, note that $r_A^*, r_D^* \in [M_R - (L_A - R_A), M_R]$, $l_D^* \in X$, and for each $n = A, D$, the following holds:

$$\begin{aligned}
\pi_R(r_n^*, R_n, l_A^*, L_A) &= 0 \\
\max_{r_n} \mathrm{E}V_{M_R}(r_n, R_n, l_A^*, l_D^*) &= \mathrm{E}V_{M_R}(r_n^*, R_n, l_A^*, l_D^*).
\end{aligned}$$

Notice that there is no l_D that candidate L_D can profitably deviate to by (B-2) and because for each $n = A, D$, $\pi_R(r_n^*, R_n, l_A^*, L_A) = 0$. Thus, l_D^* is unrestricted and we obtain the desired result.

Second, suppose $M_R - M_L > R_A - L_D$. Suppose that for each $n = A, D$, there is $\bar{d}_n > 0$ such that $\bar{d}_n = (L_A - R_n) - \frac{\left(F(M_R + \frac{L_D - R_n}{2})\right)^2}{\frac{1}{2} f(M_R + \frac{L_D - R_n}{2})}$. Fix $n = A, D$. When $\pi_R(r_n^*, R_n, l_A^*, L_A) = 0$ and $l_A^* = M_R$,

$$V_{M_L}(M_R, L_A, r_n^*, R_n) = L_A - M_R + M_L.$$

As shown in Lemma A-1, when candidate L_D maximizes $V_{M_L}(l_D, L_D, r_n^*, R_n)$, we have

$$V_{M_L}(\hat{l}_D(r_n^*), L_D, r_n^*, R_n) = R_n - r_n^* + \frac{\left(F(\frac{r_n^* + \hat{l}_D(r_n^*) + L_D - R_n}{2})\right)^2}{\frac{1}{2} f(\frac{r_n^* + \hat{l}_D(r_n^*) + L_D - R_n}{2})} + M_L.$$

Now, fix $n = A, D$. First, suppose $r_n^* \leq M_R$. Because $\bar{d}_n = (L_A - R_n) - \frac{\left(F(M_R + \frac{L_D - R_n}{2})\right)^2}{\frac{1}{2} f(M_R + \frac{L_D - R_n}{2})}$, we have

$$R_n - \left(M_R - \bar{d}_n\right) + \frac{\left(F(\frac{r_n^* + \hat{l}_D(r_n^*) + L_D - R_n}{2})\right)^2}{\frac{1}{2} f(\frac{r_n^* + \hat{l}_D(r_n^*) + L_D - R_n}{2})} = L_A - M_R.$$

By Lemma A-1, $M_L \leq \hat{l}_D(r_n^*) < r_n^* \leq M_R$ and thus $r_n^* + \hat{l}_D(r_n^*) < 2M_R$. By Assumption 1, we have

$$\max \frac{\left(F(\frac{r_n^*+\hat{l}_D(r_n^*)+L_D-R_n}{2})\right)^2}{\frac{1}{2}f(\frac{r_n^*+\hat{l}_D(r_n^*)+L_D-R_n}{2})} < F(M_R + \frac{L_D - R_n}{2}) \cdot \frac{F(M_R + \frac{L_D-R_n}{2})}{\frac{1}{2}f(M_R + \frac{L_D-R_n}{2})}.$$

Thus,

$$R_n - r_n^* + \frac{\left(F(\frac{r_n^*+\hat{l}_D(r_n^*)+L_D-R_n}{2})\right)^2}{\frac{1}{2}f(\frac{r_n^*+\hat{l}_D(r_n^*)+L_D-R_n}{2})} < L_A - M_R.$$

Thus $V_{M_L}(\hat{l}_D(r_n^*), L_D, r_n^*, R_n) < L_A - M_R + M_L$.

Second, $r_n^* > M_R$. Notice that $L_D \geq R_A - M_R + M_L$. Thus, for every $r_n \in [M_L, M_R]$,

$$\max_{l_D} V_{M_L}(\hat{l}_D(r_n), L_D, r_n, R_n) \geq L_D \geq R_n - M_R + M_L.$$

Then $V_{M_L}(\hat{l}_D(r_n), L_D, r_n, R_n) < L_A - M_R + M_L$ for $r_n \in [M_L, M_R]$ by the previous step of this proof.

Because n is arbitrary, for each $n = A, D$, $r_n^* \in [M_R - \bar{d}_n, M_R]$, $l_D^* \in X$, and the following holds:

$$\begin{aligned} \pi_R(r_n^*, R_n, l_A^*, L_A) &= 0 \\ \max_{r_n} \mathrm{E}V_{M_R}(r_n, R_n, l_A^*, l_D^*) &= \mathrm{E}V_{M_R}(r_n^*, R_n, l_A^*, l_D^*) \\ \max_{l_D} \mathrm{E}V_{M_L}(l_D, L_D, r_A^*, r_D^*) &< \mathrm{E}V_{M_L}(M_R, L_A, r_A^*, r_D^*). \end{aligned} \tag{B-2}$$

Thus, we obtain $w_A^{*L} = 1$. This completes the proof. □

Proofs of Results in Section 4

Proofs of Results in Section 4.1

Proof of Theorem 2. In a symmetric game, if $\bar{m} \notin \beta_R(\bar{m}, L_A) \cap \beta_L(\bar{m}, R_A)$, by Theorem 1, regime 1 arises. In the proof, the size of $\left|\frac{\mathrm{d}V_{M_L}(l_A^*, L_A, r_A^*, R_A)}{\mathrm{d}l_A}\right|$ is important. Thus, first we note the following. When the game is symmetric, by (12) and (13) of Theorem 1,

$$L_A - l_A^* - R_A + r_A^* = 2r_A^* \tag{B-3}$$

$$L_A + l_A^* - R_A + r_A^* = 0. \tag{B-4}$$

Thus, because the expected payoff for candidate L_A is decreasing at l_A^*,

$$\begin{aligned}
\frac{\mathrm{d}V_{M_L}(l_A^*, L_A, r_A^*, R_A)}{\mathrm{d}l_A} &= (1-\pi_R)'(L_A - l_A^* - R_A + r_A^*) - (1-\pi_R(r_A^*, R_A, l_A^*, L_A)) \\
= -F(\frac{l_A^* + r_A^* + L_A - R_A}{2}) &+ f(\frac{l_A^* + r_A^* + L_A - R_A}{2})\frac{(L_A - l_A^* - R_A + r_A^*)}{2} < 0 \\
&= -\frac{1}{2} + f(0)r_A^* < 0.
\end{aligned} \tag{B-5}$$

Step I: Computing the formula for $\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A}$, *and* $\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A}$.

By taking the derivative of (A-21), we have

$$\begin{aligned}
&\frac{\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} + \frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} + 1}{2} f(\frac{l_A^* + r_A^*}{2})(r_A^* - l_A^*) + F(\frac{l_A^* + r_A^*}{2})(1 - \frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} + \frac{\mathrm{d}r_A^*}{\mathrm{d}L_A}) \\
&= \left(\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} - \frac{\mathrm{d}l_D^*}{\mathrm{d}L_A}\right) F(\frac{l_D^* + r_A^* + L_D - R_A}{2}) \\
&\quad + (L_D + r_A^* - R_A - l_D^*) f(\frac{l_D^* + r_A^* + L_D - R_A}{2})\frac{(\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} + \frac{\mathrm{d}l_D^*}{\mathrm{d}L_A})}{2}.
\end{aligned}$$

Then by (A-20),

$$\left(\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} + \frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} + 1\right) f(0)\cdot r_A^* + \frac{1}{2}(1 - \frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} + \frac{\mathrm{d}r_A^*}{\mathrm{d}L_A}) = 2\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A}F(\frac{l_D^* + r_A^* + L_D - R_A}{2}). \tag{B-6}$$

Conversely, by taking the derivative of (A-23),

$$\begin{aligned}
&-\frac{\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} + \frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} + 1}{2} f(\frac{l_A^* + r_A^* + L_A - R_A}{2})(R_A + r_A^* - L_A - l_A^*) \\
&\quad + (1 - F(\frac{l_A^* + r_A^* + L_A - R_A}{2}))(\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} - \frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} - 1) \\
&\quad = (1 - F(\frac{l_A^* + r_D^* + L_A - R_D}{2}))(\frac{\mathrm{d}r_D^*}{\mathrm{d}L_A} - \frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} - 1) \\
&\qquad - (R_D + r_D^* - L_A - l_A^*) f(\frac{l_A^* + r_D^* + L_A - R_D}{2})\frac{\frac{\mathrm{d}r_D^*}{\mathrm{d}L_A} + \frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} + 1}{2}.
\end{aligned}$$

By applying (A-21) and (B-4),

$$\begin{aligned}
&-\left(\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} + \frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} + 1\right) f(0)r_A^* + \frac{1}{2}(\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} - \frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} - 1) \\
&\qquad = -2(1 - F(\frac{l_A^* + r_D^* + L_A - R_D}{2}))\left(\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} + 1\right).
\end{aligned} \tag{B-7}$$

By adding (B-6) and (B-7), because $F_D \equiv F(\frac{l_D^* + r_A^* + L_D - R_A}{2}) = 1 - F(\frac{l_A^* + r_D^* + L_A - R_D}{2})$, we obtain

$$\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A}\left(\frac{1}{2} - 2F_D\right) = \left(\frac{1}{2} + 2F_D\right)\left(\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} + 1\right) - \frac{1}{2}. \tag{B-8}$$

Further, by (B-7),

$$(\frac{1}{2} - f(0)r_A^*)\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} = \left(\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} + 1\right) \times X, \tag{B-9}$$

where

$$X = (\frac{1}{2} + f(0)r_A^*) - 2(1 - F(\frac{l_A^* + r_D^* + L_A - R_D}{2})) = \frac{1}{2} + f(0)r_A^* - 2F_D.$$

By (B-8) and (B-9), we obtain

$$\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A}\left(\frac{1}{2} - 2F_D - \frac{\frac{1}{2} - f(0)r_A^*}{X}\left(\frac{1}{2} + 2F_D\right)\right) = -\frac{1}{2}. \tag{B-10}$$

Thus,

$$\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} = \frac{\frac{1}{2}}{2F_D + \frac{\frac{1}{2}-f(0)r_A^*}{X}\left(\frac{1}{2} + 2F_D\right) - \frac{1}{2}}; \tag{B-11}$$

$$\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} = \frac{\frac{1}{2} - f(0)r_A^*}{X} \cdot \frac{\frac{1}{2}}{2F_D + \frac{\frac{1}{2}-f(0)r_A^*}{X}\left(\frac{1}{2} + 2F_D\right) - \frac{1}{2}} - 1; \tag{B-12}$$

$$\frac{\mathrm{d}F(\frac{l_A^*+r_A^*+L_A-R_A}{2})}{\mathrm{d}L_A} = \frac{\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} + \frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} + 1}{2} \cdot f(\frac{l_D^* + r_A^* + L_D - R_A}{2}). \tag{B-13}$$

Step II: Proving the signs of $\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A}$ and $\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A}$.
Step II-1: Showing $\frac{1}{2} - f(0)r_A^* > X > 0$. Notice that by (A-21) and (B-4),

$$0 < L_A - l_A^* - R_A + r_A^* = 2r_A^* < L_D - l_D^* - R_A + r_A^*. \tag{B-14}$$

Further, by (B-3)

$$0 = L_A + l_A^* - R_A + r_A^* > L_D + l_D^* - R_A + r_A^*. \tag{B-15}$$

Let $F_A \equiv F(\frac{l_A^*+r_A^*+L_A-R_A}{2})$, $f_D \equiv f(\frac{l_D^*+r_A^*+L_D-R_A}{2})$ and $f_A \equiv f(\frac{l_A^*+r_A^*+L_A-R_A}{2})$. Then by (B-15), $F_A > F_D$ and thus we can set $F_A = kF_D$ for some $k > 1$. Also, given the game is symmetric with respect to 0, by (B-15), we can set $f_A = mf_D$ for some $m > 1$. Further by Assumption 1 and (B-15), $\frac{F_A}{f_A} > \frac{F_D}{f_D}$ implies $k > m > 1$. By (A-21),

$$L_A - l_A^* - R_A + r_A^* : L_D - l_D^* - R_A + r_A^* = F_D : F_A = 1 : k. \tag{B-16}$$

Then, by applying (B-16) and (A-21),

$$\begin{aligned} X &= (F_A - F_D) + \left(f_A r_A^* - f_D\frac{L_D - l_D^* - R_A + r_A^*}{2}\right) \\ &= (F_A - F_D) + \left(mr_A^* - \frac{L_D - l_D^* - R_A + r_A^*}{2}\right) f_D \\ &= (k-1)\,F_D + (m-k)\,r_A^* f_D = \left((k-1)\frac{L_D - l_D^* - R_A + r_A^*}{2} + (m-k)r_A^*\right) f_D \\ &= ((k-1)k + (m-k))\,f_D r_A^* = (k^2 - 2k + m)f_D r_A^* > (k-1)^2 f_D r_A^* > 0. \end{aligned}$$

The first term in X is strictly positive by (B-14), and the second term is strictly negative. Thus,

$$\begin{aligned} X &< F(\frac{l_A^* + r_A^* + L_A - R_A}{2}) - F(\frac{l_D^* + r_A^* + L_D - R_A}{2}) \\ &< F(\frac{l_A^* + r_A^* + L_A - R_A}{2}) - f(\frac{l_A^* + r_A^* + L_A - R_A}{2})\frac{(L_A - l_A^* - R_A + r_A^*)}{2} \\ &= \frac{1}{2} - f(0)r_A^*. \end{aligned}$$

Notice that

$$2F_D + \frac{\frac{1}{2} - f(0)r_A^*}{X}\left(\frac{1}{2} + 2F_D\right) - \frac{1}{2} > 2F_D + \frac{1}{2} + 2F_D - \frac{1}{2} = 4F_D > 0. \qquad \text{(B-17)}$$

Thus,

$$\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} = \frac{\frac{1}{2}}{2F_D + \frac{\frac{1}{2} - f(0)r_A^*}{X}\left(\frac{1}{2} + 2F_D\right) - \frac{1}{2}} > 0.$$

Also, because

$$\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} + 1 = \frac{\frac{1}{2} - f(0)r_A^*}{X} \cdot \frac{\frac{1}{2}}{2F_D + \frac{\frac{1}{2} - f(0)r_A^*}{X}\left(\frac{1}{2} + 2F_D\right) - \frac{1}{2}} > 0,$$

we obtain

$$\frac{\mathrm{d}F(\frac{l_A^* + r_A^* + L_A - R_A}{2})}{\mathrm{d}L_A} = \frac{\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} + \frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} + 1}{2} \cdot f(\frac{l_D^* + r_A^* + L_D - R_A}{2}) > 0.$$

Step III: Proving the signs of $\frac{\mathrm{d}r_D^*}{\mathrm{d}L_A}$ and $\frac{\mathrm{d}l_D^*}{\mathrm{d}L_A}$ and Concluding.

Step III-1. Showing $\frac{\mathrm{d}l_D^*}{\mathrm{d}L_A} < 0$.

By taking the derivative of (A-20),

$$\begin{aligned} &\left(\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} - \frac{\mathrm{d}l_D^*}{\mathrm{d}L_A}\right)\frac{f(\frac{r_A^* + l_D^* + L_D - R_A}{2})}{2} \\ &= -\left((L_D + r_A^* - R_A - l_D^*)\frac{f'(\frac{r_A^* + l_D^* + L_D - R_A}{2})}{2} + f(\frac{r_A^* + l_D^* + L_D - R_A}{2})\right)\frac{\left(\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} + \frac{\mathrm{d}l_D^*}{\mathrm{d}L_A}\right)}{2}. \end{aligned}$$

Then

$$\begin{aligned} &\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A}\left(f(\frac{r_A^* + l_D^* + L_D - R_A}{2}) - \frac{(L_D + r_A^* - R_A - l_D^*)}{2}f'(\frac{r_A^* + l_D^* + L_D - R_A}{2})\right) \\ &= -\frac{(L_D + r_A^* - R_A - l_D^*)}{2}f'(\frac{r_A^* + l_D^* + L_D - R_A}{2})\frac{\mathrm{d}l_D^*}{\mathrm{d}L_A}. \end{aligned}$$

Because $\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} > 0$ and $f'(\frac{r_A^* + l_D^* + L_D - R_A}{2}) > 0$, $\frac{\mathrm{d}l_D^*}{\mathrm{d}L_A} < 0$.

Step III-2. Showing that there is a desired K and completing the proof.

By taking the derivative of (A-22),

$$\left(\frac{\mathrm{d}r_D^*}{\mathrm{d}L_A}-\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A}-1\right)\frac{f(\frac{r_D^*+l_A^*+L_A-R_D}{2})}{2}$$
$$=-\left((R_D+r_D^*-L_A-l_A^*)\frac{f'(\frac{r_D^*+l_A^*+L_A-R_D}{2})}{2}+f(\frac{r_D^*+l_A^*+L_A-R_D}{2})\right)\frac{\left(\frac{\mathrm{d}r_D^*}{\mathrm{d}L_A}+\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A}+1\right)}{2}.$$

Let $\Gamma=f(\frac{r_D^*+l_A^*+L_A-R_D}{2})+\frac{\left(R_D+r_D^*-L_A-l_A^*\right)}{2}f'(\frac{r_D^*+l_A^*+L_A-R_D}{2})$. Thus, we obtain

$$\frac{\mathrm{d}r_D^*}{\mathrm{d}L_A}=\frac{f(\frac{r_D^*+l_A^*+L_A-R_D}{2})-\Gamma}{f(\frac{r_D^*+l_A^*+L_A-R_D}{2})+\Gamma}\left(\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A}\right). \tag{B-18}$$

Notice that by Assumption 1, $\frac{\mathrm{d}\left(\frac{f(x)}{1-F(x)}\right)}{\mathrm{d}x}\geq 0$. Thus $\frac{f(x)}{1-F(x)}+\frac{f'(x)}{f(x)}>0$. Applying this, because

$$\begin{aligned}\Gamma&=f(\frac{r_D^*+l_A^*+L_A-R_D}{2})+\frac{(R_D+r_D^*-L_A-l_A^*)}{2}f'(\frac{r_D^*+l_A^*+L_A-R_D}{2})\\&=\left(1-F(\frac{r_D^*+l_A^*+L_A-R_D}{2})\right)\times\\&\left(\frac{f(\frac{r_D^*+l_A^*+L_A-R_D}{2})}{1-F(\frac{r_D^*+l_A^*+L_A-R_D}{2})}+\frac{\frac{\left(R_D+r_D^*-L_A-l_A^*\right)}{2}}{1-F(\frac{r_D^*+l_A^*+L_A-R_D}{2})}f'(\frac{r_D^*+l_A^*+L_A-R_D}{2})\right)\\&=\left(1-F(\frac{r_D^*+l_A^*+L_A-R_D}{2})\right)\times\left(\frac{f(\frac{r_D^*+l_A^*+L_A-R_D}{2})}{1-F(\frac{r_D^*+l_A^*+L_A-R_D}{2})}+\frac{f'(\frac{r_D^*+l_A^*+L_A-R_D}{2})}{f(\frac{r_D^*+l_A^*+L_A-R_D}{2})}\right)\\&>0,\end{aligned} \tag{B-19}$$

together with the fact that $\frac{r_D^*+l_A^*+L_A-R_D}{2}>0$ implies $f'(\frac{r_D^*+l_A^*+L_A-R_D}{2})<0$, it is held that

$$1>\frac{f(\frac{r_D^*+l_A^*+L_A-R_D}{2})-\Gamma}{f(\frac{r_D^*+l_A^*+L_A-R_D}{2})+\Gamma}>0. \tag{B-20}$$

Therefore, by (B-18), r_D^* moves in the same direction as l_A^*.

Remember that

$$\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A}=\frac{\frac{\frac{1}{2}-f(0)r_A^*}{X}\cdot\frac{1}{2}}{2F_D+\frac{\frac{1}{2}-f(0)r_A^*}{X}\left(\frac{1}{2}+2F_D\right)-\frac{1}{2}}-1.$$

By Step II and (B-17), the denominator and numerator of the first term are strictly positive. Thus $\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A}>0$ if and only if

$$2F_D\left(1+\frac{\frac{1}{2}-f(0)r_A^*}{X}\right)=2F_D\left(1+\frac{\frac{1}{2}-f(0)r_A^*}{\frac{1}{2}+f(0)r_A^*-2F_D}\right)<\frac{1}{2}.$$

Define K to satisfy $2K\left(1+\frac{\frac{1}{2}-f(0)r_A^*}{\frac{1}{2}+f(0)r_A^*-2K}\right)=\frac{1}{2}$. Then, $8K^2-6K+(\frac{1}{2}+f(0)r_A^*)=0$. Notice that $8(0)-6(0)+(\frac{1}{2}+f(0)r_A^*)>0$, and $8(\frac{1}{2})^2-6(\frac{1}{2})+(\frac{1}{2}+f(0)r_A^*)=f(0)r_A^*-\frac{1}{2}<0$. Thus, K exists in $(0,\frac{1}{2})$. Further, because the quadratic equation has two solutions and $8(1)^2-6(1)+(\frac{1}{2}+f(0)r_A^*)>0$, another solution exists in $(\frac{1}{2},1)$. Thus, a solution that is between 0 and $\frac{1}{2}$ is unique. Now, it is clear that $2F_D\left(1+\frac{\frac{1}{2}-f(0)r_A^*}{X}\right)<\frac{1}{2}$ if and only if $F_D>K$. $\square$

Proof of Theorem 3. Let $R_A-R_D=\delta$. In what follows, $(g(x))'$ denotes the first derivative of $g(x)$. We denote $r_A^S+l_D^S+L_D-R_A=x^S$ and $r_A^N+l_D^N+L_D-R_A=x^N$. We denote $r_D^S+l_A^S+L_A-R_D=y^S$ and $r_D^N+l_A^N+L_A-R_D=y^N$.

Suppose that $|f_N(x)-f_S(x)|<\bar{\epsilon}$ for some arbitrarily small $\bar{\epsilon}>0$. We will show $r_A^S<r_A^N$ and $l_A^S>l_A^N$. Then, because we can recursively repeat the same argument, we will obtain the desired result. By the equilibrium condition for regime 1, we must have

$$V_{M_R}^S(l_A^S,L_A,r_A^S,R_A)+M_R=V_{M_R}^S(l_A^S,L_A,r_D^S,R_D)+M_R. \tag{B-21}$$

Also, by symmetry,

$$V_{M_R}^S(l_A^S,L_A,r_A^S,R_A)=V_{M_R}^N(l_A^N,L_A,r_A^N,R_A)=L_A-M_R. \tag{B-22}$$

By Lemma A-1, in regime 2, for each society $s=N,S$, we must have

$$V_{M_R}^s(l_A^s,L_A,r_D^s,R_D)+M_R=l_A^s+L_A+(1-F_s(\frac{y^s}{2}))\cdot(r_D^s-l_A^s-\delta). \tag{B-23}$$

By comparing (B-22) and (B-23) for each society $s=N,S$, we must have

$$l_A^s+\frac{(1-F_s(\frac{y^s}{2}))^2}{\frac{1}{2}f_s(\frac{y^s}{2})}=l_A^s+(1-F_s(\frac{y^s}{2}))\cdot(r_D^s-l_A^s-\delta)=0. \tag{B-24}$$

By comparing (B-24) for societies S and N, we obtain

$$(F_S(\frac{y^S}{2})-F_N(\frac{y^N}{2}))\delta+F_S(\frac{y^S}{2})l_A^S-F_N(\frac{y^N}{2})l_A^N=(1-F_N(\frac{y^N}{2}))r_D^N-(1-F_S(\frac{y^S}{2}))r_D^S. \tag{B-25}$$

As $y^S>0$, property 2 implies $F_S(\frac{y^S}{2})>F_N(\frac{y^S}{2})$ and thus for some small $\epsilon_y>0$, $F_S(\frac{y^S}{2})-F_N(\frac{y^S}{2})=\epsilon_y$ and we have

$$\begin{aligned}F_S(\frac{y^S}{2})-F_N(\frac{y^N}{2})&=F_S(\frac{y^S}{2})-F_N(\frac{y^S}{2})-F_N(\frac{y^N}{2})+F_N(\frac{y^S}{2})\\&=\epsilon_y-f_N(\frac{y^N}{2})\cdot\frac{(r_D^N-r_D^S)+(l_A^N-l_A^S)}{2}.\end{aligned} \tag{B-26}$$

Consider how $F_N(\frac{r+l+\delta}{2})\cdot l$ changes when r and l change infinitesimally, denoted by $\mathrm{d}F_N(\frac{r+l+\delta}{2})\cdot l$, which equals

$$\left(\frac{f_N(\frac{r+l+\delta}{2})l}{2}\right)\mathrm{d}r+\left(\frac{f_N(\frac{r+l+\delta}{2})}{2}+F_N(\frac{r+l+\delta}{2})\right)\mathrm{d}l.$$

Thus, $F_S(\frac{y^S}{2})l_A^S - F_N(\frac{y^N}{2})l_A^N$ is given by $F_S(\frac{y^S}{2})l_A^S - F_N(\frac{y^S}{2})l_A^S = \epsilon_y l_A^S$ plus $F_N(\frac{y^S}{2})l_A^S - F_N(\frac{y^N}{2})l_A^N$, which is

$$\epsilon_y l_A^S + \left(\frac{f_N(\frac{y^N}{2})l_A^N}{2}\right)(r_D^S - r_D^N) + \left(\frac{f_N(\frac{y^N}{2})l_A^N}{2} + F_N(\frac{y^N}{2})\right)(l_A^S - l_A^N). \tag{B-27}$$

Similarly, consider how $(1 - F_N(\frac{r+l+\delta}{2})) \cdot r$ changes when r and l change infinitesimally, denoted by $\mathrm{d}(1 - F_N(\frac{r+l+\delta}{2})) \cdot r$, which equals

$$\left(\frac{-f_N(\frac{r+l+\delta}{2})r}{2} + (1 - F_N(\frac{r+l+\delta}{2}))\right)\mathrm{d}r - \frac{f_N(\frac{r+l+\delta}{2})r}{2}\mathrm{d}l.$$

Thus, $(1-F_N(\frac{y^N}{2}))r_D^N - (1-F_S(\frac{y^S}{2}))r_D^S$ is given by $(1-F_N(\frac{y^S}{2}))r_D^S - (1-F_S(\frac{y^S}{2}))r_D^S = \epsilon_y r_D^S$ plus $(1 - F_N(\frac{y^N}{2}))r_D^N - (1 - F_N(\frac{y^S}{2}))r_D^S$, which is

$$\epsilon_y r_D^S + \left(\frac{-f_N(\frac{y^N}{2})r_D^N}{2} + (1 - F_N(\frac{y^N}{2}))\right)(r_D^N - r_D^S) - \frac{f_N(\frac{y^N}{2})r_D^N}{2}(l_A^N - l_A^S). \tag{B-28}$$

Combining the three terms (B-26), (B-27), and (B-28) into (B-25), we obtain

$$\begin{aligned}&\left(\epsilon_y - f_N(\frac{y^N}{2}) \cdot \frac{(r_D^N - r_D^S) + (l_A^N - l_A^S)}{2}\right)\delta \\ &+ \epsilon_y l_A^S + \left(\frac{f_N(\frac{y^N}{2})l_A^N}{2}\right)(r_D^S - r_D^N) + \left(\frac{f_N(\frac{y^N}{2})l_A^N}{2} + F_N(\frac{y^N}{2})\right)(l_A^S - l_A^N) \\ &= \epsilon_y r_D^S + \left(\frac{-f_N(\frac{y^N}{2})r_D^N}{2} + (1 - F_N(\frac{y^N}{2}))\right)(r_D^N - r_D^S) - \frac{f_N(\frac{y^N}{2})r_D^N}{2}(l_A^N - l_A^S).\end{aligned}$$

By organizing terms, we obtain

$$\begin{aligned}\epsilon_y(\delta + l_A^S - r_D^S) = &\left(\frac{f_N(\frac{y^N}{2})}{2}\delta + \frac{f_N(\frac{y^N}{2})l_A^N}{2} - \frac{f_N(\frac{y^N}{2})r_D^N}{2} + (1 - F_N(\frac{y^N}{2}))\right)(r_D^N - r_D^S) \\ + &\left(\frac{f_N(\frac{y^N}{2})}{2}(L_A - R_D) + \frac{f_N(\frac{y^N}{2})l_A^N}{2} + F_N(\frac{y^N}{2}) - \frac{f_N(\frac{y^N}{2})r_D^N}{2}\right)(l_A^N - l_A^S).\end{aligned} \tag{B-29}$$

We can simplify the RHS because by (A-22), which implies $1-F_N(\frac{y^N}{2}) = \frac{f_N(\frac{y^N}{2})}{2}(r_D^N-\delta-l_A^N)$, the first term becomes

$$\frac{f_N(\frac{y^N}{2})}{2}\delta + \frac{f_N(\frac{y^N}{2})l_A^N}{2} - \frac{f_N(\frac{y^N}{2})r_D^N}{2} + (1 - F_N(\frac{y^N}{2})) = 0,$$

and the second term becomes

$$\frac{f_N(\frac{y^N}{2})}{2}\delta + \frac{f_N(\frac{y^N}{2})l_A^N}{2} + F_N(\frac{y^N}{2}) - \frac{f_N(\frac{y^N}{2})r_D^N}{2} = 1 - f_N(\frac{y^N}{2})(r_D^N - \delta - l_A^N).$$

Because $1 - F_N(\frac{y^N}{2}) = \frac{f_N(\frac{y^N}{2})}{2}(r_D^N - \delta - l_A^N) < \frac{1}{2}$, $f_N(\frac{y^N}{2}) \cdot (r_D^N - \delta - l_A^N) < 1$. As a result, by (B-29), we obtain

$$l_A^N - l_A^S = \frac{-\epsilon_y(r_D^S - l_A^S - \delta)}{1 - f_N(\frac{y^N}{2})(r_D^N - \delta - l_A^N)} < 0.$$

Thus, we obtain $r_A^S < r_A^N$. □

Lemma B-1. *In the symmetric game, an equilibrium policy for party R's candidate satisfies $r_A < r_D - (R_A - R_D)$.*

Proof of Lemma B-1. See Proposition 3 in Hummel (2013). □

Proof of Theorem 4. We continue to use the notation for x^S, x^N, y^S, y^N as defined in the proof of Theorem 3. Note that under both scenarios, $0 \le F_N(x) - F_S(x) < \bar{\epsilon} \cdot \bar{\epsilon}$ for every $x \in [-a, a]$ and further,

(A) Under the scenario of extremists' change, for every $x_0, x_1 \in [-a + \bar{\epsilon}, a]$, $F_N(x_0) - F_S(x_0) = F_N(x_1) - F_S(x_1)$;

(B) Under the scenario of centrists' change, for every $x \in [-a, a] \setminus [-\bar{\epsilon}, \bar{\epsilon}]$, $F_N(x) = F_S(x)$.

Because $V_{M_R}^S(l_A^S, L_A, r_A^S, R_A) = F_S(0)(L_A + l_A^S) + (1 - F_S(0))(R_A + r_A^S) - M_R$ and $l_A^S = -r_A^S$, we obtain

$$\begin{aligned}
&V_{M_L}^N(l_A^N, L_A, r_A^N, R_A) - V_{M_L}^S(l_A^S, L_A, r_A^S, R_A) \\
&= V_{M_R}^S(l_A^S, L_A, r_A^S, R_A) - V_{M_R}^N(l_A^N, L_A, r_A^N, R_A) \\
&= F_N(\frac{r_A^N + l_A^N}{2})l_A^N + (1 - F_N(\frac{r_A^N + l_A^N}{2}))r_A^N.
\end{aligned} \tag{B-30}$$

In regime 1 equilibria, the following holds:

$$\begin{aligned}
&V_{M_L}^N(l_A^N, L_A, r_A^N, R_A) - V_{M_L}^S(l_A^S, L_A, r_A^S, R_A) \\
&= V_{M_L}^N(l_D^N, L_D, r_A^N, R_A) - V_{M_L}^S(l_D^S, L_D, r_A^S, R_A) \\
&= V_{M_R}^S(l_A^S, L_A, r_D^S, R_D) - V_{M_R}^N(l_A^N, L_A, r_D^N, R_D).
\end{aligned} \tag{B-31}$$

Step I. Calculating (B-31).

Consider $\mathrm{d}(r - \delta - l)(1 - F(\frac{r+l+\delta}{2}))$ when r and l change. It is given by

$$\begin{aligned}
&\left((1 - F(\frac{r + l + \delta}{2})) - \frac{r - \delta - l}{2} f(\frac{r + l + \delta}{2})\right) \mathrm{d}r \\
&- \left((1 - F(\frac{r + l + \delta}{2})) + \frac{r - \delta - l}{2} f(\frac{r + l + \delta}{2})\right) \mathrm{d}l.
\end{aligned}$$

Similarly to (B-27) and (B-28) in the proof of Theorem 3, by applying the first-order condition (A-22), we can compute that $(r_D^S-\delta-l_A^S)(1-F_N(\frac{y^S}{2}))-(r_D^N-\delta-l_A^N)(1-F_N(\frac{y^N}{2}))$ equals

$$-\left((1-F_N(\frac{y^N}{2}))+\frac{r_D^N-\delta-l_A^N}{2}f_N(\frac{y^N}{2})\right)(l_A^S-l_A^N).$$

Let $F_N(\frac{y^S}{2})-F_S(\frac{y^S}{2})=\epsilon_1$. By property 5, $\epsilon_1>0$. The RHS of (B-30) is given by

$$\begin{aligned}
&V_{M_R}^S(l_A^S,L_A,r_D^S,R_D)-V_{M_R}^N(l_A^N,L_A,r_D^N,R_D)\\
&=l_A^S+(r_D^S-\delta-l_A^S)(1-F_S(\frac{y^S}{2}))-l_A^N-(r_D^N-\delta-l_A^N)(1-F_N(\frac{y^N}{2}))\\
&-(r_D^S-\delta-l_A^S)(1-F_N(\frac{y^S}{2}))+(r_D^S-\delta-l_A^S)(1-F_N(\frac{y^S}{2}))\\
&=l_A^S-l_A^N+(r_D^S-\delta-l_A^S)\epsilon_1-\left((1-F_N(\frac{y^N}{2}))+\frac{r_D^N-\delta-l_A^N}{2}f_N(\frac{y^N}{2})\right)(l_A^S-l_A^N)\\
&=(r_D^S-\delta-l_A^S)\epsilon_1+\left(F_N(\frac{y^N}{2})-\frac{r_D^N-\delta-l_A^N}{2}f_N(\frac{y^N}{2})\right)(l_A^S-l_A^N). \qquad \text{(B-32)}
\end{aligned}$$

Similarly, consider $\mathrm{d}(r-\delta-l)F(\frac{r+l-\delta}{2})$ when r and l change. It is given by

$$\begin{aligned}
&\left(F(\frac{r+l-\delta}{2})+\frac{r-\delta-l}{2}f(\frac{r+l-\delta}{2})\right)\mathrm{d}r\\
&-\left(F(\frac{r+l-\delta}{2})-\frac{r-\delta-l}{2}f(\frac{r+l-\delta}{2})\right)\mathrm{d}l.
\end{aligned}$$

Then, by applying the first-order condition (A-20), we can compute that $(r_A^S-\delta-l_D^S)F_N(\frac{x^S}{2})-(r_A^N-\delta-l_D^N)F_N(\frac{x^N}{2})$ equals

$$\left(F_N(\frac{x^N}{2})+\frac{r_A^N-\delta-l_D^N}{2}f_N(\frac{x^N}{2})\right)(r_A^S-r_A^N).$$

Let $F_N(\frac{x^S}{2})-F_S(\frac{x^S}{2})=\epsilon_0$. By property 5, $\epsilon_0>0$. The RHS of (B-31) is given by

$$\begin{aligned}
&V_{M_L}^S(l_D^S,L_D,r_A^S,R_A)-V_{M_L}^N(l_D^N,L_D,r_A^N,R_A)\\
&=-r_A^S+(r_A^S-\delta-l_D^S)F_S(\frac{x^S}{2})+r_A^N-(r_A^N-\delta-l_D^N)F_N(\frac{x^N}{2})\\
&-(r_A^S-\delta-l_D^S)F_N(\frac{x^S}{2})+(r_A^S-\delta-l_D^S)F_N(\frac{x^S}{2})\\
&=r_A^N-r_A^S-(r_A^S-\delta-l_D^S)\epsilon_0+\left(F_N(\frac{x^N}{2})+\frac{r_A^N-\delta-l_D^N}{2}f_N(\frac{x^N}{2})\right)(r_A^S-r_A^N). \qquad \text{(B-33)}
\end{aligned}$$

Let

$$\chi_1=1-F_N(\frac{x^N}{2})-\frac{r_A^N-\delta-l_D^N}{2}f_N(\frac{x^N}{2}) \text{ and } \chi_2=F_N(\frac{y^N}{2})-\frac{r_D^N-\delta-l_A^N}{2}f_N(\frac{y^N}{2}).$$

By the results of Step I into (B-31), equating $V^N_{M_L}(l^N_D, L_D, r^N_A, R_A) - V^S_{M_L}(l^S_D, L_D, r^S_A, R_A)$ and $V^S_{M_R}(l^S_A, L_A, r^S_D, R_D) - V^N_{M_R}(l^N_A, L_A, r^N_D, R_D)$ yields

$$(r^S_A - \delta - l^S_D)\epsilon_0 - \chi_1(r^S_A - r^N_A) = (r^S_D - \delta - l^S_A)\epsilon_1 + \chi_2(l^S_A - l^N_A). \tag{B-34}$$

Therefore,

$$r^S_A - r^N_A = \frac{\chi_2}{\chi_1}(l^S_A - l^N_A) + \frac{(r^S_D - \delta - l^S_A)\epsilon_1 - (r^S_A - \delta - l^S_D)\epsilon_0}{\chi_1}.$$

Finally, we obtain

$$r^N_A - r^S_A = \frac{\chi_2}{\chi_1}(l^N_A - l^S_A) - \frac{(r^S_D - \delta - l^S_A)(\epsilon_1 - \epsilon_0)}{\chi_1}. \tag{B-35}$$

Let $F_N(0) - F_S(0) = \epsilon_A$. Then $V^N_{M_L}(l^N_A, L_A, r^N_A, R_A) - V^S_{M_L}(l^S_A, L_A, r^S_A, R_A)$ is rewritten as

$$\begin{aligned}
&F_S(0)l^S_A + (1 - F_S(0))r^S_A - F_N(\frac{r^N_A + l^N_A}{2})l^N_A - (1 - F_N(\frac{r^N_A + l^N_A}{2}))r^N_A \\
&= F_S(0)l^S_A + (1 - F_S(0))r^S_A - F_N(\frac{r^N_A + l^N_A}{2})l^N_A - (1 - F_N(\frac{r^N_A + l^N_A}{2}))r^N_A \\
&- F_N(0)l^S_A - (1 - F_N(0))r^S_A + F_N(0)l^S_A + (1 - F_N(0))r^S_A \\
&= 2\epsilon_A r^S_A + \left(F_N(0)l^S_A - F_N(\frac{r^N_A + l^N_A}{2})l^N_A\right) + \left((1 - F_N(0))r^S_A - (1 - F_N(\frac{r^N_A + l^N_A}{2}))r^N_A\right).
\end{aligned} \tag{B-36}$$

Consider how $G(\frac{x+y}{2}) \cdot x$ when x and y change where we denote $g(x) = \frac{\mathrm{d}G(x)}{\mathrm{d}x}$. Then

$$\mathrm{d}(G(\frac{x+y}{2}) \cdot x) = (\frac{g(\frac{x+y}{2}) \cdot x}{2} + G(\frac{x+y}{2}))\mathrm{d}x + \frac{g(\frac{x+y}{2}) \cdot x}{2}\mathrm{d}y.$$

Consider how $(1 - G(\frac{x+y}{2})) \cdot y$ when x and y change. Then

$$\mathrm{d}(1 - G(\frac{x+y}{2})) \cdot y = (\frac{-g(\frac{x+y}{2}) \cdot y}{2} + (1 - G(\frac{x+y}{2})))\mathrm{d}y - \frac{g(\frac{x+y}{2}) \cdot y}{2}\mathrm{d}x.$$

Then the second term of (B-36) is

$$\begin{aligned}
&F_N(\frac{r^S_A + l^S_A}{2})l^S_A - F_N(\frac{r^N_A + l^N_A}{2})l^N_A \\
&= \left[\frac{f_N(\frac{r^S_A + l^S_A}{2}) \cdot l^S_A}{2} + F_N(\frac{r^S_A + l^S_A}{2})\right] \cdot (l^S_A - l^N_A) + \frac{f_N(\frac{r^S_A + l^S_A}{2}) \cdot l^S_A}{2} \cdot (r^S_A - r^N_A).
\end{aligned}$$

Similarly, the third term of (B-36) is

$$\begin{aligned}
&(1 - F_N(\frac{r^S_A + l^S_A}{2}))r^S_A - (1 - F_N(\frac{r^N_A + l^N_A}{2}))r^N_A \\
&= \left[(1 - F_N(\frac{r^S_A + l^S_A}{2})) - \frac{f_N(\frac{r^S_A + l^S_A}{2}) \cdot r^S_A}{2}\right] \cdot (r^S_A - r^N_A) - \frac{f_N(\frac{r^S_A + l^S_A}{2}) \cdot r^S_A}{2} \cdot (l^S_A - l^N_A).
\end{aligned}$$

Substituting the two terms into (B-36), we obtain

$$
\begin{aligned}
&V_{M_L}^N(l_A^N, L_A, r_A^N, R_A) - V_{M_L}^S(l_A^S, L_A, r_A^S, R_A)\\
&= 2\epsilon_A r_A^S + \left[\frac{f_N(0)\cdot r_A^S}{2} - F_N(0)\right]\cdot(l_A^N - l_A^S) + \frac{f_N(0)\cdot r_A^S}{2}\cdot(r_A^N - r_A^S)\\
&+\left[\frac{f_N(0)\cdot r_A^S}{2} - (1-F_N(0))\right]\cdot(r_A^N - r_A^S) + \frac{f_N(0)\cdot r_A^S}{2}\cdot(l_A^N - l_A^S)\\
&= 2\epsilon_A r_A^S + (f_N(0)r_A^S - F_N(0))\cdot(l_A^N - l_A^S) + (f_N(0)r_A^S - (1-F_N(0)))(r_A^N - r_A^S). \quad \text{(B-37)}
\end{aligned}
$$

Thus, by (B-31),

$$
\begin{aligned}
&2\epsilon_A r_A^S + (f_N(0)r_A^S - F_N(0))\cdot(l_A^N - l_A^S) + (f_N(0)r_A^S - (1-F_N(0)))(r_A^N - r_A^S)\\
&= (r_D^S - \delta - l_A^S)\epsilon_1 - \chi_2(l_A^N - l_A^S).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\chi_2\left(1 + \tfrac{f_N(0)r_A^S - F_N(0)}{\chi_2} + \tfrac{f_N(0)r_A^S-(1-F_N(0))}{\chi_1}\right)(l_A^N - l_A^S)\\
&= (r_D^S - \delta - l_A^S)\epsilon_1 - 2\epsilon_A r_A^S - \tfrac{f_N(0)r_A^S-(1-F_N(0))}{\chi_1}(r_D^S - \delta - l_A^S)(\epsilon_1 - \epsilon_0).
\end{aligned}
\quad \text{(B-38)}
$$

Step II. Completing the proof.

By applying (A-20) and (A-22),

$$\chi_1 = 1 - 2F_N(\frac{x^N}{2}) > 0 \text{ and } \chi_2 = 2F_N(\frac{y^N}{2}) - 1 > 0.$$

By the slopes of expected payoffs, $f_N(0)r_A^S - F_N(0) < 0$ and $f_N(0)r_A^S - (1-F_N(0)) < 0$. Therefore, $f_N(0)r_A^S < \frac{1}{2}$. Therefore, for some γ, we have

$$1 + \frac{f_N(0)r_A^S - F_N(0)}{\chi_2} + \frac{f_N(0)r_A^S - (1-F_N(0))}{\chi_1} = 2\cdot\frac{f_N(0)r_A^S - F_N(\frac{x^N}{2})}{\chi_1} + \gamma.$$

When $\epsilon_0 = \epsilon_1 = \epsilon_A$ by (A) under the scenario of extremists' change, the RHS of (B-38) is

$$
\begin{aligned}
&(r_D^S - \delta - l_A^S)\epsilon_1 - 2\epsilon_A r_A^S + \frac{(1-F_N(0)) - f_N(0)r_A^S}{\chi_1}(r_D^S - \delta - l_A^S)(\epsilon_1 - \epsilon_0)\\
&= \epsilon_1(r_D^S - \delta - l_A^S - 2r_A^S) = \epsilon_1(r_D^S - \delta - r_A^S) > 0,
\end{aligned}
$$

where the last inequality is due to Lemma B-1, which shows $r_D^S - \delta - r_A^S > 0$.

When $0 = \epsilon_0 = \epsilon_1 < \epsilon_A$ by (B) under the scenario of centrists' change, the RHS of (B-38) is

$$
\begin{aligned}
&(r_D^S - \delta - l_A^S)\epsilon_1 - 2\epsilon_A r_A^S + \frac{(1-F_N(0)) - f_N(0)r_A^S}{\chi_1}(r_D^S - \delta - l_A^S)(\epsilon_1 - \epsilon_0)\\
&= -2\epsilon_A r_A^S < 0.
\end{aligned}
$$

Consider the LHS of (B-38). By (A-20) and (A-21), $r_A^S = \frac{(F_S(\frac{x^S}{2}))^2}{\frac{1}{2}f_S(\frac{x^S}{2})}$. Thus, the LHS is positive if and only if $\frac{(F_S(\frac{x^S}{2}))^2}{\frac{1}{2}f_S(\frac{x^S}{2})} < \frac{F_N(\frac{x^N}{2})}{f_N(0)}$. By Assumption 1, we have $\frac{F_S(\frac{x^S}{2})}{f_S(\frac{x^S}{2})} < \frac{F_S(0)}{f_S(0)} = \frac{\frac{1}{2}}{f_S(0)}$. By property 5, we have $F_S(\frac{x^S}{2}) < F_N(\frac{x^S}{2})$. Thus, we have $\frac{(F_S(\frac{x^S}{2}))^2}{f_S(\frac{x^S}{2})} < \frac{\frac{1}{2}F_N(\frac{x^S}{2})}{f_S(0)}$. Because x^S and x^N can be arbitrarily close, we have $r_A^S = \frac{(F_S(\frac{x^S}{2}))^2}{f_S(\frac{x^S}{2})} < \frac{\frac{1}{2}F_N(\frac{x^N}{2})}{f_S(0)}$. Given $f_N(0)r_A^S < F_N(\frac{x^N}{2})$, the LHS of (B-38) is negative. Combining these observations, we obtain the desired result. □

Proofs of Results in Section 4.2

Proposition B-1. *Let F is uniformly distributed in $[-1,1]$. In a regime 1 equilibrium, the following holds.*

$$\frac{\partial(r_A^* - l_A^*)}{\partial L_A} < 0 \qquad \textit{and} \qquad \frac{\partial(r_A^* - l_A^*)}{\partial L_D} > 0.$$

Proof of Proposition B-1. From Proposition 1 we have that:

$$\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} = \frac{\left((r_A^* - R_A - a) + (L_A - L_D)\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} + L_A\right)}{\sqrt{(2(L_A - L_D)(r_A^* - R_A - a) + (L_A^2 - L_D^2))}} \tag{B-39}$$

$$\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} = -\frac{(R_A - R_D)\left(-1 - \frac{\mathrm{d}l_A^*}{\mathrm{d}L_A}\right)}{\sqrt{(2(R_A - R_D)(b - L_A - l_A^*) + (R_A^2 - R_D^2))}}. \tag{B-40}$$

Let's assume for a moment a symmetric case in a regime 1 equilibrium and let

$$\begin{aligned}\Psi &:= \left(2(L_A - L_D)(r_A^* - R_A - a) + (L_A^2 - L_D^2)\right)\\ &= \left(2(R_A - R_D)(b - L_A - l_A^*) + (R_A^2 - R_D^2)\right) \in (0, b).\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} = \Psi^{-1/2}\left((r_A^* - R_A - a) + (L_A - L_D)\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} + L_A\right) \tag{B-41}$$

$$\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} = \Psi^{-1/2}(R_A - R_D)\left(1 + \frac{\mathrm{d}l_A^*}{\mathrm{d}L_A}\right). \tag{B-42}$$

Combining (B-41) with (B-42),

$$\begin{aligned}\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} &= \Psi^{-1/2}\left((r_A^* - R_A - a) + (L_A - L_D)\Psi^{-1/2}(R_A - R_D)\left(1 + \frac{\mathrm{d}l_A^*}{\mathrm{d}L_A}\right) + L_A\right)\\ &= \frac{\Psi^{-1/2}\left((L_A - R_A) + (r_A^* - a) + (L_A - L_D)\Psi^{-1/2}(R_A - R_D)\right)}{1 - \Psi^{-1}(L_A - L_D)(R_A - R_D)}\\ &= \frac{\Psi^{-1/2}((L_A - R_A) + (r_A^* - a)) + (L_A - L_D)\Psi^{-1}(R_A - R_D)}{1 - \Psi^{-1}(L_A - L_D)(R_A - R_D)}.\end{aligned}$$

By symmetry, we obtain

$$\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} = \frac{\Psi^{-1/2}(r_A^* - a) + \Psi^{-1}(L_A - L_D)^2}{1 - \Psi^{-1}(L_A - L_D)^2} > 0.$$

Symmetrically, we obtain

$$\begin{aligned}\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} &= \frac{\Psi^{-1/2}(R_A - R_D) + \Psi^{-1}(R_A - R_D)(r_A^* - a)}{1 - \Psi^{-1}(L_A - L_D)(R_A - R_D)} \\ &= \frac{\Psi^{-1/2}(L_A - L_D) + \Psi^{-1}(L_A - L_D)(r_A^* - a)}{1 - \Psi^{-1}(L_A - L_D)^2} > 0.\end{aligned}$$

By some calculation, because

$$\frac{\Psi^{-1/2}(r_A^* - a) + \Psi^{-1}(L_A - L_D)^2}{1 - \Psi^{-1}(L_A - L_D)^2} > \frac{\Psi^{-1/2}(L_A - L_D) + \Psi^{-1}(L_A - L_D)(r_A^* - a)}{1 - \Psi^{-1}(L_A - L_D)^2}$$

we obtain $(r_A^* + b) > R_A - R_D$. Therefore, in a regime 1 equilibrium, we have that $R_A - R_D < b$ and $r_A^* \geq 0$, where the last inequality holds. It follows that $0 < \frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} < \frac{\mathrm{d}l_A^*}{\mathrm{d}L_A}$ and starting from a symmetry situation, we conclude that when L_A increases, $r_A^* - l_A^*$ decreases.

By the similar procedure, starting from a symmetric situation, we have

$$\frac{\mathrm{d}l_A^*}{\mathrm{d}L_D} = \Psi^{-1/2}\left(-(r_A^* - R_A - a) + (L_A - L_D)\frac{\mathrm{d}r_A^*}{\mathrm{d}L_D} - L_D\right) \tag{B-43}$$

$$\frac{\mathrm{d}r_A^*}{\mathrm{d}L_D} = \Psi^{-1/2}(R_A - R_D)\left(\frac{\mathrm{d}l_A^*}{\mathrm{d}L_D}\right) \tag{B-44}$$

Combining (B-43) with (B-44), and in a symmetric situation,

$$\begin{aligned}\frac{\mathrm{d}l_A^*}{\mathrm{d}L_D} &= \Psi^{-1/2}\left(-(r_A^* - R_A - a) + (L_A - L_D)\Psi^{-1/2}(R_A - R_D)\left(\frac{\mathrm{d}l_A^*}{\mathrm{d}L_D}\right) - L_D\right) \\ &= \frac{\Psi^{-1/2}(-(r_A^* - a) + (L_A - L_D))}{1 - \Psi^{-1}(L_A - L_D)^2} < 0. \end{aligned} \tag{B-45}$$

Using (B-44) into (B-45), we obtain

$$\frac{\mathrm{d}l_A^*}{\mathrm{d}L_D} < 0 \tag{B-46}$$

Plugging (B-46) into (B-44):

$$\frac{\mathrm{d}r_A^*}{\mathrm{d}L_D} = \Psi^{-1/2}(R_A - R_D)\left(\frac{\Psi^{-1/2}(-(r_A^* - a) + (L_A - L_D))}{1 - \Psi^{-1}(L_A - L_D)^2}\right)$$

Notice that

$$\begin{aligned}\Psi^{-1/2}(R_A - R_D) &= \sqrt{\frac{(R_A - R_D)^2}{2(R_A - R_D)(b - L_A - l_A^*) + (R_A^2 - R_D^2)}} \\ &= \sqrt{\frac{(R_A - R_D)}{2(b - L_A - l_A^*) + R_A + R_D}} \\ &< 1,\end{aligned}$$

if and only if $2(b-L_A-l_A^*)+R_A+R_D > (R_A-R_D)$, which implies $2b-l_A^* > 2(L_A-R_D)$. This holds in a regime 2 equilibrium where $l_A^* < 0$. Then we obtain $\frac{\mathrm{d}l_A^*}{\mathrm{d}L_D} < \frac{\mathrm{d}r_A^*}{\mathrm{d}L_D} < 0$.

But now, let's consider a new increment on L_A, which extends the analysis of a symmetric case to an asymmetric situation after the previous increment on L_A. From our previous calculations we have that,

$$\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} = \frac{\Psi_{l_A^*}^{-1/2}((L_A-R_A)+(r_A^*-a))+(L_A-L_D)\Psi_{l_A^*}^{-1/2}\Psi_{r_A^*}^{-1/2}(R_A-R_D)}{1-\Psi_{l_A^*}^{-1/2}\Psi_{r_A^*}^{-1/2}(L_A-L_D)(R_A-R_D)}$$
$$\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} = \frac{\Psi_{r_A^*}^{-1/2}(R_A-R_D)+\Psi_{l_A^*}^{-1/2}\Psi_{r_A^*}^{-1/2}(R_A-R_D)(r_A^*-a)}{1-\Psi_{l_A^*}^{-1/2}\Psi_{r_A^*}^{-1/2}(L_A-L_D)(R_A-R_D)}$$

where

$$\Psi_{l_A^*} = \left(2(L_A-L_D)(r_A^*-R_A-a)+(L_A^2-L_D^2)\right)$$
$$\Psi_{r_A^*} = \left(2(R_A-R_D)(b-L_A-l_A^*)+(R_A^2-R_D^2)\right).$$

Then, $\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} > \frac{\mathrm{d}r_A^*}{\mathrm{d}L_A}$ if and only if

$$\Psi_{l_A^*}^{-1/2}((L_A-R_A)+(r_A^*-a))+(L_A-L_D)\Psi_{l_A^*}^{-1/2}\Psi_{r_A^*}^{-1/2}(R_A-R_D)$$
$$> \Psi_{r_A^*}^{-1/2}(R_A-R_D)+\Psi_{l_A^*}^{-1/2}\Psi_{r_A^*}^{-1/2}(R_A-R_D)(r_A^*-a).$$

This inequality holds for the following reason. From the initial increment on L_A, l_A^* increases in a larger magnitude, because $\Psi_{l_A^*} < \Psi_{r_A^*} < 1$ implies $\Psi_{l_A^*}^{-1/2} > \Psi_{r_A^*}^{-1/2}$. Then $\Psi_{r_A^*}^{-1/2}(R_A-R_D)$ is still less than 1, in the asymmetric case we consider, L_A is higher than the initial symmetric. Therefore:

$$\Psi_{l_A^*}^{-1/2}(r_A^*-a) > \Psi_{l_A^*}^{-1/2}\Psi_{r_A^*}^{-1/2}(R_A-R_D)(r_A^*-a).$$

From symmetric situation we have prove that:

$$\Psi_{l_A^*}^{-1/2}(L_A-R_A)+\Psi_{l_A^*}^{-1/2}\Psi_{r_A^*}^{-1/2}(R_A-R_D) > \Psi_{r_A^*}^{-1/2}(R_A-R_D).$$

Because the current asymmetric case we study, $\Psi_{l_A^*}^{-1/2} > \Psi_{r_A^*}^{-1/2}$ and L_A is higher, the inequality also holds.

As a consequence, we obtain

$$\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A} > \frac{\mathrm{d}r_A^*}{\mathrm{d}L_A}.$$

Consider the situation where instead of increasing L_A, we increase L_D. Notice that parameters that affects l_A^* are the same for changes on L_A and on L_D. However, r_A^* is less sensitive to changes on L_D than L_A. As r_A^* is a parameter involved in the same way on $\frac{\mathrm{d}l_A^*}{\mathrm{d}L_A}$ and $\frac{\mathrm{d}l_A^*}{\mathrm{d}L_D}$, l_A^* is also less sensitive to changes on L_D than L_A. Then, the effect that a previous marginal increment of L_D has on $\Psi_{l_A^*}$ and $\Psi_{r_A^*}$ is compensated for a marginal increase in L_A.

Repeating the same exercise but with an increase on L_D we have the same result:

$$\frac{\mathrm{d}r_A^*}{\mathrm{d}L_D} = \Psi_{r_A^*}^{-1/2}(R_A - R_D)\left(\frac{\mathrm{d}l_A^*}{\mathrm{d}L_D}\right).$$

In a regime 1 equilibrium, $\frac{\mathrm{d}l_A^*}{\mathrm{d}L_D}$ is still negative. Then, as we have shown that $\Psi_{r_A^*}^{-1/2}(R_A - R_D) < 1$, we obtain $\frac{\mathrm{d}l_A^*}{\mathrm{d}L_D} < \frac{\mathrm{d}r_A^*}{\mathrm{d}L_D} < 0$. □

Proposition B-2. *Let F is uniformly distributed in $[-1, 1]$. In a regime 1 equilibrium,*

$$\frac{\partial(r_A^* - l_A^*)}{\partial L_A} = 0 \qquad \textit{and} \qquad \frac{\partial(r_A^* - l_A^*)}{\partial L_D} = 0.$$

Proof of Proposition B-2. Starting from symmetric situation such that $b \leq L_A - L_D$, the difference between advantage and disadvantage candidates is already too high and regime 1 arises. Therefore at any marginal change of L_A or L_D that preserves the inequality, the optimal l_A^* that ensures winning the election remains between $[M_L, M_R]$ and therefore the differentiable between r_A^* and l_A^* remain constant. Following the same logic, identical behaviour if after the marginal increment of L_A or L_D we increase again one of both variables. □

Proof of Proposition 1. Assuming uniform distribution on support $[a, b]$. Then, in equilibrium,

$$a \leq \frac{r_A^* + l_D^* + L_D - R_A}{2} \leq b$$
$$a \leq \frac{r_A^* + l_A^* + L_A - R_D}{2} \leq b.$$

The detailed calculations are available upon request. We provide the main steps of calculation. By (A-20),

$$L_D + r_A^* - R_A - l_D^* = r_A^* + l_D^* + L_D - R_A - 2a; \text{ and}$$
$$l_D^* = a.$$

Similarly, by (A-22),

$$R_D + r_D^* - L_A - l_A^* = 2(b - a) - (r_D^* + l_A^* + L_A - R_D - 2a); \text{ and}$$
$$r_D^* = b.$$

By substituting the uniform distribution F in (A-23), and modifying, we obtain the desired results. □

Proof of Theorem 5. The result is obtained by Propositions B-1 and B-2. □

Proof of Theorem 6. Let $a+b=k$. Using Proposition 1, taking the derivative with respect to a yields

$$\frac{\mathrm{d}l_A^*}{\mathrm{d}a}=1+\frac{\sqrt{L_A-L_D}}{\sqrt{2(r_A^*-R_A-a)+(L_A+L_D)}}\left(\frac{\mathrm{d}r_A^*}{\mathrm{d}a}-1\right)$$
$$\frac{\mathrm{d}r_A^*}{\mathrm{d}a}=-1-\frac{\sqrt{R_A-R_D}}{\sqrt{2(k-a-L_A-l_A^*)+(R_A+R_D)}}\left(-\frac{\mathrm{d}l_A^*}{\mathrm{d}a}-1\right).$$

For notational simplicity, let $L:=\frac{\sqrt{L_A-L_D}}{\sqrt{2(r_A^*-R_A-a)+(L_A+L_D)}}$ and $R:=\frac{\sqrt{R_A-R_D}}{\sqrt{2(k-a-L_A-l_A^*)+(R_A+R_D)}}$. Then we obtain

$$\frac{\mathrm{d}l_A^*}{\mathrm{d}a}=\frac{1+L(R-2)}{1-RL};$$
$$\frac{\mathrm{d}r_A^*}{\mathrm{d}a}=\frac{-1+2R-LR}{1-RL}.$$

Then $1+L(R-2)+1-2R+RL=2(1-L)(1-R)>0$. By (A-20) and (A-22), $b-L_A-l_A^*>d_D^*-L_A-l_A^*>-R_D$ and $r_A^*-R_A>l_D^*-L_D>a-L_D$. Therefore, we obtain

$$2(r_A^*-R_A-a)+(L_A+L_D)>L_A-L_D;$$
$$2(b-L_A-l_A^*)+(R_A+R_D)>R_A-R_D.$$

Thus we have $L<1$ and $R<1$. Thus, $\frac{\mathrm{d}r_A^*}{\mathrm{d}a}-\frac{\mathrm{d}l_A^*}{\mathrm{d}a}<0$. When we decrease a, we have $\frac{\mathrm{d}r_A^*}{\mathrm{d}(b-a)}-\frac{\mathrm{d}l_A^*}{\mathrm{d}(b-a)}=\left(\frac{\mathrm{d}r_A^*}{\mathrm{d}a}-\frac{\mathrm{d}l_A^*}{\mathrm{d}a}\right)\cdot(-1)>0$.

Using Proposition 1,

$$\frac{\mathrm{d}l_A^*}{\mathrm{d}b}=\frac{\sqrt{L_A-L_D}}{\sqrt{2(r_A^*-R_A-a)+(L_A+L_D)}}\frac{\mathrm{d}r_A^*}{\mathrm{d}b}$$
$$\frac{\mathrm{d}r_A^*}{\mathrm{d}b}=\left(1-\frac{\sqrt{R_A-R_D}}{\sqrt{2(b-L_A-l_A^*)+(R_A+R_D)}}\right)\left(1-\frac{\mathrm{d}l_A^*}{\mathrm{d}b}\right)$$

Because $1>R=\frac{\sqrt{R_A-R_D}}{\sqrt{2(b-L_A-l_A^*)+(R_A+R_D)}}$, by substituting $\frac{\mathrm{d}l_A^*}{\mathrm{d}b}$ into $\frac{\mathrm{d}r_A^*}{\mathrm{d}b}$, we obtain $\frac{\mathrm{d}(r_A^*-l_A^*)}{\mathrm{d}b}>0$. □

Proof of Theorem 7. By (A-20) and (A-22), we have

$$L_D+r_A^*-R_A-l_D^*=\frac{r_A^*+l_D^*+L_D-R_A}{2}-a \tag{B-47}$$
$$R_D+r_D^*-L_A-l_A^*=b-\frac{r_D^*+l_A^*+L_A-R_D}{2}. \tag{B-48}$$

Thus by modifying (B-47) and (B-48),

$$l_D^*=\frac{r_A^*+L_D-R_A}{3}+\frac{2}{3}a \tag{B-49}$$
$$r_D^*=\frac{2}{3}b+\frac{l_A^*+L_A-R_D}{3}. \tag{B-50}$$

By substituting (B-49) and (B-50) into (B-47) and (B-48),

$$L_D + r_A^* - R_A - l_D^* = \frac{2(r_A^* + L_D - R_A)}{3} - \frac{2}{3}a > 0 \tag{B-51}$$

$$R_D + r_D^* - L_A - l_A^* = \frac{2}{3}b - \frac{2(l_A^* + L_A - R_D)}{3} > 0. \tag{B-52}$$

By (B-49) and (B-50), and the property of the equilibrium we consider here,

$$r_A^* + l_D^* + L_D - R_A = \frac{4(r_A^* + L_D - R_A)}{3} + \frac{2}{3}a < 0; \tag{B-53}$$

$$l_A^* + r_D^* + L_A - R_D = \frac{2}{3}b + \frac{4(l_A^* + L_A - R_D)}{3} > 0. \tag{B-54}$$

<u>Case 1. Suppose that $l_A^* + r_A^* + L_A - R_A < 0$.</u> Then (A-21) and (A-23) are rewritten as

$$\left(\frac{l_A^* + r_A^* + L_A - R_A}{2} - a\right)^2 (L_A - R_A + r_A^* - l_A^*) = \frac{8}{27}(r_A^* + L_D - R_A - a)^3$$
$$\frac{1}{2a}\left(\frac{l_A^* + r_A^* + L_A - R_A}{2} - a\right)^2 (R_A - L_A + r_A^* - l_A^*) = \frac{1}{2b} \cdot \frac{8}{27}(b - (l_A^* + L_A - R_D))^3.$$

When L_A increases,

$$\begin{aligned}
&- \frac{3}{r_A^* + L_D - R_A - a} \\
&+ \frac{\mathrm{d}r_A^*}{\mathrm{d}L_A}\left(\frac{1}{L_A - R_A + r_A^* - l_A^*} - \frac{1}{R_A - L_A + r_A^* - l_A^*} - \frac{3}{r_A^* + L_D - R_A - a}\right) \\
&= \frac{\mathrm{d}l_A^*}{\mathrm{d}L_A}\left(\frac{1}{L_A - R_A + r_A^* - l_A^*} - \frac{1}{R_A - L_A + r_A^* - l_A^*} + \frac{3}{b - (l_A^* + L_A - R_D)}\right) \\
&- \frac{3}{b - (l_A^* + L_A - R_D)}
\end{aligned}$$

Then we have $\frac{\mathrm{d}r_A^*}{\mathrm{d}L_A} < \frac{\mathrm{d}l_A^*}{\mathrm{d}L_A}$.

When L_D increases,

$$\begin{aligned}
&\frac{\mathrm{d}r_A^*}{\mathrm{d}L_D}\left(\frac{1}{L_A - R_A + r_A^* - l_A^*} - \frac{1}{R_A - L_A + r_A^* - l_A^*} - \frac{3}{r_A^* + L_D - R_A - a}\right) \\
&= \frac{\mathrm{d}l_A^*}{\mathrm{d}L_D}\left(\frac{1}{L_A - R_A + r_A^* - l_A^*} - \frac{1}{R_A - L_A + r_A^* - l_A^*} + \frac{3}{b - (l_A^* + L_A - R_D)}\right)
\end{aligned}$$

Then we have $\frac{\mathrm{d}r_A^*}{\mathrm{d}L_D} > \frac{\mathrm{d}l_A^*}{\mathrm{d}L_D}$.

<u>Case 2. Suppose that $l_A^* + r_A^* + L_A - R_A > 0$.</u> Then (A-21) and (A-23) are rewritten as

$$\frac{1}{b}\left(b - \frac{l_A^* + r_A^* + L_A - R_A}{2}\right)^2 (L_A - R_A + r_A^* - l_A^*) = \frac{8}{27} \cdot \left(-\frac{1}{a}\right) \cdot (r_A^* + L_D - R_A - a)^3$$
$$\left(b - \frac{l_A^* + r_A^* + L_A - R_A}{2}\right)^2 (R_A - L_A + r_A^* - l_A^*) = \frac{8}{27} \cdot (b - (l_A^* + L_A - R_D))^3.$$

The proofs are done in the same way with Case 1. □

Proof of Theorem 8. The proof works in the same way with Theorem 7. First, we show $\frac{\mathrm{d}r_A^*}{\mathrm{d}a} < \frac{\mathrm{d}l_A^*}{\mathrm{d}a}$. Consider case 1 in the proof of Theorem 7. When a increases,

$$\begin{aligned}
&\frac{3}{r_A^* + L_D - R_A - a} + \frac{1}{a} \\
&+ \frac{\mathrm{d}r_A^*}{\mathrm{d}a}\left(\frac{1}{L_A - R_A + r_A^* - l_A^*} - \frac{1}{R_A - L_A + r_A^* - l_A^*} - \frac{3}{r_A^* + L_D - R_A - a}\right) \\
&= \frac{\mathrm{d}l_A^*}{\mathrm{d}a}\left(\frac{1}{L_A - R_A + r_A^* - l_A^*} - \frac{1}{R_A - L_A + r_A^* - l_A^*} + \frac{3}{b - (l_A^* + L_A - R_D)}\right).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\frac{\mathrm{d}r_A^*}{\mathrm{d}a}\left(\frac{1}{L_A - R_A + r_A^* - l_A^*} - \frac{1}{R_A - L_A + r_A^* - l_A^*} - \frac{3}{r_A^* + L_D - R_A - a}\right) \\
&< \frac{\mathrm{d}l_A^*}{\mathrm{d}a}\left(\frac{1}{L_A - R_A + r_A^* - l_A^*} - \frac{1}{R_A - L_A + r_A^* - l_A^*} + \frac{3}{b - (l_A^* + L_A - R_D)}\right).
\end{aligned}$$

Then we have $\frac{\mathrm{d}r_A^*}{\mathrm{d}a} < \frac{\mathrm{d}l_A^*}{\mathrm{d}a}$. Now, suppose $a_N < a_S$. Then, $\frac{r_A^S - r_A^N}{a_S - a_N} < \frac{l_A^S - l_A^N}{a_S - a_N}$. Thus, we obtain $r_A^N - l_A^N > r_A^S - l_A^S$. □

Proof of Proposition 2. Equations (A-21) and (A-23) are

$$\pi(L_A - l_A^S - R_A + r_A^S) = \frac{2}{3}(r_A^S + L_D - R_A - a)\pi_{LD} \tag{B-55}$$

$$(1-\pi)\left(R_A + r_A^S - L_A - l_A^S\right) = \frac{2}{3}(b - (l_A^S + L_A - R_D))\left(1 - \pi_{RD}\right). \tag{B-56}$$

In society N, because the density around a increases,

$$(\pi + \delta)(L_A - l_A^N - R_A + r_A^N) = \frac{2}{3}(r_A^N + L_D - R_A - a)(\pi_{LD} + \delta) \tag{B-57}$$

$$(1 - \pi - \delta)\left(R_A + r_A^N - L_A - l_A^N\right) = \frac{2}{3}(b - (l_A^N + L_A - R_D))\left(1 - \pi_{RD} - \epsilon\right). \tag{B-58}$$

Denote $\mathrm{d}r_A = r_A^N - r_A^S$ and $\mathrm{d}l_A = l_A^N - l_A^S$. Then by the first order approximation, we obtain

$$\delta(L_A - R_A) + \pi(\mathrm{d}r_A - \mathrm{d}l_A) = \frac{2}{3}\delta(L_D - R_A - a) + \frac{2}{3}\pi_{LD}\mathrm{d}r_A \tag{B-59}$$

$$-\delta(R_A - L_A) + (1-\pi)(\mathrm{d}r_A - \mathrm{d}l_A) = -\frac{2}{3}\delta(b - L_A + R_D) - \frac{2}{3}(1 - \pi_{RD})\mathrm{d}l_A. \tag{B-60}$$

Thus we obtain

$$\begin{aligned}
\mathrm{d}l_A &= \frac{\mathrm{d}r_A(3\pi - 2\pi_{LD}) + 3\delta(L_A - R_A) - 2\delta(L_D - R_A - a)}{3\pi} &\text{(B-61)}\\
&= \frac{\mathrm{d}r_A \cdot 3(1-\pi) - 3\delta(R_A - L_A) + 2\delta(b - L_A + R_D)}{3(1-\pi) - 2(1 - \pi_{RD})}. &\text{(B-62)}
\end{aligned}$$

Then

$$
\begin{aligned}
&\mathrm{d}r_A\left(\frac{3\pi-2\pi_{LD}}{3\pi}-\frac{3(1-\pi)}{3(1-\pi)-2(1-\pi_{RD})}\right)\\
&=\delta\cdot\left(\frac{-3(R_A-L_A)+2(b-L_A+R_D)}{3(1-\pi)-2(1-\pi_{RD})}-\frac{3(L_A-R_A)-2(L_D-R_A-a)}{3\pi}\right).
\end{aligned}
$$

Further,

$$
\begin{aligned}
&\mathrm{d}l_A\cdot 3\pi\left(\frac{3\pi-2\pi_{LD}}{3\pi}-\frac{3(1-\pi)}{3(1-\pi)-2(1-\pi_{RD})}\right)\\
&=\delta\cdot\left(\frac{3\pi-2\pi_{LD}}{3(1-\pi)-2(1-\pi_{RD})}\cdot(-3(R_A-L_A)+2(b-L_A+R_D))\right.\\
&\left.+\frac{2\pi_{LD}}{3\pi}\cdot(3(L_A-R_A)-2(L_D-R_A-a))\right).
\end{aligned}
$$

Then,

$$
\begin{aligned}
3(L_A-R_A)-2(L_D-R_A-a)&=3(L_A-R_A)+2(R_A-L_D)+2a;\\
-3(R_A-L_A)+2(b-L_A+R_D)&=3(L_A-R_A)+2(R_D-L_A)+2b.
\end{aligned}
$$

Then we obtain the desired result. □

Proof of Corollary 2. Suppose $\pi>\pi_L$ and $\frac{1}{2}<\pi<\pi_R$. Then $Y<0$. In addition, suppose $b-a>(R_A-L_D)-(R_D-L_A)$. Then because $3(1-\pi)-2(1-\pi_R)>3\pi$, we obtain $X>0$. Then $r_A^N<r_A^S$. Also, because

$$
\begin{aligned}
&\left(\frac{3(L_A-R_A)+2(R_D-L_A)+2b}{3(1-\pi)-2(1-\pi_R)}-\frac{2\pi_L}{3\pi}\cdot X\right)\\
&=\left(1-\frac{2\pi_L}{3\pi}\right)\frac{3(L_A-R_A)+2(R_D-L_A)+2b}{3(1-\pi)-2(1-\pi_R)}+\frac{3(L_A-R_A)+2(R_A-L_D)+2a}{3\pi}\\
&>\left(1-\frac{2\pi_L}{3\pi}\right)\frac{3(L_A-R_A)+2(R_D-L_A)+2b}{3(1-\pi)-2(1-\pi_R)}+\frac{3(L_A-R_A)+2(R_D-L_A)+2b}{3\pi}>0,
\end{aligned}
$$

where the last inequality is due to $\frac{L_A-R_A}{2}>R_A-R_D-b$. Thus we obtain $l_A^N<l_A^S$. □