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On Fisher Aggregation of Multi-factor Productivity Indexes*

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Abstract

The goal of this paper is to investigate the question of importance of Fisher aggregation of the Paasche and Laspeyres versions of Malmquist quantity and productivity indexes from both theoretical and empirical perspectives. We discuss existing justification and provide an alternative theoretical justification based on results from functional equations literature. We also use real data (from Kumar and Russell (2002, *American Economic Review*)) to illustrate how dramatic the differences in conclusions can be in practice depending on whether one employs Laspeyres or Paasche productivity indexes.

Keywords: Fisher, Laspeyres, Paasche, Malmquist, Productivity Indexes.

JEL Codes: D24, E22, E23, E24, O47

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1 Introduction

When computing an index, in the index number theory and practice in general, and for the case of Malmquist indexes in particular, it is now common to look at two alternative references, say s and t or 0 and 1. By convention, these references are called the base-period and the current-period, and are sometimes referred to as Laspeyres and Paasche approaches, respectively.

Typically, except for peculiar cases, an index may give different values for the two different references, sometimes may even imply qualitatively different conclusions. In the context of price indexes, Fisher (1921) was apparently among the first to focus on and concisely explain the essence of the differences between the two perspectives. Specifically, he pointed out the inherent bias of both these perspectives, stating:

"... It is also true that methods of weighting will introduce a bias. Thus, if the weights used are the values of the *base* year, they impart a *downward bias* to all the index numbers of any given year calculated thereby; while, on the other hand, if the weights used are the values of the given year itself, they impart an *upward bias*."

– Irving Fisher (1921, p.535; original emphasis).

A natural solution to such situations is to take an average of the different perspectives. Most commonly, one takes an equally weighted geometric average of the two values, in the spirit of Fisher (1921, 1922), who dubbed it as the 'ideal index' (in the context of price indexes). Eventually, many started to refer to this type of index as the 'Fisher ideal index', although Fisher himself did acknowledge that the idea of the formula goes back to earlier and apparently independent proposals by Bowley and Walsh (see discussions in Fisher (1922, p.xv and p. 241)).[1]

Such aggregation of the Laspeyres and Paasche perspectives of an index into one that is a geometric average of the two has various desirable properties. For example, for the case of the price index, Fisher aggregation allowed the index to pass the factor-reversal and the time reversal tests (Fisher (1921, 1922), Funke and Voeller (1978), Diewert (1992)). For the case of the Malmquist quantity and productivity indexes, it allowed an equivalence to be established with the corresponding Törnqvist indexes (Caves et al. (1982a)) and Fisher indexes (Diewert (1992)). More recently, for the case of a multi-factor productivity index, Diewert and Fox (2017) theoretically justified such aggregation by deriving it from three properties for the aggregation function: (i) homogeneity, (ii) symmetry and (iii) time-reversal property. One of the goals of this paper is to present another theoretical justification that makes less assumptions.

[1] For a nice exposition of the history of index numbers, see Balk (2008, Chapter 1).

In particular, using results from functional equations literature (Eichhorn (1978), Aczél (1987), Aczél (1990)), we show that if an index already has a multiplicative structure that one wants to preserve in the aggregate context, then the aggregation function must be a geometric mean. Furthermore, the symmetry condition (also assumed in Diewert and Fox (2017)) implies that the weights of the geometric mean must be equal, while the homogeneity implies that they should sum up to unity and therefore be 1/2. Thus, this is an alternative approach that does not require the imposition of time-reversal property *a priori*. Indeed, instead of imposing the time-reversal property, we use the multiplicativity nature, which is already inherent in the construction of the index and so is not an additional assumption. Moreover, we then show that the resulting index would necessarily satisfy the time-reversal test (without imposing it *a priori*).

Our second goal is to illustrate the empirical importance of going with the Fisher version of an index rather than taking only one side, be it the Laspeyres or the Paasche version or any other side including a fixed-weight index. Indeed, there appears to be a perception that, while one could construct theoretical examples where the difference between the two perspectives is large, in practice it is often not so large and perhaps can be ignored. Here we try to counter this misperception, by using real data from Kumar and Russell (2002).[2] Specifically, we estimated the Laspeyres and the Paasche versions of the multi-factor productivity indexes (along with the popular labor productivity index) and get very interesting results. In particular, we found very large (and sometimes dramatic) differences between the Laspeyres and Paasche versions of the productivity indexes for most countries in the sample. For example, if one were to take the Laspeyres version of the productivity indexes, then one would conclude that countries like Austria, Denmark, New Zealand, USA, etc. had substantial improvements in productivity, while the Paasche version suggests they had no improvement or even substantial deterioration in productivity! Overall, our theoretical reasoning and this empirical example confirm our understanding that taking one perspective, be it Laspeyres or Paasche, can be quite misleading (as also a use of any fixed-weight index) and taking the 'Fisher aggregation' approach by reconciling the two with a geometric mean seems to be indeed a more reasonable approach.

The structure of the paper is as follows: Section 2 introduces the basic notation and definitions of a technology set and its functional characterizations via Shephard's distance functions. Section 3 gives definitions of various quantity and productivity indexes. Section 4 briefly discusses the question of equality of the Laspeyres and Paasche versions of the indexes from a theoretical perspective, concluding that aggregation is needed to avoid very restrictive assumptions. Section 5 discusses alternative (and complementing) theoretical justifications

[2] Also see Henderson and Russell (2005) and Park et al. (2008).

for the aggregation of Laspeyres and Paasche versions of the indexes. Section 6 presents an empirical illustration of the importance of the aggregation. Section 7 concludes.

2 Characterization of Technology

We denote inputs by a column vector $x = (x_1, \ldots, x_N)' \in \mathbb{R}^N_+$ and outputs by a column vector $y = (y_1, \ldots, y_M)' \in \mathbb{R}^M_+$ and assume the production technology at time t is given by the set

$$\Psi^t = \{(x, y) : x \text{ can produce } y \text{ at time } t\} \tag{1}$$

which we assume meets standard regularity conditions, with weak disposability of inputs and outputs in particular.[3]

The t-period Shephard's input distance function defined with respect to Ψ^t is given by

$$D^t_i(y, x) = \sup_{\beta}\{\beta > 0 : (x/\beta, y) \in \Psi^t\}. \tag{2}$$

This function is by definition homogeneous of degree 1 in inputs, i.e.,

$$D^t_i(y, \lambda x) = \lambda D^t_i(y, x), \quad \lambda > 0. \tag{3}$$

If the technology has inputs weakly disposable, i.e.,

$$(x, y) \in \Psi^t, \; \mu \geq 1 \quad \text{then} \quad (\mu x, y) \in \Psi^t. \tag{4}$$

then the Shephard's input distance function is a function representation of technology.

Similarly, the t-period Shephard's output distance function defined with respect to Ψ^t is given by

$$D^t_o(x, y) = \inf_{\beta}\{\beta > 0 : (x, y/\beta) \in \Psi^t\}. \tag{5}$$

which, by construction, is homogeneous of degree 1 in outputs, i.e.,

$$D^t_o(x, \lambda y) = \lambda D^t_o(x, y), \quad \lambda > 0. \tag{6}$$

[3]See Färe and Primont (1995) and Sickles and Zelenyuk (2019).

Moreover, if the technology has outputs weakly disposable, i.e.,

$$(x, y) \in \Psi^t,\ \ \mu \leq 1 \quad \text{then} \quad (x, \mu y) \in \Psi^t. \tag{7}$$

then the Shephard's output distance function gives a function representation of technology. Being equivalent representations of technology, these distance functions serve as useful tools for defining various types of indexes, as discussed in the next section.

3 Quantity and Productivity Indexes

Following Caves et al. (1982), the t-period Malmquist input quantity index can be defined as

$$X(x^0, x^1, y, \Psi^t) \;\; = \;\; D_i^t(y, x^1)/D_i^t(y, x^0). \tag{8}$$

The Laspeyres and Paasche version of this index are given, respectively, by

$$X_L(x^0, x^1, y^0, \Psi^0) \;\; = \;\; D_i^0(y^0, x^1)/D_i^0(y^0, x^0) \tag{9}$$

and

$$X_P(x^0, x^1, y^1, \Psi^1) \;\; = \;\; D_i^1(y^1, x^1)/D_i^1(y^1, x^0). \tag{10}$$

Similarly, t-period Malmquist output quantity index can be defined as

$$Y(y^0, y^1, x, \Psi^t) \;\; = \;\; D_o^t(x, y^1)/D_o^t(x, y^0). \tag{11}$$

and so the Laspeyres and Paasche versions of this index are

$$Y_L(y^0, y^1, x^0, \Psi^0) \;\; = \;\; D_o^0(x^0, y^1)/D_o^0(x^0, y^0) \tag{12}$$

and

$$Y_P(y^0, y^1, x^1, \Psi^1) \;\; = \;\; D_o^1(x^1, y^1)/D_o^1(x^1, y^0). \tag{13}$$

Note that in all these cases, in the Laspeyres version, the technology is for $t = 0$ and so is the output reference vector y^0, while in the Paasche version it is $t = 1$.

While useful in their own right, these quantity indexes are often used for constructing a multi-factor productivity index, with the Laspeyres version defined as

$$DBPI_L(x^0, x^1, y^0, y^1, \Psi^0) \equiv \frac{Y_L(y^0, y^1, x^0, \Psi^0)}{X_L(x^0, x^1, y^0, \Psi^0)} = \frac{D_o^0(x^0, y^1)/D_o^0(x^0, y^0)}{D_i^0(y^0, x^1)/D_i^0(y^0, x^0)} \tag{14}$$

and its Paasche version defined as

$$DBPI_P(x^0, x^1, y^0, y^1, \Psi^1) \equiv \frac{Y_P(y^0, y^1, x^1, \Psi^1)}{X_P(x^0, x^1, y^1, \Psi^1)} = \frac{D_o^1(x^1, y^1)/D_o^1(x^1, y^0)}{D_i^1(y^1, x^1)/D_i^1(y^1, x^0)}. \tag{15}$$

Again, to reconcile the Laspeyres and Paasche perspectives, the geometric average of these two is usually taken, i.e.,

$$\begin{aligned} &DBPI(x^0, x^1, y^0, y^1, \Psi^0, \Psi^1) \equiv \\ &\quad \left[DBPI(x^0, x^1, y^0, y^1, \Psi^0) \times DBPI(x^0, x^1, y^0, y^1, \Psi^1)\right]^{\frac{1}{2}}. \end{aligned} \tag{16}$$

While this index emerged under different names at different waves of the literature and, perhaps, the name 'Malmquist–Shephard–Hicks–Moorsteen–Diewert–Bjurek productivity index' would probably be the fairest, to simplify, we will refer to it as the Diewert–Bjurek productivity index (DBPI), to credit the two authors who appear to have contributed the most to the origin of this interesting index.[4]

Another approach is to compare the whole allocations (x, y) across periods with respect to different technology references—the idea also suggested in the seminal work of Caves et al. (1982a), and named as the Malmquist Productivity Index (MPI). In the output oriented context, its Laspeyres version is defined as

$$MPI_o(x^0, x^1, y^0, y^1, \Psi^0) \equiv \frac{D_o^0(x^1, y^1)}{D_o^0(x^0, y^0)} \tag{17}$$

[4] While the roots of this index can be found in Caves et al. (1982), Diewert (1992, p.240) appears to be the first who explicitly introduced it after apparently being inspired by the geometric descriptions from Moorsteen (1961, p.460) and intuitive description of Hicks (1961, footnote 4). He dubbed it as the "Hicks–Moorsteen approach to productivity indexes" and then it often emerged as the 'Hicks–Moorsteen Productivity index' or the 'Hicks–Moorsteen TFP index' or simply the 'Hicks–Moorsteen index' in various works. About the same time, Bjurek (1996) developed this index more explicitly and generalized it to allow for technical inefficiency and called it the "Malmquist TFP index", although others started to call it the "Bjurek productivity index" or the "Bjurek TFP index" (or simply the "Bjurek index"). Professor Bert Balk suggested to us that the name of Hicks should not be there, referring to it as the "Moorsteen-Bjurek productivity index", although see explanations in Diewert and Fox (2017, p.276) on the justification of the influence from Hicks. Meanwhile, Sickles and Zelenyuk (2019, p.110) suggested that the name of Diewert (since he was the first to suggest this index explicitly), should be also added there, and also the name of Malmquist, since it is defined as the ratio of Malmquist indexes. Finally, the name of Shephard can also be justified, since his distance functions were used to define this index explicitly.

while its Paasche version is defined as

$$MPI_o(x^0, x^1, y^0, y^1, \Psi^1) \equiv \frac{D_o^1(x^1, y^1)}{D_o^1(x^0, y^0)} \tag{18}$$

and, to reconcile between the two MPIs, their geometric mean is usually taken, i.e.,

$$\begin{aligned} MPI_o(x^0, x^1, y^0, y^1, \Psi^0, \Psi^1) &\equiv \left[MPI_o(x^0, x^1, y^0, y^1, \Psi^0) \times MPI_o(x^0, x^1, y^0, y^1, \Psi^1)\right]^{\frac{1}{2}} \\ &\equiv \left[\frac{D_o^0(x^1, y^1)}{D_o^0(x^0, y^0)} \times \frac{D_o^1(x^1, y^1)}{D_o^1(x^0, y^0)}\right]^{\frac{1}{2}} \end{aligned} \tag{19}$$

Similarly, in the input oriented context, the Laspeyres version of the MPI is defined as

$$MPI_i(x^0, x^1, y^0, y^1, \Psi^0) \equiv \frac{D_i^0(y^1, x^1)}{D_i^0(y^0, x^0)} \tag{20}$$

while its Paasche version is defined as

$$MPI_i(x^0, x^1, y^0, y^1, \Psi^1) \equiv \frac{D_i^1(y^1, x^1)}{D_i^0(y^0, x^0)} \tag{21}$$

and, again, their geometric mean is usually taken to reconcile between the two, i.e.,

$$\begin{aligned} MPI_i(x^0, x^1, y^0, y^1, \Psi^0, \Psi^1) &\equiv \left[MPI_i(x^0, x^1, y^0, y^1, \Psi^0) \times MPI_i(x^0, x^1, y^0, y^1, \Psi^1)\right]^{\frac{1}{2}} \\ &\equiv \left[\frac{D_i^0(y^1, x^1)}{D_i^0(y^0, x^0)} \times \frac{D_i^1(y^1, x^1)}{D_i^1(y^0, x^0)}\right]^{\frac{1}{2}}. \end{aligned} \tag{22}$$

All these indexes are sometimes referred to as the 'true' economic indexes in the sense that they are constructed with functions that represent or characterize the hypothetical 'true' technologies underlying the economic phenomenon of interest. In practice, the true technologies are, of course, unobserved and so they need to be estimated with some suitable methods and we will discuss this in Section 6.

4 Are they Ever Equal?

A natural question is how are all these (and other) indexes related and when are they equal? This question has been extensively explored in the index number literature and so we will only briefly touch upon this here.[5]

[5] E.g., see Sickles and Zelenyuk (2019, Chapter 4), Färe et al. (2020) and references therein.

4.1 A Bilateral Case

Under what conditions on the technology do the Laspeyres and Paasche versions of those indexes coincide? To outline the crux of the problem, let us focus on this question in the context of input quantity indexes, where formally, the question is: When do we have:

$$X_L(x^0, x^1, y^0, \Psi^0) \;\; = \;\; X_P(x^0, x^1, y^1, \Psi^1)? \tag{23}$$

By equating the two indexes, and after rearranging, we have

$$D_i^1(y^1, x^1) \;\; = \;\; \frac{D_i^0(y^0, x^1) D_i^1(y^1, x^0)}{D_i^0(y^0, x^0)} \tag{24}$$

suggesting that because the left hand side is independent from variables in period 0, so should be the right hand side. Moreover, when fixing all "$0'$s", we get

$$D_i^1(y^1, x^1) \;\; = \;\; f^1(y^1) h(x^1) \tag{25}$$

where $f^1(.)$ and $h(.)$ are some suitable functions, whose properties are determined by $D_i^1(y, x)$ and $D_i^0(y, x)$, according to (24).

In the same way, one can show that

$$D_i^0(y^0, x^0) \;\; = \;\; f^0(y^0) h(x^0) \tag{26}$$

where $f^0(.)$ and $h(.)$ are some suitable functions whose properties are also determined by $D_i^1(y, x)$ and $D_i^0(y, x)$, according to (24). These conditions are also sufficient for the equality to hold.

A similar argument can be developed for the output orientation, i.e., for the Malmquist output quantity index and then, combining the two, also for the Diewert-Bjurek productivity index and the Malmquist productivity indexes. In all these cases, technology must satisfy various types of Hicks-neutrality and of homotheticity—very restrictive assumptions. Chambers and Färe (1994) appear to be among the first to thoroughly discuss technologies of the type that obey such conditions (also see Chambers (1988, p. 207) for a related discussion), which were later revisited by Färe and Grosskopf (1996), Balk (1998), Balk et al. (2003), Peyrache (2013) and more recently by Sickles and Zelenyuk (2019, p. 130-137).

4.2 A Multilateral Case

When more than two periods are compared the problem is more complicated, yet the essence is very similar. Here, the requirement of equality of the Laspeyres and Paasche versions generalizes to what is sometimes referred to as the*circularity* property of (or '*circular test*' for) indexes.

Intuitively, the circularity property for an index $\mathcal{I}_{st}$ that compares period s to t is defined as the situation when one can always 'circle back' or 'close the circuit with unity' between any arbitrary periods s and t through any other arbitrary period r. Putting this more formally, the circularity for an index $\mathcal{I}_{st}$ requires that[6]

$$\mathcal{I}_{sr} \times \mathcal{I}_{rt} \times \mathcal{I}_{ts} = 1, \ \forall r, s, t. \tag{27}$$

This type of requirement also sometimes incarnates in the so-called transitivity property for indexes (although it is a somewhat unfortunate name due to confusion with the more general transitivity property or relations in mathematic and in consumer theory). Formally, an index $\mathcal{I}_{st}$ is *transitive* if and only if

$$\mathcal{I}_{st} = \mathcal{I}_{sr} \times \mathcal{I}_{rt}, \ \ s, r, t. \tag{28}$$

Thus, the circularity property together with the so-called identity property requiring that

$$\mathcal{I}_{tt} = 1, \ \forall t. \tag{29}$$

or together with the time reversal property requiring that

$$\mathcal{I}_{st} = 1/\mathcal{I}_{ts}, \ \forall s, t. \tag{30}$$

lead to transitivity property.

In other words, the transitivity is a slightly stronger requirement than this version of the circularity property, but because the identity property is often viewed as being so basic and natural, and typically accepted without much argument, the circularity (27) and the transitivity (28) are often viewed as equivalent properties.

While the condition (29) and (30) may look reasonable at first sight, note that since the definition of circularity and transitivity are given for the *arbitrary* three periods s, r, t, by

[6] It appears that Westergaard (1890) was the first to discuss this requirement for indexes, later considered by Fisher (1911) under the name of 'test by changing base'. Also see related discussions in Balk and Althin (1996) and Sickles and Zelenyuk (2019).

logical induction, it then requires this index to be 'decomposable' in the same way into *any* number of indexes via the same formula, regardless of the path taken. For example, imposing transitivity (28) also implies that the following should hold

$$\mathcal{I}_{st} = \mathcal{I}_{s1} \times \mathcal{I}_{12} \times \mathcal{I}_{23} \times ... \times \mathcal{I}_{(t-1)t}. \quad (31)$$

even where s is the period of the beginning of the world while t goes to ∞ and includes all the possible time periods in the past (including the Stone Age, the Bronze Age, etc.) and even the yet unseen future.

Mechanically, such circularity can sometimes be achieved for the various indexes by simply requiring the fixed weights (or fixed base). However, from an economic point of view, for an index to be sensible its weights should represent some relevant characterizations of the phenomenon of interest, such as preferences in consumer theory or technology characterization in the production theory of economics. When the weights are fixed, they can severely misrepresent the phenomenon of interest and so the requirements of (27) or (28), which in turn imply (31), can be viewed as too restrictive. In fact, while being one of the first advocates of circularity for indexes (e.g., see Fisher (1911)), Fisher later realized the serious problem with requiring this property, acknowledging its pathological incoherence with reality, eloquently stating:

> "... for the only definite error which I have found among my former conclusions has to do with the so-called "circular test" which I originally, with other writers, accepted as sound, but which, in this book, I reject as theoretically unsound."
>
> — Fisher (1922, p.xii-xiii)

and

> "The only formulae which conform perfectly to the circular test are index numbers which have *constant weights*, i.e. weights which are the same for all sides of the "triangle" or segments of the "circle," i.e. for every pair of times or places compared. ... But, clearly, constant weighting is not theoretically correct... We cannot justify using the same weights for comparing the price level of 1913, not only with 1914 and 1915, but with 1860, 1776, 1492, and the times of Diocletian, Rameses II, and the Stone Age!"
>
> — Fisher (1922, p. 274-275, original emphasis retained).

This reasoning was rigorously confirmed by Eichhorn (1976) and further refined by Funke et al. (1979). Specifically, in the context of price indexes and using the functional equations approach with minimal assumptions, they proved that the only test that satisfies commensurability property and the circularity property is a geometric mean (i.e., the Cobb-Douglas) type index with *fixed weights*.[7] About the same time, in their seminal paper, Samuelson and Swamy (1974) also confirmed the high degree of restrictiveness mathematically for the context of price indexes in the consumer theory framework by deriving the exact conditions needed for the circularity to hold. These conditions included homothetic preferences that do not change over time. While homotheticity is a somewhat standard simplification in consumer theory, the requirement of no change in preferences over a long period of time can be seen as too restrictive even here.

More importantly, in the context of production economics, the analogy to the requirements of Samuelson and Swamy (1974) for quantity and productivity indexes would be the homothetic technology that does not change or changes only in the same Hicks-neutral fashion, as we indeed concluded in Section 4.1 above.[8] Such requirements on technology might be suitable for very short intervals, but might be too restrictive for even a decade when considering modern economic history. Indeed, even with an unequipped eye one can see the nature of technological change is often such that some inputs become much more or less intensive/important than others; some inputs and outputs disappear completely, while some new inputs and outputs appear and gain greater importance. Ignoring this in measurement, by dogmatically restricting to some fixed weights, may severely misrepresent the reality, and we illustrate this in our empirical example with a real data set.[9]

All in all, considering that those conditions are too strong to impose on a technology set, one may like to aggregate the Paasche and Laspeyres indexes, as suggested by Bowley, Walsh and most prominently by Fisher. In the next section we will discuss theoretical justification for such aggregation.

5 Theoretical Justifications for Fisher Aggregation

Various theoretical justifications for why the equally-weighted geometric mean should be used to reconcile the Paasche and Laspeyres indexes were offered in the literature after Bowley, Fisher and Walsh had proposed this simple, yet powerful idea.

For example, in the context of price index, Fisher (1921, 1922) wanted it to pass what he deemed as 'two supreme tests': (i) the factor-reversal test, i.e., when the formula works both ways with respect to the two factors, prices and quantities, which is sometimes referred to as

[7] Essentially the same result was much earlier derived in somewhat overlooked paper of Konüs and Byushgens (1926).

[8] See Färe and Grosskopf (1996), Sickles and Zelenyuk (2019, p. 130-137) for more details.

[9] Also, see Diewert and Fox (2017) for related discussions on the caveats of fixed-weight indexes.

the 'self-duality' property of an index, and (ii) the time reversal test, i.e., when the formula works both ways with respect to time. More rigorously, this reasoning was later developed by Funke and Voeller (1978), who proved that the Fisher price index is the only price index that satisfies time reversal, factor reversal, positivity, and quantity reversal properties. Furthermore, Diewert (1992) derived the same conclusion using a different route, while its analogue for the quantity indexes can be found in Sickles and Zelenyuk (2019, p. 222).

Moreover, in the case of the Malmquist quantity and productivity indexes, their Fisher aggregation was used by Caves et al. (1982a) to derive the equivalence with the Törnqvist productivity index. Similarly, the Fisher aggregation of theLaspeyres and Paasche quantity and productivity indexes, was used to derive the equivalence with the Malmquist quantity and productivity indexes by Diewert (1992). More recently, Diewert and Fox (2017) provided a solid theoretical justification for aggregating the Paasche and Laspeyres versions of the Diewert-Bjurek index using index numbers or the test approach. Their results required or leveraged upon three properties for the aggregation function: (i) be a homogeneous mean, (ii) be a symmetric mean and (iii) satisfy time-reversal property.[10] In this section we will outline an alternative (and complementary) theoretical justification leveraging on results from the closely related literature on functional equations.[11]

5.1 Aggregation of Two Perspectives

To abstract from a particular index and develop a general conclusion, let q_{st}^{τ} be an index measuring a change between period s and t (say $s < t$) with respect to a reference period τ and suppose that this can be constructed from (or decomposed into) two components, say z_{st}^{τ} and g_{st}^{τ}, in a multiplicative way, i.e.,

$$q_{st}^{\tau} = z_{st}^{\tau} \times g_{st}^{\tau}. \tag{32}$$

E.g., in the context of the τ-period Malmquist input quantity index, we have $q_{st}^{\tau} = Q(x^s, x^t, y^{\tau}, \Psi^{\tau})$, $z_{st}^{\tau} = D_i^{\tau}(y^{\tau}, x^t)$ and $g_{st}^{\tau} = 1/D_i^{\tau}(y^{\tau}, x^s)$.

Formally, the problem is to find some aggregation functions ψ, ψ_z and ψ_g so that we preserve the multiplicative structure (of construction or decomposition) on the aggregate level, i.e., to have

$$\psi(q_{st}^{s}, q_{st}^{t}) = \psi_z(z_{st}^{s}, z_{st}^{t}) \times \psi_g(g_{st}^{s}, g_{st}^{t}), \quad \forall s, t \tag{33}$$

[10] Also see Balk (2008), Theorem 3.11, p. 97 and the related discussion.

[11] Also note that similar reasoning applies for the additive measures, sometimes referred to as indicators, as outlined in Färe and Zelenyuk (2019). In fact, their result was derived after the result of this section, although published earlier.

for all q^s_{st}, q^t_{st}, z^s_{st}, z^t_{st} and g^s_{st}, g^t_{st} satisfying (32).

The solution to this problem is:[12]

$$\psi(q^s_{st}, q^t_{st}) = \rho \times \gamma \times (q^s_{st})^{\alpha_s} \times (q^t_{st})^{\alpha_t}, \quad \forall s, t. \tag{34}$$

and

$$\psi_z(z^s_{st}, z^t_{st}) = \gamma \times (z^s_{st})^{\alpha_s} \times (z^t_{st})^{\alpha_t}, \quad \forall s, t. \tag{35}$$

and

$$\psi_g(g^s_{st}, g^t_{st}) = \rho \times (g^s_{st})^{\alpha_s} \times (g^t_{st})^{\alpha_t}, \quad \forall s, t. \tag{36}$$

where γ, ρ, α_s and α_t are arbitrary values.

Now, if we also impose the first condition of Diewert and Fox (2017) so that the aggregation function is a homogeneous mean, i.e.,

$$\psi(\delta q^s_{st}, \delta q^t_{st}) = \delta \psi(q^s_{st}, q^t_{st}), \quad \forall s, t, \ \forall \delta > 0$$

then it will require that $\alpha_s + \alpha_t = 1$, which is the weighted geometric mean if we also set or normalize

$$\rho = \gamma = 1. \tag{37}$$

This last normalization (37) can be justified by requiring the so-called 'identity property' in the theory index numbers, i.e., requiring ψ to be such that

$$\psi(q, q) = q, \quad \forall q, \tag{38}$$

which is sometimes also referred to as 'agreement property' (Aczél (1990, p.24)). Note that this 'agreement property' is also imposed by Diewert and Fox (2017, p.279), who referred to it as 'mean property' since they require the aggregating function to be a mean. Moreover, together with the symmetry assumption on the mean, which was the second condition in Diewert and Fox (2017), it follows that the only solution would be a geometric mean, which is the result that goes back to at least Aczél (1990, p.35-36).[13]

To see how this unfolds in our context of indexes, recall the second condition of Diewert and Fox (2017) is that the aggregation function is a symmetric mean, i.e.,

$$\psi(q^s_{st}, q^t_{st}) = \psi(q^t_{st}, q^s_{st}), \quad \forall s, t \tag{39}$$

[12]See Eichhorn (1978, p.94), Aczél (1987, p.151), Aczél (1990, p.27) and Färe and Zelenyuk (2005).

[13]We thank Robert Chambers for pointing this out.

and so it follows that $\rho = \gamma = 1$ and we have only three possibilities. The first possibility is the trivial one, when $q_{st}^{s} = 0$ or $q_{st}^{t} = 0$, which is only possible when at least one of their components is zero (i.e., $z_{st}^{\tau} = 0$ or $g_{st}^{\tau} = 0$), which is a degenerate case of little practical value.

To derive the second possibility, impose (39) and this will imply

$$(q_{st}^{s})^{\alpha_s} \times (q_{st}^{t})^{\alpha_t} = (q_{st}^{s})^{\alpha_t} \times (q_{st}^{t})^{\alpha_s}, \quad \forall s,t \tag{40}$$

and therefore

$$(q_{st}^{s}/q_{st}^{t})^{\alpha_s} = (q_{st}^{s}/q_{st}^{t})^{\alpha_t}, \quad \forall s,t \tag{41}$$

or

$$(q_{st}^{s}/q_{st}^{t})^{\alpha_s - \alpha_t} = 1, \quad \forall s,t \tag{42}$$

thus requiring that if $\alpha_s \neq \alpha_t$ then

$$q_{st}^{s} = q_{st}^{t},$$

which is a very strong assumption as we discussed in the section above.

Finally, the third possibility, and the only one that allows for any q_{st}^{s} and any q_{st}^{t}, requires that

$$\alpha_s = \alpha_t. \tag{43}$$

To see this, note that since (40) must hold for all q_{st}^{t}, take for example $q_{st}^{s} = 1$ and thus

$$(q_{st}^{t})^{\alpha_t} = (q_{st}^{t})^{\alpha_s}, \quad \forall s,t.$$

and so

$$(q_{st}^{t})^{\alpha_t - \alpha_s} = 1, \quad \forall s,t.$$

or $\alpha_t - \alpha_s = 0, \quad \forall s,t$ implying (43). Combining this with $\alpha_s + \alpha_t = 1$, we get $\alpha_s = \alpha_t = 1/2$.

Altogether, we have

$$\psi(q_{st}^{s}, q_{st}^{t}) = \left(q_{st}^{s} \times q_{st}^{t}\right)^{1/2}, \quad \forall s,t$$

and

$$\psi_z(z_{st}^{s}, z_{st}^{t}) = \left(z_{st}^{s} \times z_{st}^{t}\right)^{1/2}, \quad \forall s,t$$

and

$$\psi_g(g_{st}^{s}, g_{st}^{t}) = \left(g_{st}^{s} \times g_{st}^{t}\right)^{1/2}, \quad \forall s,t.$$

Last, yet not least, note that

$$\left(q_{st}^{s} \times q_{st}^{t}\right)^{1/2} = 1/\left(1/q_{st}^{s} \times (1/q_{st}^{t})\right)^{1/2}, \quad \forall s,t$$

meaning that it satisfies the time-reversal test and therefore we do not need to impose it *a priori* as another condition. This fact of imposing less assumptions makes it an attractive alternative justification to complement the existing justifications and makes the conclusion more robust in the sense that the more alternative theoretical justifications are found to support a conclusion, the more robust is the conclusion.

5.2 Aggregation of Many Perspectives

Similar logic can be applied to generalize to cases of many periods or many individuals. Indeed, suppose we have more than two references $\tau = 1, ..., T$, which could be time periods or states or, more generally, perspectives, to compute the index, denoted as q_{st}^{τ}. Then, from the functional equations literature, we know that if we want to preserve the multiplicative structure of each q_{st}^{τ}, then it will imply that ψ must be of the form

$$\psi(q_{st}^{1}, ..., q_{st}^{T}) = \gamma(q_{st}^{1})^{\alpha_1} \times ... \times (q_{st}^{T})^{\alpha_T}$$

where the arbitrary constant γ reduces to 1 if we impose the identity property and $\sum_{\tau=1}^{T} \alpha_\tau = 1$ if we impose the homogeneity property. Finally, the equal weights are obtained by imposing symmetry property as we did above. In fact, note that only pair-wise symmetry is needed here, which will imply the overall symmetry. Overall, we have

$$\psi(q_{st}^{1}, ..., q_{st}^{T}) = \left(\prod_{\tau=1}^{T}(q_{st}^{\tau})\right)^{1/T}.$$

Finally, further extensions are also possible using the same logic, for example to the context of multiple individuals as well as when the indexes need to be 'transitivised', e.g., according to approaches suggested by Caves et al. (1982b) and further elaborated on by Balk and Althin (1996), which are in turn an adaptation of the so-called GEKS approach from Gini (1931), Eltetö and Köves (1964) and Szulc (1964).

6 Empirical Importance

One of important questions we received as feedback from the earlier versions of this paper is 'How important is the question of averaging the Laspeyres and Paasche indexes in practice?'

Indeed, it appears that there is a perception that the difference is usually not so substantial and that perhaps one could ignore it and just use either the Laspeyres or the Paasche indexes. In fact, some statistical agencies are still using Laspeyres versions for some measurements. In academic literature, the use of averaging of the Laspeyres or the Paasche perspectives is more common these days, yet still some may do sojust as a tradition or, perhaps, to avoid criticism of referees.

Moreover, there is also apparently a recent increase of advocacy for the so-called fixed-weight or fixed-base indexes (e.g., which satisfy the so-called transitivity or circularity property). Such indexes, rather than reconciling with an average, require taking just one side (be it Laspeyres or Paasche or any other perspective) and sticking to it. A problem of this latter approach is the possibility of radically different conclusions, depending on the weights, possibly implying radically different policy implications. It is relatively easy to construct a synthetic, textbook-style example to illustrate the point. In fact, the seminal work of Moorsteen (1961) is based on an example like that. What perhaps is more interesting now is to give an empirical example based on real data to support this reasoning, and this is one of the goals of this section.

In this study we explored several data sets and *all* led to the same conclusions confirming the importance of the Fisher aggregation rather than relying on either Lasperes or Paashe perspectives alone, as well as the danger of relying on the fixed-weight indexes in particular. To get our point across vividly and concisely, here we limit our discussion for the dataset from Kumar and Russell (2002) about productivity and economic growth of various countries.[14]

6.1 Data in Brief

One of the reasons for our choice of data is because it is on a topic of wide interest and considerable attention beyond academic economists. Moreover, being published in the *American Economic Review* also gives it special status and the associated scrutiny and attention that develops a good degree of confidence around it. Remarkably, this data set indeed reveals how dramatic the differences could be depending on whether one takes the Laspeyres or Paasche perspectives or the Fisher approach to average over those two, potentially leading to different judgements on the impact of economic reforms on productivity.

[14]This is in fact also the very first data set we tried for our illustration. We also tried newer versions of Penn World Tables and also with more sophisticated measurement of inputs, e.g., by accounting for human capital as in Henderson and Russell (2005) or/and financial capital as in Badunenko and Romero-Avila (2013), as well as tried micro data (on farmers) and these data sets generally confirmed the main conclusions we discussed here. Also, for somewhat related evidence in different contexts, see Balk (2008, Chapter 3) who used artificial data for the contexts of price indexes and Balk and Zofio (2018) for the context of comparing various decompositions of productivity indexes for banking data.

Also, recall that in their seminal work, Kumar and Russell (2002) focused on the labor productivity index and its decomposition and did not present explicitly the estimates of the multi-factor productivity indexes and so it seems worth presenting them here and contrasting them to the estimates and conclusions from the labor productivity index (LPI).[15]

The data originally came from the Penn World Tables (version 5.6). In brief, the sample consists of observations on 57 countries in 1965 and 1990, for the following variables: real GDP, which we use as the single aggregate output, as well as employment and capital stock, which we use as the aggregate labor input and the aggregate capital input of each country, respectively. The monetary values (real GDP and capital) are measured in 1985 international prices. A bit more of details of the data can be found in Kumar and Russell (2002), while more details are in Summers and Heston (1991).

6.2 Estimation Strategy

Many different methods can be used to estimate the MPI and DBPI, which typically resort to estimating each distance function in their formulas. Here, we use the same approach as Kumar and Russell (2002)—Data Envelopment Analysis (DEA).[16] To be more precise, the estimates of the output and input distance functions for an allocation (x^{sj}, y^{tj}), with respect to technology of period τ is obtained by solving the following optimization problems, respectively[17]

$$\begin{aligned} (\widehat{D}_o^{\tau}(x^{sj}, y^{tj}))^{-1} &\equiv \max_{\lambda, z^1, ..., z^n} \{\lambda \\ & \quad \text{such that} \\ \sum_{k=1}^{n} z^k y_m^{\tau k} &\geq \lambda y_m^{tj}, m = 1, ..., M, \\ \sum_{k=1}^{n} z^k x_l^{\tau k} &\leq x_l^{sj}, l = 1, ..., N, \\ \lambda &\geq 0, \ (z^1, ..., z^n) \in \mathcal{Z}\}, \end{aligned} \tag{44}$$

[15]Their estimates of the MPI can be recovered from their estimates of efficiency change and technology change.

[16]Other popular methods include Stochastic Frontier Analysis or SFA (parametric, semi-parametric and non-parametric) and their symbiosis, such as Stochastic DEA, etc. See Kumbhakar and Lovell (2002) and Sickles and Zelenyuk (2019) for a comprehensive textbook style discussion and references therein for more details.

[17]All estimations were done in Matlab by the authors and then independently checked in R by one of our research assistants. For Matlab, we adapted some DEA codes from Leopold Simar, while for R we used the 'Benchmarking' package of Bogetoft and Otto (https://cran.r-project.org/web/packages/Benchmarking/Benchmarking.pdf) and 'deaR' package of Coll-Serrano, Bolos and Suarez (https://cran.r-project.org/web/packages/deaR/deaR.pdf).

and

$$
\begin{aligned}
(\widehat{D}_i^{\tau}(y^{tj}, x^{sj}))^{-1} &\equiv \min_{\theta, z^1, ..., z^n} \{\theta \\
&\quad \text{such that} \\
\sum_{k=1}^{n} z^k y_m^{\tau k} &\geq y_m^{tj}, m = 1, ..., M, \\
\sum_{k=1}^{n} z^k x_l^{\tau k} &\leq \theta x_l^{sj}, l = 1, ..., N, \\
\theta &\geq 0, \ (z^1, ..., z^n) \in \mathcal{Z}\},
\end{aligned} \tag{45}
$$

where $\mathcal{Z}$ is the permissible set for the intensity variables, $(z^1, ..., z^n)$, that impose shape restrictions on the estimated technology set, such as returns to scale, convexity, etc. In particular, to check if the conclusions are sensitive to the shape constraints, we consider the following three popular specifications in the literature:

(i) constant returns to scale (CRS), free disposability of inputs and outputs and additivity of activities (and therefore convexity), i.e.,[18]

$$
\mathcal{Z} = \{(z^1, ..., z^n) \ : \ z^k \geq 0, \ k = 1, ..., n\},
$$

(ii) non-increasing returns to scale (NIRS), free disposability of inputs and outputs and sub-additivity of activities (and therefore convexity)

$$
\mathcal{Z} = \{(z^1, ..., z^n) \ : \ z^k \geq 0, \ k = 1, ..., n; \ \sum_{k=1}^{n} z^k \leq 1\},
$$

(iii) variable returns to scale (VRS), free disposability of inputs and outputs and sub-additivity of activities (and therefore convexity)

$$
\mathcal{Z} = \{(z^1, ..., z^n) \ : \ z^k \geq 0, \ k = 1, ..., n; \ \sum_{k=1}^{n} z^k = 1\}.
$$

It is worth noting that under certain regularity conditions, all these estimators are consistent estimators of the 'true frontier' with hypothesized assumptions about the shape-constraints, but the rate of convergence depends on the dimension and on the type of the estimator. In particular, it is $O_p(n^{-\kappa})$, where $\kappa = 2/(N + M + 1)$ for the DEA-NIRS and

[18]In this case, the input and output oriented Shephard's distance functions are reciprocal to each other in general and for DEA in particular, i.e. $\widehat{D}_i^{\tau}(y^{tj}, x^{sj}) = (\widehat{D}_o^{\tau}(x^{sj}, y^{tj}))^{-1}$.

DEA-VRS and $\kappa = 2/(N+M+1)$ for the DEA-CRS.[19] That is, for our data, this means $O_p(n^{-1/2})$ and $O_p(n^{-2/3})$, which is the same and better, respectively, than the standard parametric rate.[20]

Furthermore, recall that Kumar and Russell (2002) paid considerable attention when discussing the phenomenon referred to as 'technology implosion'—a situation when the *estimated* technology frontier deteriorated in some of it regions (typically at low capital-labor ratio). Numerically, this leads to the indexes indicating technological regress for countries in that region of the technology set, although this can simply be a drop in efficiency of those countries that was impossible to reveal due to a lack of observations in that region of the frontier.

While Kumar and Russell (2002) did not control for this phenomenon in the estimation, some of the subsequent literature tried to do so by adopting ideas from Diewert (1980) by including all the available observations up to period t to estimate the frontier of period t. That is, while (44) is used for estimating the distances to $\tau = s$ (here 1965) frontier, to prevent the implosion phenomenon the following optimization problems are used for estimating the distance functions for period $\tau = t$ (1990 in this data). Formally,

$$\begin{aligned}
(\widehat{D}_o^t(x^{sj}, y^{tj}))^{-1} &\equiv \max_{\lambda, z^1, \ldots, z^n} \{\lambda \\
&\quad \text{such that} \\
\sum_{\tau=s}^{t} \sum_{k=1}^{n} z^{\tau k} y_m^{\tau k} &\geq \lambda y_m^{tj}, m = 1, \ldots, M, \\
\sum_{\tau=s}^{t} \sum_{k=1}^{n} z^{\tau k} x_l^{\tau k} &\leq x_l^{sj}, l = 1, \ldots, N, \\
\lambda &\geq 0, \ (z^1, \ldots, z^n) \in \mathcal{Z}\},
\end{aligned} \tag{46}$$

[19]E.g., see Simar and Wilson (2015) for a comprehensive review of the statistical aspects of these estimators and references therein.

[20]It is also possible to consider the so-called Free Disposal Hull (FDH) approach, which only imposes free disposability of inputs and output, by requiring $\mathcal{Z} = \{(z^1, \ldots, z^n) \ : \ z^k \in \{0,1\}, \ k = 1, \ldots, n; \ \sum_{k=1}^{n} z^k = 1\}$, which is somewhat an outsider in the literature, partly because it has even slower rate of convergence, $O_p(n^{-1/(N+M)})$, and often suffers from the overfitting problem and low discriminative power in relatively small samples so we do not pursue it here.

and

$$
\begin{aligned}
(\widehat{D}_i^t(y^{tj}, x^{sj}))^{-1} &\equiv \max_{\theta, z^1,...,z^n} \{\theta \\
& \quad \text{such that} \\
\sum_{\tau=s}^{t} \sum_{k=1}^{n} z^{\tau k} y_m^{\tau k} &\geq \theta y_m^{tj}, m = 1, ..., M, \\
\sum_{\tau=s}^{t} \sum_{k=1}^{n} z^{\tau k} x_l^{\tau k} &\leq x_l^{sj}, l = 1, ..., N, \\
\theta &\geq 0, \ (z^1, ..., z^n) \in \mathcal{Z}\},
\end{aligned} \tag{47}
$$

where the definitions of $\mathcal{Z}$ are modified accordingly. Again, we try both approaches to check if the conclusions are sensitive to this phenomenon.

6.3 Discussion of Results

The results are presented in Tables 1-6 and, for convenience, we structure our discussion in separate sub-sections below. Similar to Kumar and Russell (2002), we will focus our discussion on the CRS case (Table 1), which is a common assumption in cross-countries analysis of economic growth and productivity. We will then mention if and how the results change when we relax the CRS assumption to NIRS (Table 2) and VRS (Table 3).[21]

Unlike Kumar and Russell (2002), and more like Henderson and Russell (2005) and Badunenko and Romero-Avila (2013) and other recent works in this stream of the literature, the tables in the main text summarize the results for the estimation approach that prevents the technology implosion. We will then briefly mention if and how the results change when we allow otherwise and the tables for such results are in the appendix (Table 4, 5, 6). We also included the results for the LPI and will discuss them, contrasting to the MPI and DBPI, at the end of this section.

[21] As is well-known, the DEA may give infeasible solutions for MPI under non-CRS assumptions. In our data, this happened only for six countries (with relatively small GDP) and only for DEA-VRS where the implosion of technology was allowed.

Table 1: Productivity Estimates: LPI, MPI and DBPI under CRS, no implosion (% change)

COUNTRY	$(LPI-1)$	(MPI_0-1)	(MPI_1-1)	$(MPI-1)$	$(DBPI_0-1)$	$(DBPI_1-1)$	$(DBPI-1)$
ARGENTINA	4.59	-33.20	-35.26	-34.24	-33.20	-35.26	-34.24
AUSTRALIA	42.67	42.67	13.51	27.26	42.67	13.51	27.26
AUSTRIA	95.15	11.88	-13.04	-1.36	11.88	-13.04	-1.36
BELGIUM	78.36	60.77	23.87	41.12	60.77	23.87	41.12
BOLIVIA	32.71	-18.33	-10.30	-14.41	-18.33	-10.30	-14.41
CANADA	54.55	54.55	18.07	35.08	54.55	18.07	35.08
CHILE	16.57	-21.86	-22.90	-22.38	-21.86	-22.90	-22.38
COLOMBIA	68.78	11.66	8.81	10.22	11.66	8.81	10.22
DENMARK	39.08	18.02	-5.85	5.41	18.02	-5.85	5.41
DOMINICAN REP.	51.81	-28.86	-16.84	-23.09	-28.86	-16.84	-23.09
ECUADOR	80.89	-7.99	-8.53	-8.26	-7.99	-8.53	-8.26
FINLAND	96.23	91.64	43.62	65.90	91.64	43.62	65.90
FRANCE	78.29	36.23	7.33	20.92	36.23	7.33	20.92
GERMANY, WEST	70.75	52.75	15.07	32.58	52.75	15.07	32.58
GREECE	129.46	15.62	5.16	10.26	15.62	5.16	10.26
GUATEMALA	28.54	-5.30	1.82	-1.81	-5.30	1.82	-1.81
HONDURAS	22.87	-5.00	0.50	-2.29	-5.00	0.50	-2.29
HONG KONG	251.08	128.42	122.34	125.36	128.42	122.34	125.36
ICELAND	66.42	-7.14	-12.60	-9.91	-7.14	-12.60	-9.91
INDIA	80.52	29.33	37.23	33.22	29.33	37.23	33.22
IRELAND	133.08	21.20	13.90	17.49	21.20	13.90	17.49
ISRAEL	86.13	41.29	36.46	38.86	41.29	36.46	38.86
ITALY	117.45	63.82	34.90	48.66	63.82	34.90	48.66
IVORY COAST	15.00	-12.11	-26.58	-19.67	-12.11	-26.58	-19.67
JAMAICA	-3.56	-2.38	-2.63	-2.50	-2.38	-2.63	-2.50
JAPAN	208.52	36.01	3.99	18.93	36.01	3.99	18.93
KENYA	35.29	43.10	40.71	41.90	43.10	40.71	41.90
KOREA, REP.	424.45	32.10	48.80	40.20	32.10	48.80	40.20
LUXEMBOURG	78.47	78.47	55.57	66.62	78.47	55.57	66.62
MADAGASCAR	-29.68	-34.94	-33.44	-34.19	-34.94	-33.44	-34.19
MALAWI	43.85	-0.12	-8.84	-4.58	-0.12	-8.84	-4.58
MAURITIUS	56.99	8.61	18.52	13.45	8.61	18.52	13.45
MEXICO	47.47	-10.17	-12.85	-11.52	-10.17	-12.85	-11.52
MOROCCO	52.89	34.07	39.19	36.61	34.07	39.19	36.61
NETHERLANDS	51.45	32.79	6.52	18.93	32.79	6.52	18.93
NEW ZEALAND	7.42	7.42	-15.12	-4.52	7.42	-15.12	-4.52
NIGERIA	40.58	18.80	2.25	10.22	18.80	2.25	10.22
NORWAY	69.72	69.72	66.63	68.17	69.72	66.63	68.17
PANAMA	32.87	-26.56	-27.46	-27.02	-26.56	-27.46	-27.02
PARAGUAY	63.25	25.44	0.00	12.00	25.44	0.00	12.00
PERU	-16.11	-30.65	-31.60	-31.13	-30.65	-31.60	-31.13
PHILIPPINES	43.84	15.99	21.81	18.87	15.99	21.81	18.87
PORTUGAL	168.82	19.01	22.64	20.81	19.01	22.64	20.81
SIERRA LEONE	-5.80	-36.90	-37.46	-37.18	-36.90	-37.46	-37.18
SPAIN	111.74	0.35	-14.54	-7.39	0.35	-14.54	-7.39
SRI LANKA	72.07	5.59	7.09	6.34	5.59	7.09	6.34
SWEDEN	36.03	25.72	-3.97	9.88	25.72	-3.97	9.88
SWITZERLAND	38.68	38.68	27.11	32.77	38.68	27.11	32.77
SYRIA	107.90	51.50	52.83	52.17	51.50	52.83	52.17
TAIWAN	318.96	27.24	20.11	23.62	27.24	20.11	23.62
THAILAND	194.68	30.65	60.08	44.62	30.65	60.08	44.62
TURKEY	129.27	11.69	23.06	17.24	11.69	23.06	17.24
U.K.	60.74	-3.13	-7.25	-5.21	-3.13	-7.25	-5.21
U.S.A.	31.29	31.29	0.00	14.58	31.29	0.00	14.58
YUGOSLAVIA	88.10	-13.55	-5.72	-9.72	-13.55	-5.72	-9.72
ZAMBIA	-33.86	-15.15	-20.88	-18.07	-15.15	-20.88	-18.07
ZIMBABWE	11.38	43.98	37.35	40.62	43.98	37.35	40.62
mean	75.06	18.50	9.88	13.87	18.50	9.88	13.87
s.e.	10.57	4.42	3.95	4.04	4.42	3.95	4.04

6.3.1 Laspeyres vs. Paasche Perspectives of MPI

First of all, note that for the CRS case, the output oriented MPI and input oriented MPI are reciprocal to each other and so we only presented the former as it is the most popular in practice. Furthermore, for the CRS case we see very similar (in fact almost identical) estimates of MPI and DBPI, whether taking Laspeyres or Paasche or Fisher perspectives, when the implosion phenomenon is not allowed (Table 1). Hence, what we say about MPI also will describe DBPI for that case.

On average, the absolute difference between the Laspeyres and Paasche versions of productivity indexes (whether for the MPI or the DBPI) is quite substantial: the average of the absolute difference is about 13% (with standard error of about 1.6%). Moreover, for many countries in the sample the difference is quite dramatic and we discuss a few examples below.

For example, for Australia (where part of this research was carried out) the Laspeyres version suggests about 43% growth in productivity, while the Paasche version suggests it was only about 14% growth, with the Fisher version reconciling it at around 27% (see Table 1). Meanwhile, when we allowed for the technology implosion phenomenon (see Table 4), these estimates were about 43%, 6% and 23%, respectively, i.e., the same conclusion. Similar conclusions are reached for Australia when we relax CRS to NIRS or VRS (Table 2, 3 5, 6).

Since we remarked on Australia, we feel obliged to also remark on its long term friend and rival (e.g., in rugby), New Zealand, especially because it had an even more extreme story: indeed, the Laspeyres version suggests it had about 7% *improvement* in productivity, while the Paasche version suggests it had a *deterioration* of about 15%! Similar estimates are obtained for NIRS and VRS without allowing for implosion and even more extreme difference is revealed when we allow for the implosion (7% vs. -21%). Which perspectives should one rely on here—Laspeyres or Paasche? Clearly, the choice between the two will determine very different economic histories for this interesting country. The Fisher version gives yet another estimate to choose from, which seems more reasonable, as a reconciliation of the two extremes, it is slightly dependent on whether CRS, NIRS or VRS is used: around 5-6% of deterioration of productivity if the implosion is not allowed and around 8-9% otherwise.

Part of this research was also done in the USA and we see quite interesting results for this country too: the Laspeyres version suggests that it had about 31% improvement in productivity, while the Paasche version suggests productivity stagnated, showing 0% change when CRS is used without implosion and even deterioration of about 8% when the implosion is allowed. Meanwhile, the Fisher version reconciled with what seems to be a more plausible estimate of around 15% of improvement when no technology implosion was allowed and 10% otherwise. Remarkably, the difference here gets much larger when we relax the CRS assumption to NIRS or VRS: the Laspeyres version jumped to show about 100% (instead of 31%)

Table 2: Productivity Estimates: LPI, MPI and DBPI under NIRS, no implosion (%change)

COUNTRY	$(LPI-1)$	(MPI_0-1)	(MPI_1-1)	$(MPI-1)$	$(DBPI_0-1)$	$(DBPI_1-1)$	$(DBPI-1)$
ARGENTINA	4.59	-33.20	-33.48	-33.34	-33.20	-38.44	-35.87
AUSTRALIA	42.67	42.67	14.08	27.58	42.67	14.06	27.57
AUSTRIA	95.15	11.88	-13.05	-1.37	11.88	-13.11	-1.40
BELGIUM	78.36	60.77	24.15	41.28	60.77	24.13	41.27
BOLIVIA	32.71	-19.32	-10.29	-14.92	-18.33	-10.30	-14.41
CANADA	54.55	54.55	19.71	36.02	54.55	19.71	36.02
CHILE	16.57	-21.86	-22.91	-22.39	-21.86	-22.94	-22.40
COLOMBIA	68.78	11.66	12.00	11.83	11.66	9.64	10.64
DENMARK	39.08	18.02	-5.85	5.41	18.02	-5.85	5.41
DOMINICAN REP.	51.81	-31.55	-16.84	-24.55	-28.86	-21.24	-25.15
ECUADOR	80.89	-7.99	-8.53	-8.26	-7.99	-8.54	-8.26
FINLAND	96.23	91.64	46.16	67.36	91.64	46.06	67.31
FRANCE	78.29	35.94	3.53	18.63	36.23	3.02	18.47
GERMANY, WEST	70.75	52.75	16.42	33.35	52.75	16.22	33.24
GREECE	129.46	15.62	5.13	10.25	15.62	5.12	10.24
GUATEMALA	28.54	-7.25	1.86	-2.80	-7.53	-3.18	-5.38
HONDURAS	22.87	-5.00	0.50	-2.29	-5.00	0.50	-2.29
HONG KONG	251.08	128.42	122.34	125.36	128.42	122.25	125.31
ICELAND	66.42	-7.14	-12.60	-9.91	-7.14	-12.60	-9.91
INDIA	80.52	-2.54	-2.54	-2.54	-18.69	-27.09	-23.00
IRELAND	133.08	21.20	13.90	17.49	21.20	13.89	17.49
ISRAEL	86.13	41.29	36.46	38.86	41.29	36.46	38.86
ITALY	117.45	63.40	29.97	45.73	63.82	29.39	45.59
IVORY COAST	15.00	-15.06	-30.94	-23.41	-32.46	-58.92	-47.32
JAMAICA	-3.56	-2.38	-2.63	-2.50	-2.38	-2.63	-2.50
JAPAN	208.52	5.30	-19.80	-8.10	10.29	-27.96	-10.87
KENYA	35.29	72.80	80.95	76.83	49.38	56.88	53.09
KOREA, REP.	424.45	3.22	18.65	10.67	8.44	-19.30	-6.45
LUXEMBOURG	78.47	78.47	55.57	66.62	78.47	55.57	66.62
MADAGASCAR	-29.68	-32.41	-31.85	-32.13	-37.77	-33.44	-35.64
MALAWI	43.85	3.79	-9.67	-3.17	-0.12	-56.93	-34.41
MAURITIUS	56.99	8.61	18.52	13.45	8.61	18.52	13.45
MEXICO	47.47	-14.13	-14.10	-14.11	-12.76	-24.22	-18.69
MOROCCO	52.89	34.74	35.07	34.90	21.98	15.30	18.59
NETHERLANDS	51.45	32.79	6.52	18.93	32.79	6.00	18.64
NEW ZEALAND	7.42	7.42	-15.12	-4.52	7.42	-15.12	-4.52
NIGERIA	40.58	6.61	0.00	3.25	-16.43	-47.73	-33.91
NORWAY	69.72	69.72	69.89	69.81	69.72	69.72	69.72
PANAMA	32.87	-26.56	-27.46	-27.02	-26.56	-27.46	-27.02
PARAGUAY	63.25	48.16	0.00	21.72	25.44	-60.04	-29.20
PERU	-16.11	-30.65	-31.02	-30.83	-30.65	-31.62	-31.13
PHILIPPINES	43.84	18.01	24.01	20.97	4.19	1.99	3.09
PORTUGAL	168.82	12.49	19.73	16.05	14.55	14.25	14.40
SIERRA LEONE	-5.80	-11.65	-13.20	-12.43	-35.11	-68.69	-54.92
SPAIN	111.74	-3.73	-18.28	-11.31	-3.31	-26.57	-15.74
SRI LANKA	72.07	2.78	6.78	4.76	5.59	7.04	6.31
SWEDEN	36.03	25.72	-3.27	10.27	25.72	-3.30	10.26
SWITZERLAND	38.68	38.68	30.82	34.70	38.68	30.80	34.68
SYRIA	107.90	51.50	52.83	52.17	51.50	52.83	52.17
TAIWAN	318.96	14.78	12.44	13.60	19.54	-15.68	0.40
THAILAND	194.68	-7.01	5.67	-0.87	-20.97	-33.04	-27.26
TURKEY	129.27	-7.25	-5.06	-6.16	-9.20	-24.16	-17.01
U.K.	60.74	-4.47	-10.49	-7.53	-4.47	-18.52	-11.77
U.S.A.	31.29	100.01	0.00	41.42	31.29	-33.90	-6.84
YUGOSLAVIA	88.10	-28.79	-17.45	-23.33	-28.02	-33.85	-30.99
ZAMBIA	-33.86	1.05	-9.24	-4.23	-15.15	-20.88	-18.07
ZIMBABWE	11.38	50.95	41.64	46.22	43.98	37.35	40.62
mean	75.06	17.85	7.71	12.31	13.58	-2.47	4.69
s.e.	10.57	4.78	3.92	4.13	4.65	4.67	4.52

Table 3: Productivity Estimates: LPI, MPI and DBPI under VRS, no implosion (% change)

COUNTRY	$(LPI-1)$	(MPI_0-1)	(MPI_1-1)	$(MPI-1)$	$(DBPI_0-1)$	$(DBPI_1-1)$	$(DBPI-1)$
ARGENTINA	4.59	-33.19	-33.48	-33.34	-33.18	-38.44	-35.86
AUSTRALIA	42.67	42.26	13.58	27.11	42.67	13.79	27.42
AUSTRIA	95.15	12.76	-12.52	-0.68	14.03	-12.36	-0.03
BELGIUM	78.36	61.90	24.01	41.69	62.89	24.07	42.16
BOLIVIA	32.71	-19.21	-10.39	-14.92	-17.80	-10.09	-14.03
CANADA	54.55	54.27	19.39	35.72	54.55	19.57	35.94
CHILE	16.57	-21.92	-22.91	-22.42	-21.88	-22.94	-22.41
COLOMBIA	68.78	11.64	12.00	11.82	11.68	9.64	10.66
DENMARK	39.08	19.23	-5.46	6.17	20.51	-5.12	6.93
DOMINICAN REP.	51.81	-31.47	-17.05	-24.60	-28.39	-21.09	-24.82
ECUADOR	80.89	-8.11	-8.42	-8.26	-4.20	-7.99	-6.11
FINLAND	96.23	93.86	45.72	68.08	95.47	45.70	68.76
FRANCE	78.29	36.06	3.53	18.68	36.52	3.02	18.59
GERMANY, WEST	70.75	52.85	16.42	33.40	52.98	16.22	33.34
GREECE	129.46	16.47	5.68	10.94	19.40	6.09	12.55
GUATEMALA	28.54	-7.25	1.80	-2.83	-6.88	-3.18	-5.05
HONDURAS	22.87	-5.61	-0.11	-2.90	-4.37	0.63	-1.90
HONG KONG	251.08	128.20	122.31	125.23	128.47	122.25	125.34
ICELAND	66.42	18.94	-0.62	8.72	66.42	3.32	31.13
INDIA	80.52	-2.54	-2.54	-2.54	-18.69	-27.09	-23.00
IRELAND	133.08	23.80	15.48	19.57	30.23	16.14	22.98
ISRAEL	86.13	43.45	37.71	40.55	51.43	38.37	44.75
ITALY	117.45	63.56	30.05	45.84	64.25	29.49	45.84
IVORY COAST	15.00	-15.06	-30.94	-23.41	-32.46	-58.92	-47.32
JAMAICA	-3.56	-3.10	-3.06	-3.08	-2.41	-2.64	-2.52
JAPAN	208.52	5.30	-19.80	-8.10	10.47	-27.96	-10.79
KENYA	35.29	72.80	80.95	76.83	49.38	56.88	53.09
KOREA, REP.	424.45	3.37	18.77	10.80	11.85	-19.14	-4.90
LUXEMBOURG	78.47	68.71	27.86	46.87	78.47	39.38	57.72
MADAGASCAR	-29.68	-32.41	-31.85	-32.13	-37.77	-33.41	-35.63
MALAWI	43.85	3.79	-9.67	-3.17	-12.72	-56.93	-38.69
MAURITIUS	56.99	6.71	14.47	10.52	9.75	18.75	14.16
MEXICO	47.47	-14.13	-14.10	-14.11	-12.74	-24.22	-18.68
MOROCCO	52.89	34.74	35.07	34.90	21.98	15.30	18.59
NETHERLANDS	51.45	33.38	6.76	19.33	34.01	6.34	19.37
NEW ZEALAND	7.42	6.08	-16.04	-5.63	7.24	-15.02	-4.53
NIGERIA	40.58	6.61	0.00	3.25	-16.43	-47.73	-33.91
NORWAY	69.72	68.58	69.89	69.24	69.72	69.72	69.72
PANAMA	32.87	-26.98	-27.58	-27.28	-12.79	-25.90	-19.62
PARAGUAY	63.25	48.16	0.00	21.72	27.15	-60.04	-28.72
PERU	-16.11	-30.68	-31.02	-30.85	-30.64	-31.62	-31.13
PHILIPPINES	43.84	18.01	24.01	20.97	4.19	1.99	3.09
PORTUGAL	168.82	12.53	19.73	16.07	14.79	14.25	14.52
SIERRA LEONE	-5.80	-11.65	-13.20	-12.43	-35.11	-68.69	-54.92
SPAIN	111.74	-3.55	-18.19	-11.17	-2.75	-26.46	-15.43
SRI LANKA	72.07	2.80	6.78	4.77	5.58	7.07	6.33
SWEDEN	36.03	26.53	-3.40	10.55	27.16	-3.38	10.84
SWITZERLAND	38.68	38.38	29.96	34.11	38.68	30.06	34.30
SYRIA	107.90	51.33	52.86	52.09	51.52	53.28	52.40
TAIWAN	318.96	15.15	12.67	13.91	24.69	-15.36	2.73
THAILAND	194.68	-7.01	5.67	-0.87	-20.93	-33.04	-27.24
TURKEY	129.27	-7.25	-5.06	-6.16	-9.14	-24.16	-16.99
U.K.	60.74	-4.40	-10.45	-7.47	-4.30	-18.47	-11.67
U.S.A.	31.29	100.01	0.00	41.42	31.29	-33.90	-6.84
YUGOSLAVIA	88.10	-28.79	-17.45	-23.33	-27.96	-33.85	-30.97
ZAMBIA	-33.86	1.05	-9.34	-4.29	-15.90	-21.00	-18.49
ZIMBABWE	11.38	50.47	41.64	45.99	43.28	37.19	40.20
mean	75.06	18.24	7.37	12.30	15.85	-2.31	5.81
s.e.	10.57	4.74	3.83	4.05	4.76	4.61	4.53

improvement in productivity, while the Paasche version shows 0% change without implosion and about -28% when implosion was allowed. Clearly, Fisher aggregation becomes even more valuable when such radically different estimates are suggested by the two extremes—here it reconciles with estimates of about 41% of improvement when no technology implosion was allowed and 20% otherwise. A similar story, where Laspeyres version suggests improvements while the Paasche warns about deterioration, can also be observed for a few other countries, such as Austria, Paraguay, Sweden and we leave it to the reader to check the details.

It is also worth recalling the explanation from Fisher (1921) about the biases of Laspeyres and Paasche perspectives for measuring price changes, which we quoted in the introduction, and notice that the opposite is likely to be true here for the productivity indexes. Indeed, for most of the countries (and on average) the Laspeyres version gave more (and sometimes much more) optimistic estimates of productivity than the Paasche version, for this sample. A notable (though not dramatic) exception is South Korea, where under CRS the index showed about 32% vs. 49%, respectively, when no implosion is allowed and 32% vs. 61% when implosion is allowed. Interestingly, both numbers changed substantially when we relax CRS to NIRS or VRS: around 3% vs. 19% without implosion or around 3% vs. 38% with implosion. A similar phenomenon is observed for Turkey and Thailand and to a lesser extent for a few other countries in the sample. This suggests a possibly much more complex nature of the biases of Laspeyres and Paasche perspectives of productivity indexes relative to what we know about the price indexes, and this could open an interesting avenue for future research.

Overall, the bottomline of this empirical exercise for this widely recognized data is the confirmation of the importance of not relying on one particular side, whether it is Laspeyres or Paasche, as they may lead to dramatically different conclusions and corresponding policy implications.

6.3.2 The Case of Fixed-base Indexes

Importantly, the same reasoning as in the sub-section above applies to the so-called fixed-base (or fixed-weight) indexes that satisfy transitivity, also illustrating the possibility of dramatic dependence on the choice of the base.

Indeed, as described in Section 4.2, note that if we were to use the so-called fixed-base index with the reference being year 1965, then no matter how many periods of data we have in between 1965 and 1990 (and, in fact, and somewhat ironically, regardless of what the data in those periods were), by transitivity we would have to eventually arrive at the results we have for our Laspeyres version of the productivity indexes when comparing 1965 and 1990 (due to (31)). Similarly, if we were to use 1990 as the reference, then we would arrive at the results we have for our Paasche version of the productivity indexes when comparing 1965 and 1990.

Importantly, note that this conclusion is robust whether we assume CRS or relax to NIRS or VRS. Moreover, and remarkably, when we allowed for the technology implosion phenomenon, while some estimates did change (as expected), generally the same conclusions are maintained, in fact often so with even more dramatic differences.

Here, it is worth recalling another quote from Fisher (in the context of price indexes):

> "... theoretically the circular test ought not to be fulfilled, but that practically it is fulfilled by the best index numbers, and our evidence is the infinitesimal gap worked out [for the 'Ideal Index'] and the other curves in the "superlative" group." Fisher (1922, p. 297.)

Our empirical example, for the context of productivity indexes, vividly shows that the difference between the fixed-weight indexes and their Fisher aggregate could be very far from what one may want to call an 'infinitesimal gap'.

Overall, by showing that these two fixed perspectives may lead to very contradictory conclusions about productivity change for many countries in the sample, this example therefore supports the view that one should be very cautious about the results of fixed weight indexes and at least check their sensitivity to the choice of weights. Indeed, in the era of 'Fake News' and polarized societies, one should be able to imagine how one version of the index can be used among the evidence to support claims that certain economic reforms in some countries (e.g., as for Austria, New Zealand, Paraguay, Sweden, USA in this sample) were successful, while the other version could be as effectively used to suggest that exactly the same reforms were complete disasters. Clearly, some reasonable reconciliation is needed here and, as we discussed throughout this paper, Fisher aggregation provides a well-justified candidate for such a reconciliation.

6.3.3 MPI vs. DBPI

We have already noted that for the DEA-CRS case without implosion we see almost identical estimates of MPI and DBPI, whether taking Laspeyres or Paasche or Fisher perspectives despite using quite different philosophies in their constructions. This phenomenon of similarity of MPI and DBPI is of course not general (although not uncommon in practice) and certain restrictions on technology are needed to guarantee it to hold exactly.[22] Importantly, note that the high similarity under CRS disappears when the implosion is allowed.

When we relax CRS to NIRS or VRS then the differences in the estimates from these two productivity indexes appear, and a bit more so for VRS than NIRS. Interestingly, for

[22] See Färe et al. (2020) for a more detailed and recent discussion and earlier references.

the majority of countries, the difference is relatively minor (e.g., under 5%), yet sometimes it is substantial in quantity and for a few countries it becomes even different in terms of the sign, i.e. implying qualitatively different conclusions.

For example, the most striking difference is for the USA: DEA-NIRS and DEA-VRS without implosion for the Fisher version of MPI suggested about 41% improvement, while the Fisher version of DBPI suggested about 7% deterioration! The difference reduces to some extent when we allow for the implosion phenomenon (due to fall in MPI), becoming around 20% vs. -6%, respectively, yet still showing the two multi-factor productivity indexes implying radically different conclusions. Here it is also worth recalling that the Fisher versions of both MPI and DBPI from DEA-CRS in one voice suggested about 15% improvement when no implosion was allowed and 10% when the implosion was allowed, which seems much more plausible estimates. The latter phenomenon (DBPI falling dramatically from 15% or 10% to -7% or -6%) can be viewed as a reason to favor CRS assumption for the estimation of MPI and DBPI.

A similar story is observed for Paraguay: about 22% improvement vs. about 29% deterioration when implosion is not allowed, although allowing for the technology implosion both indexes suggest deterioration, albeit at very different magnitudes: about -7% vs. -32% (i.e., the MPI fell dramatically, from 22% to -7% while DBPI remained roughly where it was). Again recall that the Fisher versions of both MPI and DBPI from DEA-CRS without implosion coherently suggested 12% improvement, which seem more plausible, although note this also changes dramatically when the implosion is allowed: both suggesting -15% . The latter phenomenon (MPI and DBPI fall dramatically) is an example of the importance of preventing the implosion of technology in the estimation. The examples described in the last two paragraphs also present empirical evidence in favor of assuming CRS and preventing the technology implosion in the estimation of productivity indexes.

6.3.4 Labor Productivity index vs. MPI or DBPI

It seems also worth noting the dramatic difference observed between the estimates of multi-factor productivity measures like MPI or DBPI (whether the Laspeyres or Paasche versions or their average) and the estimates of the labor productivity index (LPI). Such comparison is particularly important because LPI appears to be by far the most popular productivity metric in wider literature, and especially in the media. It is therefore useful to employ a real data example like this one, to explore the radical difference and potential misrepresentation of productivity dynamics by LPI relative to the multi-factor productivity indexes, and vice versa.

For example, at Table 1 suggests that while for very few countries LPI practically coincided with the Laspeyres version of the MPI (e.g., for Australia, Canada, Luxembourg, New Zealand, Norway, Switzerland, USA), it was quite different qualitatively for most other countries in the sample, and quite dramatic for some. Importantly, the qualitative differences persisted whether CRS or NIRS or VRS were assumed or whether the implosion was allowed or not, although those assumptions, of course, mattered for the quantitative differences.

For example, for Argentina, the labor productivity index suggested improvement of about 5%, while the Fisher version of MPI and DBPI suggested it was a deterioration of over 33% in all the approaches. On the other hand, for Austria, the labor productivity index suggested a staggering improvement of about 95%, while the Fisher version of MPI and DBPI suggested it slightly deteriorated, in the range of -2-0%, depending on the assumptions. Similar stories are observed for many other countries in the sample.

The most dramatic difference is observed for South Korea and Taiwan: the LPI shows a skyrocketing improvement of nearly 424% and 319%, while the Fisher version of MPI and DBPI suggested it was much more modest (although still substantial). Indeed, the DEA-CRS version of MPI suggested about 40% and 24% improvement, respectively, when not allowing for the implosion and otherwise 46% and 26%, respectively. As was noted in a different context above, the estimates change when we relax CRS: e.g., the estimates of MPI and DBPI for South Korea then drop substantially, to about 11% and -5-(-7)%, respectively, when implosion is prevented and otherwise 19% and -5-(-7)%, respectively. We view these as examples in favor of assuming CRS and for preventing the technology implosion in the estimation.

Somewhat smaller yet still dramatic differences are also observed for Greece, Hong Kong, Ireland, Japan, Portugal, Thailand, Turkey, to mention a few.

7 Which Productivity Index to Use?

The previous section illustrated vividly that the difference between LPI and different versions of multi-factor productivity indexes can be fairly large and often quite dramatic. Moreover, even for just the two indexes we focused on, MPI and DBPI, we see there are many variations: different perspectives (Laspeyres, Paasche or their average), fixed base vs. changing base, type of estimator used (DEA, SFA (parametric, nonparametric, semi-parametric), etc.), assumptions imposed (CRS, VRS, etc.), with or without implosion, etc. Besides all these (and other possible) variations for these two indexes, there are also other productivity indexes, e.g. those based on the classical index numbers, which are typically defined as the ratio of an output quantity index to an input quantity index. And, for these, we also have many alternatives: Laspeyres, Paasche, Fisher, Törnqvist, Walsh, etc. and probably more to come.

With such a myriad of productivity indexes, a natural question is: Which productivity index is the best and should be preferred over all others? Our short answer is: None! And, very likely there will never be one! Indeed, similarly as in the context of price indexes, none of the productivity indexes appears as the best. There is, indeed, no panacea for measuring productivity or its changes for general contexts and searching for such a panacea or the 'best index' would be like chasing a *Fata Morgana*.[23] Instead, a more worthy approach is to recognize and acknowledge that all productivity indexes have their advantages and limitations or caveats, which may vary across contexts or data, as already documented in many works. [24]

For example, an important advantage of the LPI is that it has a unique perspective, and so does not need an aggregation of the often disagreeing Laspeyres and Paasche perspectives. It is also very simple to compute, without the need to estimate a frontier and thus also immune from the implosion phenomenon. It is also very easy to interpret (although possibly as easy to misinterpret too). These are the advantages that will probably continue to keep LPI on the front pages of various media as well as many academic outlets.

On the other hand, a key disadvantage of LPI (and the key reason for the substantial difference with the multi-factor productivity indexes) is that, as any single-factor productivity index, LPI measures changes in GDP only relative to the changes in one of the inputs, labor.[25] As a result, and despite its popularity in the wide audience, our empirical example gave yet another confirmation that LPI *alone* may misrepresent the productivity dynamics. Metaphorically speaking, it is like a photograph from one particular angle—sometimes it can be a perfect shot, or good enough or not at all. Hence, while being an intuitive, easy to compute and useful productivity metric, LPI better be complemented with estimates of the multi-factor productivity indexes that account for all major inputs. But, again, which one? MPI or DBPI or any other? Their Laspeyres or Paasche of Fisher versions? Under CRS, NIRS or VRS? With or without allowing for technology implosion in the estimation? All of them?Any answer to these questions are likely to be based, at least to some extent, on various judgement calls, and their adequacy may differ across contexts and data.

While it seems impossible to give a one-size-fits-all advice or rule on which productivity index to prefer, general advice on how to proceed might be useful for practitioners. Based on our theoretical discussion and empirical evidence in this and other cited papers, and our

[23]See related discussions in Keynes (1930) and Balk (2008, p. 28-30).

[24]E.g., see see Diewert (1992); Diewert and Nakamura (1993), Balk (1998, 2008), Sickles and Zelenyuk (2019, Chapter 4 and 7) and many references therein.

[25]Indeed, the gap between LPI and MPI is fully explained by the contributions from the other inputs: e.g. in the data set we used in Section 6, it can be explained by the capital-deepening contribution (Kumar and Russell (2002)), while in Henderson and Russell (2005) data set it can be explained by physical and human capital deepening.

understanding of the literature in general, we can recommend for practitioners the following steps:

Step 1. Depending on the context of study and on the available data, use economic logic (mathematical or intuitive) to justify a suitable multi-factor productivity index (or a range of them) to measure the economic phenomenon of interest. E.g., in this paper, we advocated the use of the indexes based on economic fundamentals (complete characterizations of underlying or hypothetical technologies), such as MPI or DBPI, and then taking their Fisher's versions, which can be theoretically justified in the various ways we discussed above.

Step 2. Depending on the data, select a suitable estimator, where:

- unless there is a strong reason to believe otherwise, incorporate the assumption of no technology implosion, adopting the ideas from Diewert (1980), as we illustrated in our empirical example;

- especially for cross-country studies, use the CRS assumption as the main approach (as this is where MPI and DBPI and other productivity indexes often yield coherent conclusions) to estimate a multi-factor productivity index and present its estimates along with LPI (when feasible), as it is an important and intuitive productivity metric.[26] Contrast the LPI and the multi-factor productivity index(es) to find some robustness and possibly get new insights from emerging differences.

Step 3. Check the sensitivity of conclusions when the assumption of CRS is relaxed to NIRS or VRS or FDH, whichever seem most appropriate to the question of interest, the context or the data, to confirm robustness of conclusions or to gain new insights from identifying the differences.

8 Concluding Remarks

In this paper we investigated the question of Fisher aggregation of the Paasche and Laspeyres versions, as applied to the Malmquist quantity indexes and related productivity indexes. We did so from both theoretical and empirical perspectives.

For the theoretical perspective, we considered existing theoretical justifications and then provided an alternative (and complementary) theoretical justification based on results from functional equations literature.

For the empirical perspective, we used a data set from Kumar and Russell (2002) to vividly illustrate how dramatic the differences in conclusions can be in practice depending

[26] Here, the assumption of CRS shall not be dogmatically understood as a belief that the 'true' technology is CRS, rather as a selection of a relevant common benchmark/reference, e.g. coherent with the socially optimal scale and the notion of maximal average productivity.

on whether one takes Laspeyres or Paasche productivity indexes. This conclusion was robust to different assumptions on the estimation and to the data sets we tried besides the one presented here.

In particular, our empirical illustration also shows that the imposition of fixed-base (or fixed-weight) on multi-factor productivity indexes for ensuring the so-called transitivity or circularity, while it may seem innocent at a glance, can imply a very high 'price' in the sense of implying dramatically different conclusions.

Needless to elaborate yet important to mention, very different estimates of such important economic concepts as productivity may potentially suit radically different political views, and be used to justify them by empirically 'confirming' them, potentially contributing to polarization of society on alternative economic policies, and therefore should be avoided or, at least, reconciled with a suitable aggregation, such as the Fisher aggregation we focused on in this paper.

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Table 4: Productivity Estimates: LPI, MPI and DBPI under CRS, with implosion (% change)

COUNTRY	$(LPI-1)$	(MPI_0-1)	(MPI_1-1)	$(MPI-1)$	$(DBPI_0-1)$	$(DBPI_1-1)$	$(DBPI-1)$
ARGENTINA	4.59	-33.20	-35.26	-34.24	-33.20	-35.26	-34.24
AUSTRALIA	42.67	42.67	6.43	23.23	42.67	6.43	23.23
AUSTRIA	95.15	11.88	-13.00	-1.34	11.88	-13.00	-1.34
BELGIUM	78.36	60.77	18.16	37.83	60.77	18.16	37.83
BOLIVIA	32.71	-18.33	-10.30	-14.41	-18.33	-10.30	-14.41
CANADA	54.55	54.55	9.51	30.09	54.55	9.51	30.09
CHILE	16.57	-21.86	-22.90	-22.38	-21.86	-22.90	-22.38
COLOMBIA	68.78	11.66	8.81	10.22	11.66	8.81	10.22
DENMARK	39.08	18.02	-7.82	4.31	18.02	-7.82	4.31
DOMINICAN REP.	51.81	-28.86	-16.84	-23.09	-28.86	-16.84	-23.09
ECUADOR	80.89	-7.99	-2.98	-5.52	-7.99	-2.98	-5.52
FINLAND	96.23	91.64	33.42	59.90	91.64	33.42	59.90
FRANCE	78.29	36.23	7.33	20.92	36.23	7.33	20.92
GERMANY, WEST	70.75	52.75	10.05	29.65	52.75	10.05	29.65
GREECE	129.46	15.62	10.22	12.88	15.62	10.22	12.88
GUATEMALA	28.54	-5.30	1.82	-1.81	-5.30	1.82	-1.81
HONDURAS	22.87	-5.00	0.50	-2.29	-5.00	0.50	-2.29
HONG KONG	251.08	128.42	122.34	125.36	128.42	122.34	125.36
ICELAND	66.42	-7.14	-7.63	-7.39	-7.14	-7.63	-7.39
INDIA	80.52	29.33	30.94	30.13	29.33	30.94	30.13
IRELAND	133.08	21.20	20.52	20.86	21.20	20.52	20.86
ISRAEL	86.13	41.29	44.56	42.91	41.29	44.56	42.91
ITALY	117.45	63.82	36.40	49.49	63.82	36.40	49.49
IVORY COAST	15.00	-12.11	-50.54	-34.07	-12.11	-50.54	-34.07
JAMAICA	-3.56	-2.38	-2.63	-2.50	-2.38	-2.63	-2.50
JAPAN	208.52	36.01	3.99	18.93	36.01	3.99	18.93
KENYA	35.29	43.10	40.95	42.02	43.10	40.95	42.02
KOREA, REP.	424.45	32.10	60.87	45.78	32.10	60.87	45.78
LUXEMBOURG	78.47	78.47	51.18	64.26	78.47	51.18	64.26
MADAGASCAR	-29.68	-34.94	-33.44	-34.19	-34.94	-33.44	-34.19
MALAWI	43.85	-0.12	-54.68	-32.72	-0.12	-54.68	-32.72
MAURITIUS	56.99	8.61	18.52	13.45	8.61	18.52	13.45
MEXICO	47.47	-10.17	-12.61	-11.40	-10.17	-12.61	-11.40
MOROCCO	52.89	34.07	39.19	36.61	34.07	39.19	36.61
NETHERLANDS	51.45	32.79	3.42	17.19	32.79	3.42	17.19
NEW ZEALAND	7.42	7.42	-20.91	-7.83	7.42	-20.91	-7.83
NIGERIA	40.58	18.80	-26.89	-6.80	18.80	-26.89	-6.80
NORWAY	69.72	69.72	66.63	68.17	69.72	66.63	68.17
PANAMA	32.87	-26.56	-24.53	-25.55	-26.56	-24.53	-25.55
PARAGUAY	63.25	25.44	-41.81	-14.57	25.44	-41.81	-14.57
PERU	-16.11	-30.65	-31.60	-31.13	-30.65	-31.60	-31.13
PHILIPPINES	43.84	15.99	21.81	18.87	15.99	21.81	18.87
PORTUGAL	168.82	19.01	22.64	20.81	19.01	22.64	20.81
SIERRA LEONE	-5.80	-36.90	-71.70	-57.74	-36.90	-71.70	-57.74
SPAIN	111.74	0.35	-12.03	-6.05	0.35	-12.03	-6.05
SRI LANKA	72.07	5.59	7.09	6.34	5.59	7.09	6.34
SWEDEN	36.03	25.72	-9.16	6.87	25.72	-9.16	6.87
SWITZERLAND	38.68	38.68	25.25	31.79	38.68	25.25	31.79
SYRIA	107.90	51.50	59.36	55.38	51.50	59.36	55.38
TAIWAN	318.96	27.24	24.52	25.87	27.24	24.52	25.87
THAILAND	194.68	30.65	60.08	44.62	30.65	60.08	44.62
TURKEY	129.27	11.69	23.06	17.24	11.69	23.06	17.24
U.K.	60.74	-3.13	-1.60	-2.37	-3.13	-1.60	-2.37
U.S.A.	31.29	31.29	-7.81	10.02	31.29	-7.81	10.02
YUGOSLAVIA	88.10	-13.55	-5.72	-9.72	-13.55	-5.72	-9.72
ZAMBIA	-33.86	-15.15	-20.88	-18.07	-15.15	-20.88	-18.07
ZIMBABWE	11.38	43.98	37.35	40.62	43.98	37.35	40.62
mean	75.06	18.50	6.70	11.85	18.50	6.70	11.85
s.e.	10.57	4.42	4.49	4.25	4.42	4.49	4.25

Table 5: Productivity Estimates: LPI, MPI and DBPI, under NIRS, with implosion (% change)

COUNTRY	$(LPI-1)$	(MPI_0-1)	(MPI_1-1)	$(MPI-1)$	$(DBPI_0-1)$	$(DBPI_1-1)$	$(DBPI-1)$
ARGENTINA	4.59	-33.20	-32.87	-33.03	-33.20	-44.65	-39.19
AUSTRALIA	42.67	42.67	6.97	23.54	42.67	4.02	21.82
AUSTRIA	95.15	11.88	-14.45	-2.17	11.88	-23.52	-7.50
BELGIUM	78.36	60.77	18.43	37.99	60.77	13.87	35.30
BOLIVIA	32.71	-19.32	-9.16	-14.39	-18.33	-10.30	-14.41
CANADA	54.55	54.55	7.85	29.11	54.55	0.84	24.84
CHILE	16.57	-21.86	-21.97	-21.92	-21.86	-26.94	-24.44
COLOMBIA	68.78	11.66	26.04	18.63	11.66	-2.16	4.52
DENMARK	39.08	18.02	-7.82	4.31	18.02	-7.82	4.31
DOMINICAN REP.	51.81	-31.55	-16.01	-24.18	-28.86	-22.06	-25.54
ECUADOR	80.89	-7.99	-2.98	-5.52	-7.99	-3.93	-5.98
FINLAND	96.23	91.64	35.79	61.31	91.64	35.70	61.26
FRANCE	78.29	35.94	-19.21	4.80	36.23	-23.03	2.39
GERMANY, WEST	70.75	52.75	-8.37	18.31	52.75	-12.66	15.50
GREECE	129.46	15.62	7.00	11.22	15.62	5.47	10.43
GUATEMALA	28.54	-7.25	6.62	-0.56	-7.53	-3.56	-5.57
HONDURAS	22.87	-5.00	0.50	-2.29	-5.00	0.50	-2.29
HONG KONG	251.08	128.42	122.34	125.36	128.42	98.98	113.19
ICELAND	66.42	-7.14	-7.63	-7.39	-7.14	-7.63	-7.39
INDIA	80.52	-2.54	-17.99	-10.60	-18.69	-27.09	-23.00
IRELAND	133.08	21.20	20.52	20.86	21.20	20.00	20.60
ISRAEL	86.13	41.29	44.56	42.91	41.29	44.56	42.91
ITALY	117.45	63.40	2.40	29.35	63.82	-2.30	26.51
IVORY COAST	15.00	-15.06	-36.71	-26.68	-32.46	-68.60	-53.95
JAMAICA	-3.56	-2.38	-2.63	-2.50	-2.38	-2.63	-2.50
JAPAN	208.52	5.30	-36.84	-18.45	10.29	-44.03	-21.44
KENYA	35.29	72.80	89.79	81.10	49.38	61.55	55.34
KOREA, REP.	424.45	3.22	38.14	19.41	8.44	-20.39	-7.09
LUXEMBOURG	78.47	78.47	51.18	64.26	78.47	51.18	64.26
MADAGASCAR	-29.68	-32.41	-31.96	-32.19	-37.77	-33.44	-35.64
MALAWI	43.85	3.79	-39.31	-20.64	-0.12	-60.82	-37.44
MAURITIUS	56.99	8.61	18.52	13.45	8.61	18.52	13.45
MEXICO	47.47	-14.13	-11.62	-12.88	-12.76	-26.10	-19.71
MOROCCO	52.89	34.74	33.69	34.21	21.98	13.80	17.82
NETHERLANDS	51.45	32.79	2.57	16.71	32.79	-9.25	9.78
NEW ZEALAND	7.42	7.42	-20.91	-7.83	7.42	-20.91	-7.83
NIGERIA	40.58	6.61	-31.77	-14.71	-16.43	-47.73	-33.91
NORWAY	69.72	69.72	69.89	69.81	69.72	69.72	69.72
PANAMA	32.87	-26.56	-24.53	-25.55	-26.56	-24.53	-25.55
PARAGUAY	63.25	48.16	-41.81	-7.15	25.44	-63.48	-32.32
PERU	-16.11	-30.65	-24.26	-27.53	-30.65	-33.29	-31.98
PHILIPPINES	43.84	18.01	38.64	27.91	4.19	7.13	5.65
PORTUGAL	168.82	12.49	18.98	15.69	14.55	-1.44	6.25
SIERRA LEONE	-5.80	-11.65	-71.70	-50.00	-35.11	-71.70	-57.14
SPAIN	111.74	-3.73	-31.92	-19.04	-3.31	-43.97	-26.40
SRI LANKA	72.07	2.78	12.01	7.29	5.59	1.12	3.33
SWEDEN	36.03	25.72	-8.50	7.26	25.72	-10.12	6.30
SWITZERLAND	38.68	38.68	28.91	33.71	38.68	28.89	33.70
SYRIA	107.90	51.50	59.36	55.38	51.50	59.36	55.38
TAIWAN	318.96	14.78	12.99	13.88	19.54	-29.08	-7.92
THAILAND	194.68	-7.01	30.67	10.23	-20.97	-23.33	-22.16
TURKEY	129.27	-7.25	15.96	3.71	-9.20	-14.64	-11.96
U.K.	60.74	-4.47	-26.29	-16.08	-4.47	-30.49	-18.51
U.S.A.	31.29	100.01	-28.44	19.63	31.29	-32.28	-5.71
YUGOSLAVIA	88.10	-28.79	-3.14	-16.95	-28.02	-34.67	-31.42
ZAMBIA	-33.86	1.05	-8.57	-3.88	-15.15	-20.88	-18.07
ZIMBABWE	11.38	50.95	48.66	49.80	43.98	37.35	40.62
mean	75.06	17.85	4.03	9.60	13.58	-7.24	1.78
s.e.	10.57	4.78	4.62	4.25	4.65	4.59	4.38

Table 6: Productivity Estimates: LPI, MPI and DBPI, under VRS, with implosion (% change)

COUNTRY	$(LPI-1)$	(MPI_0-1)	(MPI_1-1)	$(MPI-1)$	$(DBPI_0-1)$	$(DBPI_1-1)$	$(DBPI-1)$
ARGENTINA	4.59	-33.19	-32.87	-33.03	-33.18	-44.65	-39.18
AUSTRALIA	42.67	42.26	6.91	23.32	42.67	4.02	21.82
AUSTRIA	95.15	12.76	-13.96	-1.50	14.03	-22.89	-6.23
BELGIUM	78.36	61.90	18.33	38.41	62.89	13.87	36.19
BOLIVIA	32.71	-19.21	-9.51	-14.50	-17.80	-9.60	-13.79
CANADA	54.55	54.27	7.85	28.99	54.55	0.84	24.84
CHILE	16.57	-21.92	-21.97	-21.95	-21.88	-26.94	-24.45
COLOMBIA	68.78	11.64	26.04	18.62	11.68	-2.16	4.53
DENMARK	39.08	19.23	-7.78	4.86	20.51	-7.35	5.66
DOMINICAN REP.	51.81	-31.47	-17.56	-24.83	-28.39	-21.54	-25.04
ECUADOR	80.89	-8.11	-2.96	-5.57	-4.20	-2.71	-3.46
FINLAND	96.23	93.86	35.11	61.84	95.47	35.31	62.63
FRANCE	78.29	36.06	-19.21	4.84	36.52	-23.03	2.50
GERMANY, WEST	70.75	52.85	-8.37	18.34	52.98	-12.66	15.59
GREECE	129.46	16.47	7.22	11.75	19.40	5.99	12.50
GUATEMALA	28.54	-7.25	6.42	-0.65	-6.88	-3.56	-5.24
HONDURAS	22.87	-5.61	-36.95	-22.85	-4.37	0.94	-1.75
HONG KONG	251.08	128.20	122.22	125.19	128.47	98.98	113.22
ICELAND	66.42	18.94	-Inf	-100.00+Infi	66.42	-32.20	6.22
INDIA	80.52	-2.54	-17.99	-10.60	-18.69	-27.09	-23.00
IRELAND	133.08	23.80	20.92	22.35	30.23	21.52	25.80
ISRAEL	86.13	43.45	39.42	41.42	51.43	44.48	47.91
ITALY	117.45	63.56	2.40	29.42	64.25	-2.29	26.68
IVORY COAST	15.00	-15.06	-36.71	-26.68	-32.46	-68.60	-53.95
JAMAICA	-3.56	-3.10	-4.08	-3.59	-2.41	-2.34	-2.37
JAPAN	208.52	5.30	-36.84	-18.45	10.47	-44.03	-21.37
KENYA	35.29	72.80	89.79	81.10	49.38	61.55	55.34
KOREA, REP.	424.45	3.37	38.14	19.50	11.85	-20.39	-5.64
LUXEMBOURG	78.47	68.71	-Inf	-100.00+Infi	78.47	38.98	57.49
MADAGASCAR	-29.68	-32.41	-31.96	-32.19	-37.77	-34.70	-36.25
MALAWI	43.85	3.79	-Inf	-100.00+Infi	-12.72	-54.36	-36.89
MAURITIUS	56.99	6.71	-Inf	-100.00+Infi	9.75	6.14	7.93
MEXICO	47.47	-14.13	-11.62	-12.88	-12.74	-26.10	-19.70
MOROCCO	52.89	34.74	33.69	34.21	21.98	13.80	17.82
NETHERLANDS	51.45	33.38	2.75	17.07	34.01	-9.06	10.40
NEW ZEALAND	7.42	6.08	-21.41	-8.69	7.24	-20.63	-7.74
NIGERIA	40.58	6.61	-31.77	-14.71	-16.43	-47.73	-33.91
NORWAY	69.72	68.58	69.89	69.24	69.72	69.72	69.72
PANAMA	32.87	-26.98	-31.16	-29.10	-12.79	-21.81	-17.42
PARAGUAY	63.25	48.16	-Inf	-100.00+Infi	27.15	-63.48	-31.86
PERU	-16.11	-30.68	-24.26	-27.54	-30.64	-33.29	-31.98
PHILIPPINES	43.84	18.01	38.64	27.91	4.19	7.13	5.65
PORTUGAL	168.82	12.53	18.98	15.71	14.79	-1.44	6.37
SIERRA LEONE	-5.80	-11.65	-Inf	-100.00+Infi	-35.11	-71.70	-57.14
SPAIN	111.74	-3.55	-31.92	-18.97	-2.75	-43.97	-26.19
SRI LANKA	72.07	2.80	12.01	7.31	5.58	1.24	3.39
SWEDEN	36.03	26.53	-8.60	7.54	27.16	-10.12	6.91
SWITZERLAND	38.68	38.38	28.33	33.26	38.68	28.41	33.45
SYRIA	107.90	51.33	58.97	55.10	51.52	60.23	55.82
TAIWAN	318.96	15.15	12.99	14.07	24.69	-28.99	-5.91
THAILAND	194.68	-7.01	30.67	10.23	-20.93	-23.33	-22.14
TURKEY	129.27	-7.25	15.96	3.71	-9.14	-14.64	-11.93
U.K.	60.74	-4.40	-26.29	-16.05	-4.30	-30.49	-18.44
U.S.A.	31.29	100.01	-28.44	19.63	31.29	-32.28	-5.71
YUGOSLAVIA	88.10	-28.79	-3.14	-16.95	-27.96	-34.67	-31.40
ZAMBIA	-33.86	1.05	-8.91	-4.06	-15.90	-13.63	-14.77
ZIMBABWE	11.38	50.47	48.66	49.56	43.28	36.82	40.01
mean	75.06	18.24	-Inf	-1.24+Infi	15.85	-7.73	2.48
s.e.	10.57	4.74	-	-	4.76	4.52	4.35

Table 7: Supplement: Kumar and Russell (2002) Data Used in the Empirical Example

COUNTRY	L65	K65	Y65	L90	K90	Y90
ARGENTINA	8723.37	48458293.20	111816094.00	11346.21	127576819.00	152107332.00
AUSTRALIA	4729.19	95761323.80	100476324.00	8122.69	307476364.00	246215025.00
AUSTRIA	3257.91	26379294.50	44574720.00	3666.81	126732304.00	97903840.00
BELGIUM	3584.29	54864755.30	63764552.00	4156.42	152316326.00	131883344.00
BOLIVIA	1290.88	3449239.10	5169986.00	2237.29	12799514.20	11891176.00
CANADA	7664.20	141228266.00	170490192.00	13247.89	566280869.00	455462306.00
CHILE	2770.98	14029479.90	28178112.00	4820.69	46003856.50	57144474.00
COLOMBIA	5606.28	38375001.50	33576024.00	10545.11	133395677.00	106590000.00
DENMARK	2235.50	31625685.90	40138488.00	2863.57	94855706.50	71506169.00
DOMINICAN REP.	1072.69	1749550.80	4874285.00	2221.26	13376456.10	15322284.00
ECUADOR	1688.19	10017702.00	8429118.00	3217.12	51525318.00	29056985.00
FINLAND	2133.01	36175852.80	29729896.00	2563.00	117301029.00	70098174.00
FRANCE	20914.65	257020191.00	356113824.00	25985.55	925085696.00	788843440.00
GERMANY, WEST	26836.80	406765334.00	463793528.00	30728.98	1540013490.00	906781430.00
GREECE	3396.30	22313706.10	26222850.00	3867.05	90782785.20	68512464.00
GUATEMALA	1392.71	2955341.02	8055463.00	2631.07	9595518.94	19562019.00
HONDURAS	707.84	1917532.06	2571574.00	1574.73	6508350.09	7029585.00
HONG KONG	1923.77	12946963.30	12508344.00	3711.11	47361206.50	84713545.00
ICELAND	80.33	623338.41	1205710.00	136.41	2984295.01	3407310.00
INDIA	204235.54	160529134.00	365990087.00	331927.96	645931816.00	1073786960.00
IRELAND	1114.51	7718000.39	11504000.00	1350.35	29248672.60	32486822.00
ISRAEL	931.64	11346385.80	11902572.00	1822.06	39088737.30	43328680.00
ITALY	20889.50	251802057.00	295858017.00	23381.19	739780913.00	720070568.00
IVORY COAST	2399.48	674252.88	6416200.00	4695.00	4831155.34	14437126.00
JAMAICA	686.48	2430143.66	3663064.00	1165.18	4044342.29	5996020.00
JAPAN	60566.34	319305732.00	444132954.00	78253.57	2854690200.00	1770408750.00
KENYA	4289.53	4645558.78	5906680.00	11814.15	10715433.30	22009760.00
KOREA, REP.	9842.69	20600750.20	30069418.00	17854.50	321291770.00	286064837.00
LUXEMBOURG	134.36	3676951.93	2853477.00	164.08	8530622.07	6218960.00
MADAGASCAR	3004.20	4350087.47	6669333.00	5047.58	8772696.96	7879275.00
MALAWI	1919.26	197683.54	1623692.00	3627.88	1552733.71	4415133.00
MAURITIUS	357.24	714125.52	2320640.00	615.40	2366213.30	6275850.00
MEXICO	13029.28	79452538.80	150305754.00	27992.34	361101231.00	476205748.00
MOROCCO	3757.58	5568734.48	16638567.00	7972.04	15872340.50	53970741.00
NETHERLANDS	4407.91	65087239.50	90926424.00	6235.50	201905611.00	194809608.00
NEW ZEALAND	1006.36	18363970.20	23808352.00	1523.56	50399350.10	38718219.00
NIGERIA	24644.84	6432303.23	36499008.00	45975.98	32275136.10	95721985.00
NORWAY	1501.47	56534888.10	25874850.00	2161.32	104035133.00	63214284.00
PANAMA	443.95	2753820.82	2672578.00	873.01	12979869.90	6983184.00
PARAGUAY	677.04	138115.88	2647221.00	1425.89	1300411.70	9101456.00
PERU	3537.62	22962683.90	28874045.00	6874.29	60466245.00	47068256.00
PHILIPPINES	12097.01	31053033.70	40234667.00	22656.61	83784157.50	108389240.00
PORTUGAL	3550.41	12025247.20	21973503.00	4435.47	52422812.50	73792904.00
SIERRA LEONE	1045.22	70029.67	2759378.00	1498.41	334144.56	3726536.00
SPAIN	11791.54	69711591.80	146816480.00	14161.13	386598917.00	373344097.00
SRI LANKA	3933.42	16394475.20	13125807.00	6202.95	55423341.30	35617328.00
SWEDEN	3484.19	55078092.70	72715068.00	4450.60	175393521.00	126347958.00
SWITZERLAND	2759.70	84744735.20	65294400.00	3376.25	248016050.00	110781560.00
SYRIA	1402.75	13613665.20	10708575.00	2974.99	44606986.60	47216052.00
TAIWAN	4770.71	14145141.80	20962480.00	8914.46	229297663.00	164106239.00
THAILAND	15218.53	14975037.50	34880880.00	29843.76	146592538.00	201564740.00
TURKEY	14991.71	35455401.60	56443800.00	24312.17	184505029.00	209862618.00
U.K.	25086.73	219308215.00	417568662.00	28361.10	600659703.00	758801187.00
U.S.A.	80692.51	1410000000.00	2260000000.00	122928.83	4266245120.00	4520216090.00
YUGOSLAVIA	8792.79	21559918.90	46777638.00	10820.76	92647335.70	108283332.00
ZAMBIA	1285.98	3005325.00	4007100.00	2691.15	3630354.71	5546450.00
ZIMBABWE	1890.27	10536368.10	4135912.00	4755.65	18180835.80	11589510.00