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Data Envelopment Analysis

and

Sequences in Hilbert Spaces

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Abstract

We introduce a new class of Data Envelopment Analysis modeling, which we call 'sequential DEA'. This new approach allows for analyzing efficiency of the decision making units that consist of a sequence of sub-DMUs (e.g., branches of banks, hospital holding company running a number of hospitals at different locations, hotel chains, etc.). The approach is embedded in the Hilbert sequence space (ℓ^2) and therefore it allows for potentially different numbers of the sub-DMUs as well as different numbers of inputs and outputs used by different decision making units. This approach opens up a new stream of literature in the sense that many existing variations from the already rich literature on Data Envelopment Analysis can be adapted to this approach.

Keywords: Data Envelopment Analysis, Efficiency.

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1 Introduction

Data Envelopment Analysis (DEA) is one of the most popular approaches for analyzing the performance (productivity and efficiency) of various economic and business systems. This approach is based on the economic theory foundation found in seminal works of Koopmans (1951), Debreu (1951), Farrell (1957) and Afriat (1972) and the generalized and empowered by seminal works in the operations research literature (Charnes et al. (1978), Banker et al. (1984), Banker (1993)) and statistics/econometrics literature (Korostelev et al. (1995), Gijbels et al. (1999), Kneip et al. (2015)).

In standard DEA models, the Decision Making Units (DMUs) consists of a single input/output vector (Charnes et al. (1978)). These DMUs are used in computing their relative Farrell (1957) efficiency scores. In a more general framework one may consider the case when a DMU consists of a *sequence* of sub-DMUs. An example would be a bank that has more than one (and potentially many) branches or a company that owns a chain of food stores or pharmacies or restaurants, etc.

The purpose of this paper is to develop a DEA approach explicitly handling sub-DMUs. This will allow us to compare, for example, banks or hospitals or hotel chains with many branches considered explicitly rather than just the 'aggregate' decision making units that own the branches.

To the best of our knowledge, the models we introduce here appear to be new to the DEA literature. Indeed, the only closest (though still very distant) work we found as related to this new approach is that of Shephard and Färe (1978) on Dynamic Production Correspondences, where they developed production theory in Banach spaces.[1] In this paper our model is embedded in the Hilbert sequence space, ℓ^2, which is an example (perhaps one of the most popular) of the Banach spaces, and is a sequence space "similar" to the commonly used Euclidean space. This allows for a potentially different number of the sub-DMUs as well as

[1] See Luenberger (1969) on Banach spaces in the context of optimization theory. Also see Shephard and Färe (1980).

different numbers of inputs and outputs used by different decision making units. Overall, it appears that this new approach may open a new stream of literature within the DEA literature, in the sense that many existing variations from the already rich literature on Data Envelopment Analysis can be adapted to this approach.

The rest of the paper is structured as follows. In the next section we give a simple example to explain the essence of our approach and the notation. Then, in Section 3, we present a general model of what we call 'sequential DEA'. In Section 4 we give concluding remarks.

2 An Introductory Example

Assume there are two DMUs, indexed by $k = 1, 2$, and that they each have two sub-DMUs. Further assume that each sub-DMU uses one input to produce one output. The input for sub-DMU s that belongs to the DMU k is denoted by x_k^s. Meanwhile, the output has the same form of notation, i.e., y_k^s is the output produced by sub-DMU s that belongs to DMU k. In general, the notation would be x_k^s and y_k^s mean that, respectively, the input used and output produced by sub-DMU s belongs to DMU k. for $k = 1, 2, ..., K,\ s = 1, 2, ..., S$.

Then, for the simple 2 by 2 example, the data for the two DMUs is denoted as:

$$\begin{aligned} &\text{DMU 1: } x_1 = (x_1^1, x_1^2),\ y_1 = (y_1^1, y_1^2) \\ &\text{DMU 2: } x_2 = (x_2^1, x_2^2),\ y_2 = (y_2^1, y_2^2) \end{aligned} \tag{1}$$

Many types of efficiency measures can be estimated with DEA. Here we will focus on one of the most popular efficiency measures in practice, often referred to as the Farrell input oriented efficiency measure, due to the seminal work of Farrell (1957).[2] The DEA model for

[2]Also see Debreu (1951) for the economic theory roots of this measure and Färe et al. (2019) for various generalizations.

estimating the efficiency score of DMU 1 of this measure can be calculated as

$$\begin{aligned}
&\min \lambda \\
&s.t.\ \ z_1 y_1 + z_2 y_2 \geqq y_1 \\
&\qquad z_1 x_1 + z_2 x_2 \leqq \lambda x_1 \\
&\qquad z_1, z_2 \geqq 0
\end{aligned} \tag{2}$$

where z_1 and z_2 are the intensity variables.[3]

Inserting the data (1) into (2), yields

$$\begin{aligned}
&\min \lambda \\
&s.t.\ \ z_1(y_1^1, y_1^2) + z_2(y_2^1, y_2^2) \geqq (y_1^1, y_1^2) \\
&\qquad z_1(x_1^1, x_1^2) + z_2(x_2^1, x_2^2) \leqq \lambda(x_1^1, x_1^2) \\
&\qquad z_1, z_2 \geqq 0
\end{aligned} \tag{3}$$

which can be written as

$$\begin{aligned}
&\min \lambda \\
&s.t.\ \ (z_1 y_1^1, z_1 y_1^2) + (z_2 y_2^1, z_2 y_2^2) \geqq (y_1^1, y_1^2) \\
&\qquad (z_1 x_1^1, z_1 x_1^2) + (z_2 x_2^1, z_2 x_2^2) \leqq (\lambda x_1^1, \lambda x_1^2) \\
&\qquad z_1, z_2 \geqq 0
\end{aligned} \tag{4}$$

[3] To save space, in this paper we will also focus on the DEA model with constant returns to scale and input orientation, often referred to as the CCR model due to the seminal work of Charnes et al. (1978). Other variants can be developed by analogy.

Since addition of vectors is done componentwise, we have

$$
\begin{aligned}
&\min \lambda \\
&s.t.\ \ (z_1 y_1^1 + z_2 y_2^1, z_1 y_1^2 + z_2 y_2^2) \geqq (y_1^1, y_1^2) \\
&\qquad (z_1 x_1^1 + z_2 x_2^1, z_1 x_1^2 + z_2 x_2^2) \leqq (\lambda x_1^1, \lambda x_1^2) \\
&\qquad z_1, z_2 \geqq 0
\end{aligned}
\tag{5}
$$

Finally, since vector inequalities are defined componentwise we find that (5) can be written as

$$
\begin{aligned}
&\min \lambda \\
&s.t.\ \ z_1 y_1^1 + z_2 y_2^1 \geqq y_1^1 \\
&\qquad z_1 y_1^2 + z_2 y_2^2 \geqq y_1^2 \\
&\qquad z_1 x_1^1 + z_2 x_1^2 \leqq \lambda x_1^1 \\
&\qquad z_1 x_1^2 + z_2 x_2^2 \leqq \lambda x_1^2 \\
&\qquad z_1, z_2 \geqq 0
\end{aligned}
\tag{6}
$$

and therefore we have two DEA problems stated jointly, with the restriction that the contracting factor λ and the intensity variable z_1, z_2 are common optimization variables, i.e.,

$$\begin{aligned}
&\min \lambda \\
&s.t. \quad z_1 y_1^1 + z_2 y_2^1 \geqq y_1^1 \\
&\qquad z_1 x_1^1 + z_2 x_2^1 \leqq \lambda x_1^1 \\
&\qquad z_1 y_1^2 + z_2 y_2^2 \geqq y_1^2 \\
&\qquad z_1 x_1^2 + z_2 x_2^2 \leqq \lambda x_1^2 \\
&\qquad z_1, z_2 \geqq 0
\end{aligned} \tag{7}$$

These two problems allow us to compare DMU 1 and DMU 2 without any aggregation.

If we were to undertake our comparison using aggregate data then from (3) we would have

$$\begin{aligned}
&\min \lambda \\
&s.t. \quad z_1(y_1^1 + y_1^2) + z_2(y_2^1 + y_2^2) \geqq (y_1^1 + y_1^2) \\
&\qquad z_1(x_1^1 + x_1^2) + z_2(x_2^1 + x_2^2) \leqq \lambda(x_1^1 + x_1^2) \\
&\qquad z_1, z_2 \geqq 0
\end{aligned} \tag{8}$$

Next, we show that the two models, the sequential (7) and the aggregate (8) may yield the same efficiency score. Given data

$$\begin{aligned}
&\text{DMU 1: } x_1 = (1,2),\ y_1 = (1,2) \\
&\text{DMU 2: } x_2 = (2,1),\ y_2 = (2,1)
\end{aligned} \tag{9}$$

The aggregate model for computing the input oriented efficiency for DMU 1 becomes

$$\begin{aligned} &\min \lambda \\ &s.t.\ \ z_1 3 + z_2 3 \geqq 3 \\ &\qquad z_1 3 + z_2 3 \leqq \lambda 3 \\ &\qquad z_1, z_2 \geqq 0 \end{aligned} \tag{10}$$

yields $\lambda = 1$.

Meanwhile, the sequential model is

$$\begin{aligned} &\min \lambda \\ &s.t.\ \ z_1 1 + z_2 2 \geqq 1 \\ &\qquad z_1 1 + z_2 2 \leqq \lambda 1 \\ \\ &\qquad z_1 2 + z_2 1 \geqq 2 \\ &\qquad z_1 2 + z_2 1 \leqq \lambda 2 \\ &\qquad z_1, z_2 \geqq 0 \end{aligned} \tag{11}$$

yields $\lambda = 1$, i.e., the two models produce the same efficiency score. Next change the data into:

$$\begin{aligned} &\text{DMU 1: } x_1 = (2,2),\ y_1 = (1,2) \\ &\text{DMU 2: } x_2 = (2,1),\ y_2 = (2,1) \end{aligned} \tag{12}$$

then the aggregate model for computing the input oriented efficiency for DMU 1 becomes

$$\begin{aligned}
&\min \lambda \\
&s.t. \quad z_1 3 + z_2 3 \geqq 3 \\
&\qquad\ z_1 4 + z_2 3 \leqq \lambda 4 \\
&\qquad\ z_1, z_2 \geqq 0
\end{aligned} \tag{13}$$

yields $\lambda = 3/4$.

On the other hand, the sequential model gives

$$\begin{aligned}
&\min \lambda \\
&s.t. \quad z_1 1 + z_2 2 \geqq 1 \\
&\qquad\ z_1 2 + z_2 2 \leqq \lambda 2 \\
&\qquad\ z_1 2 + z_2 1 \geqq 2 \\
&\qquad\ z_1 2 + z_2 1 \leqq \lambda 2 \\
&\qquad\ z_1, z_2 \geqq 0
\end{aligned} \tag{14}$$

yielding $\lambda = 1$. Thus, the two models do not produce the same efficiency score.

3 General Model

Here we think of a DMU k, $k = 1, ..., K$, having $n = 1, ..., N$ different inputs which produces $m = 1, ..., M$ different outputs. We may think of the individual inputs and outputs as sequences, sub-DMUs, and take them as element in the Hilbert sequence space (ℓ^2).[4] On the other hand, our finite sequences can easily be amended by zeros to fit into ℓ^2.

[4]See Luenberger (1969) on this topic in the context of optimization theory.

Thus, given that each DMU consists of x_n, $n = 1, ..., N$ input sequences and y_m, $m = 1, ..., M$ output sequences we may consider the production technology

$$\begin{aligned} T = &\{(x, y) \ : \ x_n = (x_n^1, x_n^2, x_n^3, ...), \ n = 1, ..., N \\ &\text{can produce } y_m = (y_m^1, y_m^2, y_m^3, ...), \ m = 1, ..., M\}. \end{aligned}$$

We like to formalize this technology as a DEA/Activity Analysis technology and compute Farrell (here) technical efficiency scores for each DMU $k = 1, ..., K$.

To make our theoretical model operational, we again make use of the addition of sequences, which is done componentwise, and the comparisons are also done componentwise.

The input data we anticipate having available is

$$\begin{aligned} x_{kn}^s \in \mathbb{R}_+, \ & k = 1, ..., K \ \text{(DMUs)} \\ & n = 1, ..., N \ \text{(inputs)} \\ & s = 1, ..., S_{kn} \ \text{(sub-DMUs)} \end{aligned} \tag{15}$$

The output data is denoted similarly,

$$\begin{aligned} y_{km}^s \in \mathbb{R}_+, \ & k = 1, ..., K \ \text{(DMUs)} \\ & n = 1, ..., M \ \text{(inputs)} \\ & s = 1, ..., S_{km} \ \text{(sub-DMUs)} \end{aligned} \tag{16}$$

To simplify the notation, we note that since we can always add zeros to our observations, we may simply use S rather than S_{kn} and S_{km} and so we will do this from now on. For the same reason we also use common N and M rather than N_s and M_s.[5]

[5]Shephard and Färe (1978) describe their theory as having inputs and outputs in Banach spaces and their theoretical foundation can be potentially used (with some adaptation) to further generalize our approach, to represent DEA models in sequences of the Banach space.

Some regularity conditions on the data are also need, namely:

$$i. \qquad \sum_{s=1}^{S} x_{kn}^{s} > 0, \quad k = 1, .., K; \; n = 1, .., N. \tag{17}$$

$$ii. \qquad \sum_{s=1}^{S} y_{km}^{s} > 0, \quad k = 1, .., K; \; m = 1, .., M. \tag{18}$$

$$iii. \qquad \sum_{n=1}^{N} x_{kn}^{s} > 0, \quad k = 1, .., K; \; s = 1, ..., S. \tag{19}$$

$$iv. \qquad \sum_{n=1}^{M} y_{km}^{s} > 0, \quad k = 1, .., K; \; s = 1, ..., S. \tag{20}$$

These conditions are in the spirit of conditions discussed by Karlin (1959).[6] In particular, the first two conditions ((17) and (18)) require that at least one sub-DMU employs a positive magnitude of *each* input as well as produces a positive magnitude of *each* output. Similarly, the third and forth conditions ((19) and (20)) require, respectively, that at least one input must be used by each sub-DMU and at least one output must be produced by each sub-DMU.

Given the data (x_{kn}^{s}, y_{km}^{s}) we may compare the Farrell input technical efficiency for DMU k' as

$$\begin{aligned} & \min \lambda \\ & s.t. \;\; \sum_{k=1}^{K} z_k (y_{km}^{1}, ..., y_{km}^{S}) \geqq (y_{k'm}^{1}, ..., y_{k'm}^{S}), \; m = 1, ..., M \\ & \qquad \sum_{k=1}^{K} z_k (x_{kn}^{1}, ..., y_{kn}^{S}) \geqq \lambda (x_{k'n}^{1}, ..., x_{k'n}^{S}), \; n = 1, ..., N \\ & \qquad z_k \geqq 0, \; k = 1, ..., K. \end{aligned} \tag{21}$$

[6] Also see Sickles and Zelenyuk (2019) for related discussions.

Our aggregate model derived from above is

$$\begin{aligned}
&\min \lambda \\
&s.t. \quad \sum_{k=1}^{K} z_k \left(\sum_{s=1}^{S} y_{km}^s \right) \geqq \sum_{s=1}^{S} y_{k'm}^s, \ m = 1, ..., M \\
&\qquad \sum_{k=1}^{K} z_k \left(\sum_{s=1}^{S} x_{kn}^s \right) \leqq \lambda \sum_{s=1}^{S} x_{k'n}^s, \ n = 1, ..., N \\
&\qquad z_k \geqq 0, \ k = 1, ..., K.
\end{aligned} \tag{22}$$

Following our idea from the introductory example, we can derive the sequential model from expression (21) as "S" different DEA models using component addition and comparing

$$\begin{aligned}
&\min \lambda \\
&s.t. \quad \sum_{k=1}^{K} z_k y_{km}^s \geqq y_{k'm}^s, \ m = 1, ..., M \\
&\qquad \sum_{k=1}^{K} z_k x_{kn}^s \leqq \lambda x_{k'n}^s, \ n = 1, ..., N \\
&\qquad z_k \geqq 0, \ k = 1, ..., K \\
&\qquad s = 1, ..., S.
\end{aligned} \tag{23}$$

Thus, we have $s = 1, ..., S$ different DEA problems. Note that if a DMU has only one sub-DMU then our expression (23) becomes the standard DEA input oriented problem.

There are a few points to note here. First, since the Farrell measure is the reciprocal of the input distance function, we can use our computational model for estimating Malmquist (input) quantity indexes as well as Malmquist productivity indexes (Caves et al. (1982)).

Another interesting thing to observe is the data, here for DMU k:

$$\begin{aligned}
x_{kn} &= x_{kn}^1, x_{kn}^2, ..., x_{kn}^S, \ n = 1, ..., N \\
y_{km} &= y_{km}^1, y_{km}^2, ..., y_{km}^S, \ m = 1, ..., M
\end{aligned}$$

and we note that a DMU is modeled as a matrix, rather than a vector, as in the usual case. Hence, our model allows us to estimate Farrell efficiency scores for matrices..

4 Concluding Remarks

In this paper we introduced a new class of DEA models and proposed to name it 'sequential DEA'. The idea of this approach is to facilitate analysis of the efficiency of DMUs that consist of a sequence of sub-DMUs. Examples of such DMUs are banks with many branches, or hospital holding companies, each running a numbers of hospitals at different locations, hotel and restaurant chains, etc. To facilitate the modeling of such environments with varying number of branches and of inputs and outputs in each of them, we cast it in the Hilbert sequence space (ℓ^2).

We believe this approach may open a new stream of literature in within the rich and growing DEA literature, since it allows for adapting the myriad of existing versions of DEA in this more general framework. In particular, some fruitful directions of research would be to adapt various variants of the network DEA (e.g., following Färe and Grosskopf (1996) and Kao (2009a,b, 2014)), of modeling of congestion of inputs and undesirable outputs (e.g., following Färe and Svensson (1980),Färe and Grosskopf (2009), Dakpo et al. (2017)), for various types of efficiency measures (e.g., see Färe et al. (2019)) and to develop statistical properties of the this new class of DEA models, adapting results from Korostelev et al. (1995), Gijbels et al. (1999), Kneip et al. (2015) and in the aggregation context as in Simar and Zelenyuk (2018), to mention a few.

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References

Afriat, S.N., 1972. Efficiency estimation of production functions. International Economic Review 13, 568–598.

Banker, R.D., 1993. Maximum likelihood, consistency and data envelopment analysis: A statistical foundation. Management Science 39, 1265–1273.

Banker, R.D., Charnes, A., Cooper, W.W., 1984. Some models for estimating technical and scale inefficiencies in data envelopment analysis. Management Science 30, 1078–1092.

Caves, D.W., Christensen, L.R., Diewert, W.E., 1982. The economic theory of index numbers and the measurement of input, output, and productivity. Econometrica 50, 1393–1414.

Charnes, A., Cooper, W., Rhodes, E., 1978. Measuring the efficiency of decision making units. European Journal of Operational Research 2, 429–444.

Dakpo, K.H., Jeanneaux, P., Latruffe, L., 2017. Modelling pollution-generating technologies in performance benchmarking: Recent developments, limits and future prospects in the nonparametric framework. European Journal of Operational Research 250, 347–359.

Debreu, G., 1951. The coefficient of resource utilization. Econometrica 19, 273–292.

Färe, R., Grosskopf, S., 1996. Intertemporal Production Frontiers: With Dynamic DEA. Norwell, MA: Kluwer Academic Publishers.

Färe, R., Grosskopf, S., 2009. A comment on weak disposability in nonparametric production analysis. American Journal of Agricultural Economics 91, 535–538.

Färe, R., He, X., Li, S.K., Zelenyuk, V., 2019. A unifying framework for Farrell efficiency measurement. Operations Research 67, 183–197.

Färe, R., Svensson, L., 1980. Congestion of production factors. Econometrica: Journal of the Econometric Society 48, 1745–1753.

Farrell, M.J., 1957. The measurement of productive efficiency. Jornal of the Royal Statistical Society. Series A (General) 120, 253–290.

Gijbels, I., Mammen, E., Park, B.U., Simar, L., 1999. On estimation of monotone and concave frontier functions. Journal of the American Statistical Association 94, 220–228.

Kao, C., 2009a. Efficiency decomposition in network data envelopment analysis: A relational model. European Journal of Operational Research 192, 949–962.

Kao, C., 2009b. Efficiency measurement for parallel production systems. European Journal of Operational Research 196, 1107–1112.

Kao, C., 2014. Network data envelopment analysis: A review. European Journal of Operational Research 239, 1–16.

Karlin, S., 1959. Mathematical methods and theory in games, programming, and economics. volume 2. Reading, MA: Addison Wesley.

Kneip, A., Simar, L., Wilson, P.W., 2015. When bias kills the variance: Central limit theorems for DEA and FDH efficiency scores. Econometric Theory 31, 394–422.

Koopmans, T.C., 1951. Analysis of production as an efficienct combination of activities. Activity Analysis of Production and Allocation 13, 33–37.

Korostelev, A., Simar, L., Tsybakov, A.B., 1995. Efficient estimation of monotone boundaries. Annals of Statistics 23, 476–489.

Luenberger, D.G., 1969. Optimization by vector space methods / [by] David G. Luenberger. Series in decision and control, Wiley, New York.

Shephard, R., Färe, R., 1980. Dynamic Theory of Production Correspondences. Mathematical systems in economics, Verlag Anton Hain.

Shephard, R.W., Färe, R., 1978. Dynamic Theory of Production Correspondences. Part I, II, III. Technical Report.

Sickles, R., Zelenyuk, V., 2019. Measurement of Productivity and Efficiency: Theory and Practice. New York, NY: Cambridge University Press.

Simar, L., Zelenyuk, V., 2018. Central limit theorems for aggregate efficiency. Operations Research 166, 139–149.