Speculative Trade and Market Newcomers[*,†]

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Picturing that in the real world inexperienced investors may be prey for veteran traders, we give a formal sufficient condition for a speculative bubble of this type in a simple stationary model. The condition is simply that some relatively inexperienced cohort belongs to the most optimistic group but another more experienced cohort does not. This agreement to disagree leads to a perpetual bubble, in which the price overshoots the most optimistic fundamental valuation. Our condition allows the most experienced to be among the most optimistic. As in a fraction of the uniform-experience literature, lack of short-selling makes room for the success of such bubble schemes. This previous literature did not allow for persistent effects of experience on beliefs and, instead, relied on more direct assumptions of belief heterogeneity. Although we map experience into beliefs in a specific way, the intuition behind the perpetual bubble involves the above-mentioned disagreement patterns, not belief formation itself.

Keywords: speculative trade; price bubble; experience; optimism; belief heterogeneity; non-Bayesian learning; short-selling

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1. Introduction

Young or immigrant-turned investors are inexperienced compared to incumbent traders. On the one hand, the veteran investors may simply dominate the market and act as

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price-stabilizers in an environment with inexperienced traders potentially predisposed to form bubbles (*Dufwenberg et al.*, 2005). On the other hand, the experienced investors may systematically take advantage of newcomers and be themselves willing to pay a premium for an opportunity to resell the asset to a novice at a bubbly price. We present an intuitive sufficient condition for a perpetual bubble of this type in a model generating stationary disagreement about fundamentals across experience levels. Our sufficient condition is simply that some relatively inexperienced cohort (exemplified by Jerry) belongs to the most optimistic group but another more experienced cohort (exemplified by Larry) does not.

We argue that this sufficient condition extends beyond our deliberately simple tractable model. For instance, as grounds for the disagreement about fundamentals in terms of the primitives, we use short-sales constraints that in equilibrium permit newcomers' prior beliefs to differ from the incumbents' already updated beliefs. Also, in case one insists on somehow modeling rather than arbitrarily assuming differences in beliefs prior to trade, we do so in terms of possibly coarse and nonstatistical learning about the past and probabilistic learning while trading. Similar differentiation between learning about the past and from experience, but for different purposes, appeared in theoretical models of *Schraeder* (2015), *Collin-Dufresne et al.* (2016), and *Ehling et al.* (2017). To support our claim of generality, we proceed to giving an intuition independent of this specific way of obtaining belief heterogeneity (à la *Harrison and Kreps*, 1978; *Morris*, 1996; *Scheinkman and Xiong*, 2003; *Werner*, 2018).

For that, we first need to observe that the asset price will be at least as high as the fundamental valuation of that most optimistic, but relatively inexperienced, trader Jerry. She should be able to anticipate that a currently less experienced trader (exemplified by Harry) will one day also become the most optimistic like Jerry now but at that time Jerry will be already relatively pessimistic like Larry now. Jerry could sell the asset above her fundamental valuation to Harry then. This anticipation make today's the most optimistic Jerry willing to pay above her fundamental valuation already today. The result is a perpetual bubble in the sense that the asset price is always strictly greater than the most optimistic fundamental valuation (Jerry's) of this asset.

Supporting empirical evidence of persistent age differences in investors' expectations exists. When stock prices were reaching two-year lows during the last week of March 2001, investors above age 60 were more likely to consider the stocks to be overvalued (39% vs. 25%) according to *Dreman et al.* (2001). They also compare this downturn period's survey measures of investors' expectations with those from a period of rapid rise in stock prices in 1998 and do not find significant differences.

Admittedly, perpetual bubbles fueled by repeated trade between veteran investors and novices, as under our condition, are only one aspect of the interaction between different experience cohorts in financial markets. We think that such perpetual bubbles may coexist with a significant degree of price stabilization and bubble bursting attributable to actions of relatively experienced investors, as in a fraction of the previous literature. For instance, experience as a source of pessimism helped explain the dotcom bubble burst—a record number of IPO lockups expired, bringing relatively pessimistic asset-holders back to the market (*Ofek and Richardson*, 2003). However, this conclusion does not imply that a bubble cannot have a perpetual component with boom-and-bust phases, which we do not model, on top of it. This also reconciles the perpetual bubbles with the experimental findings that inexperienced traders' entry into the market does not cause "bubble-crash phenomena" (*Dufwenberg et al.*, 2005; *Xie and Zhang*, 2016). Such a causal effect is theoretically possible, though, if the inexperienced behave like positive-feedback-investment strategists (noise traders that buy high and sell low) in *de Long et al.* (1990).

Our question about trade between veteran investors and novices is indeed similar to the one of rational speculators versus noise traders in *de Long et al.* (1990). Unlike them, we focus on perpetual bubbles rather than temporary ones and do not assume that newcomers follow the noise traders' "really dumb" positive-feedback strategies. Nevertheless, a part of how we model disagreement between veterans and novices—possibly coarse and nonstatistical learning about history—likewise helped *de Long et al.* (1990) justify noise traders' long-run relevance:

> By the time the new bubble comes along, many investors have forgotten the old one or have been replaced by younger investors who have never experienced the old one at all.

To model experience acquisition and learning, we adopt (and adapt) the framework of *Harrison and Kreps* (1978), where all traders, unlike in *de Long et al.* (1990), react to current prices. In the former model, traders' fully rational consideration of all available information may not prevent overpricing if their prior beliefs are suitably heterogeneous and short-selling is limited. Our approach is to introduce dispersed timing of market entry and to assume that newcomers form prior beliefs over the future at time of entry but not over the entire history, learning about the past not via updating. This accommodates heterogeneous beliefs (at a given time) even though traders form identical beliefs at time of entry—the histories prior to entry are identical—and learn identically as time goes by: The source of disagreement in our symmetric-information setting is merely the variation in the duration of market participation and probabilistic learning. Our assumption of identical newcomers permits the formal identification of optimism as a function of cohort with that as a function of a single trader's experience.

2. Model

Despite being similar in spirit to the papers in the framework of *Harrison and Kreps* (1978), with recent contributions by *Steiner and Stewart* (2015) and *Werner* (2018), our setting is novel and one of the most tractable. We model time as continuous with infinite forward and backward horizons. We identify traders with their market-entry times (one trader per time point). They are risk-neutral and maximize expected discounted returns of holding the asset under uncertainty about the dividend. In Section 2.1 below, we start describing the model in detail from time 0.

2.1. Asset, Traders, Beliefs, and Market Participation

At time 0, a trader enters the market for an asset and forms beliefs that the asset pays a dividend of \$1 per unit at uncertain time $\theta \in (0, \infty)$, but for simplicity the truth is that this asset does not pay dividends at all. Our model applies when the asset pays a dividend, but we need to reinterpret many things (for example, the bubble) as conditional on the dividend being unpaid. The total number of units of the asset is finite and strictly positive.

The beliefs of this time-0 entrant are that the dividend time is a random variable whose values are strictly positive and which has a continuous distribution function F with range $[0, 1)$. For an illustration of the most important scenarios, one can think of the gamma distribution.

We adopt the interpretation according to which these beliefs incorporate information from the past and current times $(-\infty, 0]$, but we neither model this information processing nor require it to be rational in any sense. In particular, we do not view these beliefs as a Bayesian update of longer-horizon-supported beliefs. Since this is the moment the trader just enters the market, the idea that the trader's Bayesian statistical model does not cover the past is hard to dismiss without finding a grain of truth. This spares the trader forming probabilistic beliefs about no longer random, at least to the extent that they have occurred, events. Admitting that rationality and timing of prior formation are philosophical questions, we remind the reader that the intuition behind the bubbles in question relies on the induced valuations, not on belief formation per se.

To define, among others, the trader's infinite-horizon valuation as a measure of the trader's willingness to pay for the asset if obliged to hold it forever, we assume that the trader can borrow and lend at a constant rate $r \in (0, \infty)$. The infinite-horizon valuation is a function of time, through probabilistic learning. Since the trader learns that the asset does not pay, the (conditional) infinite-horizon valuation of the asset at time $t \geq 0$ is the (per-unit) expected discounted dividend

$$V_0(t) = \frac{1}{1 - F(t)} \int_t^{\infty} e^{-r(\theta - t)} dF(\theta) \in (0, 1), \tag{1}$$

less than the dividend amount but strictly positive $(F(t) < 1)$.

At each time point $\tau \in \mathbb{R} \backslash \{0\}$, an identical trader enters the market after past entrants observe that the asset does not pay the dividend at this time τ and forms beliefs that the asset pays at uncertain time $\theta \in (\tau, \infty)$. This trader's uncertainty is analogous to the time-0 entrant's. Since we will look at a steady-state price (Section 2.3), all traders will find themselves in exactly time-0 entrant's position at entry. This allows us to assume that they form their beliefs and valuations in the same way, starting the same probabilistic learning at entry. For every time-τ entrant, who can also borrow and lend at the rate r, the infinite-horizon valuation of the asset at time $t \geq \tau$, in $t - \tau$ time units from entry, is

$$V_\tau(t) = V_0(t - \tau),$$

equal to the time-0 entrant's in $t-\tau$ time units from entry. This infinite-horizon valuation gives their willingness to pay for the asset in $t-\tau$ units from entry if obliged to hold the asset forever.

Traders' infinite-horizon valuations depend on time only through experience, and their infinite-horizon valuation as a function of experience coincides with the time-0 entrant's infinite-horizon-valuation function V_0. To accommodate disagreement across experience levels, we simply need V_0 to be nonconstant for some choices of the beliefs F about the dividend time. Loosely speaking, to furnish an example, the function V_0 must be strictly increasing where the distribution function F is flat and thus learning that the asset does not pay reduces the expected waiting time without changing the beliefs. Reversing this intuition suggests that V_0 should be strictly decreasing where F is infinitely steep (almost jumps), in which case the news that the asset does not pay is particularly surprising. To make these sufficient conditions for local strict monotonicity precise and to verify them when F has density f over and continuous on $(0,\infty)$, note that (it should be easy to see) at every time $t>0$ the derivative of V_0 is

$$V_0'(t) = rV_0(t) - \frac{f(t)}{1-F(t)}(1-V_0(t)).$$

Firstly, this shows that V_0 is indeed strictly increasing on an open ball around t if $F'(t) = f(t) = 0$. Secondly, the function V_0 will be strictly decreasing on a nonempty initial segment of $(0,\infty)$ if $\lim_{t\to 0^+} f(t) = \infty$, as when f is $\mathrm{Gamma}(\alpha,\beta)$ with $\alpha<1$ (the formula is

$$f(\theta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \theta^{\alpha-1} e^{-\frac{\theta}{\beta}} \tag{2}$$

for all $\theta>0$). These examples confirm that our model has room for the heterogeneity of infinite-horizon valuations, and thus beliefs at a given time, prior to trading, if any (Section 2.3), at that time. If traders were sufficiently short-lived in the second example, the most experienced traders would always be the least optimistic under the infinite-horizon criterion, which is an instance of our claimed condition for a perpetual bubble. Below we restrict the life duration to be finite to have sets of potential asset-holders compact, allowing them to use the infinite-horizon criterion too, for instance for the sake of paternalistic bequest. Our simple model does not capture intergenerational transfers explicitly though.

We assume that all traders have a fixed lifetime $T\in(0,\infty)$: Every time-τ entrant exits at $\tau+T$ after the new trader enters, giving us the compact set $[\tau,\tau+T]$ of potential asset-holders at $\tau+T$. As indicated above, the traders' planning horizon may differ from their lifetime T. All traders have a fixed planning horizon $H\in(0,\infty]$, which can be shorter or longer than the lifetime T. While the latter case accommodates paternalistic bequest, the former is somewhat opposite and we interpret the horizon $H\le T$ as the market-participation duration, or the (early) retirement age. For the sake of smooth interpretation, it is convenient to think of traders planning beyond lifetimes as being too paternalistic to retire early so that in this case the market participation-duration is

$T \leq H$. Our unifying symbol for the market-participation duration is $M = \min\{H, T\}$. Planning for finite horizons $H < \infty$ always involves the consideration of resale proceeds, which we introduce together with prices in Section 2.2 below.

2.2. Prices and Long- versus Short-term Valuations

We focus on a steady-state nominal per-unit price $p \in \mathbb{R}$, which is compelling for our simple stationary set-up and our emphasis on perpetual bubbles. We allow traders to plan to resell, only conditional on the dividend being unpaid for the asset to be still valuable, the asset at future time points. We need to distinguish between long- and short-term valuations for the purpose of defining fundamental value in a robust way as the most optimistic long-term valuation. Both long- and short-term valuations are expected discounted returns of holding the asset, which we define formally next.

The time-0 entrant's expected discounted return of holding a unit of the asset between arbitrary $t \geq 0$ and $s > t$ is the sum $R_0(t, s)$ of the expected discounted dividend

$$D_0(t, s) = \frac{1}{1 - F(t)} \int_t^s e^{-r(\theta - t)} dF(\theta) \tag{3}$$

and resale proceeds

$$S_0(t, s) = \frac{1 - F(s)}{1 - F(t)} e^{-r(s-t)} p, \tag{4}$$

i.e.,

$$R_0(t, s) = D_0(t, s) + S_0(t, s). \tag{5}$$

Since with probability

$$\frac{1 - F(s)}{1 - F(t)}$$

the asset remains valuable and the trader swaps the updated infinite-horizon valuation $V_0(s)$ for the price p at the selling time s, we can rewrite the expected discounted return $R_0(t, s)$ as

$$R_0(t, s) = V_0(t) + \frac{1 - F(s)}{1 - F(t)} e^{-r(s-t)} (p - V_0(s)). \tag{6}$$

This representation of the return $R_0(t, s)$ remains true if we replace the infinite-horizon valuations with the respective expected discounted returns of holding the asset until an arbitrary $s' > s$, i.e., we have

$$R_0(t, s) = R_0(t, s') + \frac{1 - F(s)}{1 - F(t)} e^{-r(s-t)} \left(p - R_0(s, s')\right). \tag{7}$$

Note that the return $R_0(t,s)$ is smaller than the one-shot randomly timed positive dividend payment or the price p, whichever is greater, and either positive or above p, i.e., for $p \neq 0$ we have

$$\min\{p,0\} < R_0(t,s) < \max\{1,p\}. \tag{8}$$

As the price is constant, for every time-τ entrant at every time $t \geq \tau$ the expected discounted return of holding a unit of the asset to an arbitrary $s > t$ is

$$R_\tau(t,s) = R_0(t-\tau, s-\tau),$$

defined using the time-0 entrant in $t-\tau$ and $s-\tau$ time units from entry. This means that the expected discounted returns depend on time only through experience at buying and selling times, just like infinite-horizon valuations do (but only at buying times). As a natural alternative notation for every time-τ entrant's infinite-horizon valuation $V_\tau : [\tau,\infty) \to \mathbb{R}$, we also denote it by $R_\tau(\cdot,\infty)$ so that every time $t \geq \tau$ satisfies

$$R_\tau(t,\infty) = V_\tau(t),$$

the expected discounted return of holding forever. This notation is consistent with the conventions $F(\infty) = 1$ and $e^{-r\cdot\infty} = 0$, which we adopt.

A special case is the expected discounted return (of any trader) of holding the asset until reaching the ultimate experience level H (in time units) as a function of already accumulated experience (less than H). Our long-term valuation is precisely this function, which we call $V : [0,H) \to \mathbb{R}$, formally defined by

$$V(x) = R_0(x,H).$$

Notice that when traders become aged and experienced the long-term future from their perspective may be very close to today, for $H < \infty$, relative to a young trader's perspective. For the most clear-cut version of our main result, one can think of the case $H = \infty$, in which the long-term valuation is simply the expected discounted return of holding the asset forever as a function of accumulated experience.

For the sake of terminology, short-term valuations are expected discounted returns of holding and selling the asset before acquiring the ultimate experience level H. We need them to introduce our notion of equilibrium à la *Harrison and Kreps* (1978) in Section 2.3 below.

2.3. Trade, Equilibrium Prices, and Their Existence

As done in other papers in the framework of *Harrison and Kreps* (1978), we assume from the outset that traders cannot sell the asset short. This allows us to make every trader's demand for the asset, when the trader is in the market, a simple function of the trader's reservation prices, which are selected (long- or short-term) valuations.

While we formulated every traders' expected discounted returns for all time points from entry, we model trade (more realistically) as discrete with period $\Delta \in (0,H)$:

Trade occurs every Δ time units at time points $t \in \Delta\mathbb{Z}$ after new traders enter but before the old traders leave the market (retire) at those points t, this within-t timing giving us compact sets $[t - M, t]$ of market participants. The technical benefit of discrete trade is in its closed constraint sets $[t + \Delta, \infty)$ for selling times, as opposed to open ones (t, ∞) or constraint sets $[t, \infty)$ allowing for somewhat paradoxical instant selling (Section 4.3.3). Our results will require sufficiently many trading rounds in a trader's professional lifetime though. For the sake of brevity, we reserve the right to implicitly assume that the period Δ between trades is sufficiently short (relatively frequent trade) when making semi-formal statements about our results. Wherever possible, we make things continuous for simplicity.

When the market is open, every participating trader has separate reservation prices for different durations, within the trader's planning horizon, of holding the asset. For every time-τ entrant and at every trading time $t \in [\tau, \tau + M] \cap (\Delta\mathbb{Z})$ the reservation price for holding to an arbitrary $s \in [t + \Delta, \tau + H] \cap (\Delta\mathbb{Z})$, if any, is

$$P_\tau(t, s) = R_\tau(t, s),$$

and the reservation price for holding forever, if $H = \infty$, is

$$P_\tau(t, \infty) = R_\tau(t, \infty).$$

The domains of the reservation prices, unlike those of the expected discounted returns, take into account the planning horizon and the discreteness of trade. Since a trader has a reservation price only when the next trading time is still within the trader's planning horizon, the trader may cease activity shortly before market exit (retirement). For instance, the trader that enters at $-M$ is still formally a market participant at 0 but no longer trades if $M = H \leq T$, unlike in some paternalistic scenarios with $M = T < H$. While such inactive traders may appear to be inessential details of discrete-trade analysis, they are important for ensuring that our notions of fundamental value are well-defined (Sections 2.5–2.8).

Every active trader also has separate, but identical as functions of reservation prices, simple demand schedules for these different durations of holding the asset. The demand is zero above, arbitrary at, and infinite below the corresponding reservation price. For market clearing, only the total across durations and market participants matters. Thus, the price p clears the market at trading time $t \in \Delta\mathbb{Z}$ if and only if p is the maximum of the cross section of the reservation prices

$$\begin{aligned} &\{P_\tau(t, s) : \tau \in [t - M, t], s \in [t + \Delta, \tau + H] \cap ((\Delta\mathbb{Z}) \cup \{\infty\})\} \\ &\qquad = \{R_0(x, x + y) : x \in [0, M], y \in [\Delta, H - x] \cap ((\Delta\mathbb{Z}) \cup \{\infty\})\}, \end{aligned} \tag{9}$$

representable in terms of t-independent, but p-dependent, long- or short-term valuations. Due to this time independence, which is a consequence of the stationarity of the model, the price p clears the market at 0 if and only if p clears the market at all $t \in \Delta\mathbb{Z}$. Any such price is called a *Harrison-Kreps equilibrium price*.

For a fixed-point version of this definition, notice that the maximization of reservation prices is well-behaved regardless of H once we view the holding durations y in (9) as elements of the one-point compactification $\mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$ (see, for example, *Aliprantis and Border*, 2006): Most importantly, maxima of the reservation prices exist, and we have a continuous value function $v_\Delta : \mathbb{R} \to \mathbb{R}$ defined by

$$v_\Delta (p) = \max_{\substack{x \in [0,M] \\ y \in [\Delta, H-x] \cap ((\Delta\mathbb{Z}) \cup \{\infty\})}} R_0 (x, x+y) , \tag{10}$$

as seen from the Berge Maximum Theorem. Thus, the price p is a Harrison-Kreps equilibrium price if and only if p is a fixed point of this continuous function v_Δ, whose values are the maximum reservation prices. The fixed point cannot be too high or too low, because for $p \neq 0$ reservation prices have bounds in the sense of belonging to the interval

$$(\min \{p, 0\} , \max \{1, p\})$$

according to the bounds for expected discounted returns in (1) and (8): For $p < 0$ ($p \geq 1$) the price is still (already) strictly smaller (strictly greater) than all the reservation prices, respectively, including their maximum value $v_\Delta (p)$. Since v_Δ is continuous, it is immediate that a Harrison-Kreps equilibrium price exists and must belong to the interval

$$[0, 1) . \tag{11}$$

2.4. Most Experienced Traders

To formalize our claimed condition for a perpetual bubble, we need notions of most experienced traders and their optimism relative to less experienced ones. In this process, we will both discard, to keep intermediate notions well-defined, and include, to make use of simple continuous-trade approximations, inactive market participants (Section 2.3). Formally, the longest active trader at trading time $t \in \Delta\mathbb{Z}$ is the trader with maximum experience over the set of all active traders' experience levels, which is

$$\begin{aligned} \{t - \tau : \tau \in [t - M, t] , [t + \Delta, \tau + H] \cap ((\Delta\mathbb{Z}) \cup \{\infty\}) \neq \emptyset\} \\ = \{x \in [0, M] : [\Delta, H - x] \cap ((\Delta\mathbb{Z}) \cup \{\infty\}) \neq \emptyset\} , \end{aligned} \tag{12}$$

t-independent like the cross section (9) of reservation prices. We denote the experience level of the longest active trader by $\overline{x} (\Delta)$ and observe that the set (12) of considered experience levels is of course the interval

$$[0, \overline{x} (\Delta)] \subset [0, T] ,$$

within the lifetime T. The equality $[0, \overline{x} (\Delta)] = [0, T]$ holds when $H = \infty$, because the next trading time is always within everyone's planning horizon H if it is infinite, in which case traders remain active and acquire experience throughout lifetime T. Even when $H \leq T$, the longest active traders' experience level $\overline{x} (\Delta)$ falls at least somewhat short of H, meaning $\overline{x} (\Delta) < H$, as at least those with experience $M = H$ cease activity. As a function of Δ, which we will assume to be sufficiently small for our results, the maximum experience $\overline{x}(\cdot)$ is (weakly) decreasing: The longer the period Δ between trades the fewer plan to hold on till the next trading time. However, there are always many active traders (those with experience levels $[0, \overline{x}(\Delta)]$) in the sense that $\overline{x}(\Delta) > 0$, because the bound $H > \Delta$ ensures that at least sufficiently inexperienced ones hold on. To facilitate exposition, we are able to state our sufficient condition (Section 2.9) using the longest active traders' experience level in the continuous-trade limit

$$\lim_{\Delta \to 0^+} \overline{x}(\Delta) = M, \tag{13}$$

where all market participants are active.

2.5. Constrained Fundamental Value

We are liberal in defining fundamental value to make it hard for the equilibrium price to overshoot. Our approach here is to define fundamental value implicitly as the most optimistic long-term valuation rather than the most experienced trader's long-term valuation. Formally, to begin with, the price p is a Δ-*constrained fundamental value* if p is the maximum of the set of all active traders' long-term valuations, i.e.,

$$p = \max_{x \in [0, \overline{x}(\Delta)]} V(x),$$

where the right-hand side is a function of p, suppressed in the notation for convenience. A fixed-point version of this definition is that the price p is a Δ-constrained fundamental value if and only if p is a fixed point of the function $w_\Delta : \mathbb{R} \to \mathbb{R}$ defined by

$$w_\Delta(p) = \max_{x \in [0, \overline{x}(\Delta)]} V(x). \tag{14}$$

We see that a Δ-constrained fundamental value exists and must belong to $[0, 1)$ in the same way as with the fixed point of v_Δ, whose values are the maximum reservation prices as opposed to the maximum long-term valuations (Section 2.3). The uniqueness of Δ-constrained fundamental value is easy to establish, which we do in Section 2.6 below in (continuing) parallel with the uniqueness of Harrison-Kreps equilibrium price. The clear-cut uniqueness of the former in the case $H = \infty$, when the Δ-constrained fundamental value is just the (Δ-independent) most optimistic infinite-horizon valuation

$$\max_{x \in [0,T]} V_0(x),$$

is among the things we alluded to in Section 2.2.

2.6. Uniqueness

Fundamental value (as a fixed point) is unique, because the function w in question is in fact a contraction (every trader discounts a price increase due to the need to wait for the reopening of the market): For distinct prices p, q with $p > q$ we can use a maximizer x in the problem (14) when the price is p to obtain the contraction property

$$|w_\Delta(p) - w_\Delta(q)| \leq \frac{1 - F(H)}{1 - F(\overline{x}(\Delta))} e^{-r(H - \overline{x}(\Delta))} |p - q| \leq e^{-r\Delta} |p - q|$$

of this increasing w_Δ (the higher the price the higher the resale proceeds) by the calculation

$$\begin{aligned}
w_\Delta(q) &= w_\Delta(p + q - p) \\
&\geq \frac{1}{1 - F(x)} \int_x^H e^{-r(\theta - x)} dF(\theta) + \frac{1 - F(H)}{1 - F(x)} e^{-r(H - x)} (p + q - p) \\
&= w_\Delta(p) + \frac{1 - F(H)}{1 - F(x)} e^{-r(H - x)} (q - p) \\
&\geq w_\Delta(p) + \frac{1 - F(H)}{1 - F(\overline{x}(\Delta))} e^{-r(H - \overline{x}(\Delta))} (q - p).
\end{aligned}$$

In the same way and with essentially the same intuition, we infer that Harrison-Kreps equilibrium price is unique by the contraction property of the value function v_Δ with modulus of contraction $e^{-r\Delta}$. In a sense, the period Δ between trades constrains both this Harrison-Kreps equilibrium price and this fundamental value. We denote the unique (Δ-constrained) Harrison-Kreps equilibrium price and Δ-constrained fundamental value by $p^*(\Delta)$ and $\overline{p}(\Delta)$ respectively.

For understanding the effect of Δ, it is important to observe that for $p > \overline{p}(\Delta)$ the maximum long-term valuation $w_\Delta(p) \geq w_\Delta(\overline{p}(\Delta)) = \overline{p}(\Delta)$ is strictly below the diagonal in $\mathbb{R}^2$ due to the modulus of contraction $e^{-r\Delta} < 1$ as

$$w_\Delta(p) - w_\Delta(\overline{p}(\Delta)) \leq e^{-r\Delta}(p - \overline{p}(\Delta)) < p - \overline{p}(\Delta) = p - w_\Delta(\overline{p}(\Delta)). \tag{15}$$

Now we see that, as a function of Δ, the constrained fundamental value $\overline{p}(\cdot)$ is decreasing, because the longer the period between trades the fewer active traders (Section 2.4), and thus the relatively flat w_Δ shifts downward. In other words, the more frequent the trade, the (weakly) higher the Δ-constrained fundamental value is, but it is Δ-independent when $H = \infty$ (Section 2.5).

2.7. Unconstrained Fundamental Value

As we are thinking of frequent trade, we will approximate the Δ-constrained fundamental value, like the longest active traders' experience level $\overline{x}(\Delta)$, with its continuous-trade limit to simplify the exposition of our results. This gives us the Δ-independent notion of unconstrained fundamental value. Formally, the *unconstrained fundamental value* is the limit

$$\overline{\overline{p}} = \lim_{\Delta \to 0^+} \overline{p}(\Delta). \tag{16}$$

This $\overline{\overline{p}}$ is itself a fixed point in the limit in the sense that

$$\overline{\overline{p}} = \lim_{\Delta \to 0^+} w_\Delta \left(\overline{\overline{p}}\right), \tag{17}$$

as the Δ-constrained fundamental value

$$\overline{p}(\Delta) = w_\Delta (\overline{p}(\Delta)) \le w_\Delta \left(\overline{\overline{p}}\right) \le \overline{\overline{p}}$$

squeezes the maximum long-term valuation $w_\Delta \left(\overline{\overline{p}}\right)$, which stays weakly below the diagonal by (15), to the unconstrained fundamental value $\overline{\overline{p}}$. When $H = \infty$, the unconstrained and Δ-constrained fundamental values coincide, because the latter is Δ-independent.

2.8. Fundamental Valuation

To finalize the formalization of our claimed condition for a perpetual bubble, we need to settle two questions on a measure of optimism across experience levels. While the long-term valuation V is our only candidate, it is price-dependent, so our first question is to choose the most appropriate price. Since from the beginning we have been thinking of optimism about fundamentals under frequent trade, a natural resale price for the calculation of optimism is the unconstrained fundamental value $p = \overline{\overline{p}}$. Our second question is to measure the longest active traders' optimism in the continuous-trade limit for the ultimate purpose of our approximation (Sections 2.4 and 2.7). The problem is that in the limit their experience M can be equal to $H \le T$, for which case we have not defined the long-term valuation. In such a case, we will use the limit

$$\lim_{x \to H^-} V(x) = p = \overline{\overline{p}},$$

i.e., our early retirees eventually learn enough about long-term expected discounted returns to agree with the unconstrained fundamental value $\overline{\overline{p}}$. Note that this need not be so when $H > T = M$, as our example with the gamma distribution (2) illustrates for $H = \infty$, a sufficiently short lifetime T, and a shape parameter $\alpha < 1$, in which case

$$\overline{\overline{p}} = V(0) > V(T).$$

Having settled the two questions, we call the resulting optimism measure $W : [0, M] \to \mathbb{R}$ as a (continuous) function of experience the *fundamental valuation* and note that its defining formula is

$$W(x) = \frac{1}{1 - F(x)} \int_x^H e^{-r(\theta - x)} dF(\theta) + \frac{1 - F(H)}{1 - F(x)} e^{-r(H - x)} \overline{\overline{p}}.$$

When $H = \infty$, this fundamental valuation $W : [0, T] \to \mathbb{R}$ is price-independent and agrees with the infinite-horizon valuation $V_0 : [0, \infty) \to \mathbb{R}$.

2.9. Statement of Sufficient Condition

Our claimed sufficient condition (SC) for a perpetual bubble under frequent trade is formally that:

(SC) some (relatively low) experience level $x^* \in [0, M)$ maximizes the fundamental valuation $W : [0, M] \to \mathbb{R}$ but another (higher) experience level $x \in (x^*, M)$ does not.

Under (SC), some relatively inexperienced traders are always among the most optimistic but some more experienced traders are never among the most optimistic (the intuition generalizes beyond stationarity). We are ready to prove formally that this condition is sufficient for a perpetual bubble under frequent trade.

3. Existence of Perpetual Bubble

The Harrison-Kreps equilibrium price $p^* (\Delta)$ is (perpetually) bubbly in our model if and only if $p^* (\Delta)$ is strictly above the unconstrained fundamental value $\overline{\overline{p}}$, a deliberately high target we chose to aim at. To visualize this comparison problem, first observe that for $p > p^* (\Delta)$ the maximum reservation price $v_\Delta (p) \geq v_\Delta (p^* (\Delta)) = p^* (\Delta)$ is also strictly below the diagonal in $\mathbb{R}^2$ due to the same modulus of contraction $e^{-r\Delta} < 1$ as in (15). More intuitively, as one would expect, an above-equilibrium price is too high for the asset to be in demand. This simple fact reduces the problem to one of comparing the unconstrained fundamental value $\overline{\overline{p}}$, viewed as a point on the diagonal, with the maximum reservation price $v_\Delta \left(\overline{\overline{p}}\right)$, because

$$v_\Delta \left(\overline{\overline{p}}\right) > \overline{\overline{p}} \iff \overline{\overline{p}} < p^* (\Delta) \tag{18}$$

(excess demand implies $\overline{\overline{p}}$ is below equilibrium and vice versa). In other words, it suffices to check if there is excess demand $\left(v_\Delta \left(\overline{\overline{p}}\right) > \overline{\overline{p}}\right)$ at the price equal to the unconstrained fundamental value $\overline{\overline{p}}$. According to our hypothesis, that is the case under (SC) if the period Δ between trades is sufficiently short. Sufficiently frequent trade here ensures, among other things, that, in view of the limit (13), traders with the maximizing relatively low experience x^* in (SC) are active, in the sense that

$$x^* \in [0, \overline{x} (\Delta)] . \tag{19}$$

This is so no matter which value we fix for Δ as long as it is less than an x^*-dependent threshold obtained from the limit (13). In this case, the maximum fundamental valuation $W (x^*)$ coincides with the maximum long-term valuation $w_\Delta \left(\overline{\overline{p}}\right)$ at the unconstrained fundamental value $\overline{\overline{p}}$ by definition (14) of w_Δ. The insensitivity of property (19) to decreasing Δ further implies that the maximum long-term valuation $w_\Delta \left(\overline{\overline{p}}\right)$ as a function of Δ becomes constant at the level $W (x^*)$ of the maximum fundamental valuation. Since these constant values $w_\Delta \left(\overline{\overline{p}}\right)$ of the maximum long-term valuation converge to the unconstrained fundamental valuation $\overline{\overline{p}}$ by the fixed-point property (17), we obtain the valuable quadruple identity

$$\overline{\overline{p}} = w_{\Delta}\left(\overline{\overline{p}}\right) = \overline{p}\left(\Delta\right) = W\left(x^*\right)$$

under (SC) and (19). Most importantly, the Δ-constrained and unconstrained fundamental values both coincide with the maximum fundamental valuation $W\left(x^*\right)$. This gives us some freedom in checking whether there is excess demand at the price equal to the fundamental value to verify the sufficiency of (SC) for a perpetual bubble via (18).

As a first indication of such a bubble-confirmatory excess demand, Proposition 1, below, identifies a short-term valuation, not necessarily a reservation price, in excess of that price $p = \overline{\overline{p}}$. It is an immediate corollary of the representation (6)–(7) of shorter-term expected discounted returns of holding the asset in terms of longer-term ones, for which in this case we use the long-term valuation V.

Proposition 1. *If* (SC) *and* (19) *hold, then the price* $p = \overline{\overline{p}}$ *satisfies*

$$\begin{aligned} R_0\left(x^*, x\right) &= W\left(x^*\right) + \frac{1 - F\left(x\right)}{1 - F\left(x^*\right)} e^{-r\left(x - x^*\right)} \left(\overline{\overline{p}} - W\left(x\right)\right) \\ &= \overline{\overline{p}} + \frac{1 - F\left(x\right)}{1 - F\left(x^*\right)} e^{-r\left(x - x^*\right)} \left(W\left(x^*\right) - W\left(x\right)\right) \\ &> \overline{\overline{p}}. \end{aligned}$$

It is a relatively inexperienced maximally optimistic trader that could profit from holding the asset until somebody else becomes maximally optimistic and could, in turn, profit from holding until a third trader becomes maximally optimistic. Formally, a trader with the maximizing experience x^* could profit from holding until reaching the higher experience x, when another trader has the maximizing experience x^*. For such trades to be at least approximately feasible, we again need the market to open frequently relative to the traders' planning horizon H so that the period Δ between trades can be short. If it is sufficiently short, then traders with experience x^* will have a feasible holding duration $y \in [\Delta, H - x^*] \cap (\Delta\mathbb{Z})$ in a suitable (profitable) neighborhood of $x - x^*$ such that

$$\overline{\overline{p}} < R_0\left(x^*, x^* + y\right) \leq v_{\Delta}\left(\overline{\overline{p}}\right).$$

In other words, this price $p = \overline{\overline{p}}$ indeed leads to excess demand, as in our intermediate characterization (18) of a perpetual bubble, proving our main result, stated below as Proposition 2.

Proposition 2. *If* (SC) *holds and the period* Δ *between trades is sufficiently short, then the Harrison-Kreps equilibrium price* $p^*\left(\Delta\right)$ *has the bubble property* $p^*\left(\Delta\right) > \overline{\overline{p}}$*, i.e., the equilibrium price overshoots the unconstrained fundamental value.*

4. Example

In this example, the planning horizon is infinite $H = \infty$ and the beliefs of the time-0 entrant about the dividend time $\theta \in (0, \infty)$ are $\text{Gamma}(\alpha, \beta)$. It has hazard function $h : (0, \infty) \to \mathbb{R}$ defined by

$$h(t) = \frac{f(t)}{1 - F(t)}.$$

4.1. Optimal Selling

Every trader's reservation prices reflect the trade-offs apparent in (5) between planning to hold the asset longer to increase the expected discounted dividend (3) and shorter to raise the expected discounted resale proceeds (4). Looking at the time-0 entrant at any time $t \geq 0$ illustrates how the marginal expected discounted dividend and resale proceeds, which in this case, at every selling time $s > t$, are the derivatives

$$\frac{\partial D_0(t,s)}{\partial s} = \frac{1}{1 - F(t)} e^{-r(s-t)} f(s) \tag{20}$$

and

$$\frac{\partial S_0(t,s)}{\partial s} = -\frac{\partial D_0(t,s)}{\partial s} p - \frac{1 - F(s)}{1 - F(t)} e^{-r(s-t)} rp, \tag{21}$$

are interestingly interrelated: A 1¢ gain in the expected discounted dividend goes hand in hand approximately with at least p¢ loss in the expected discounted resale proceeds, because the payout of the dividend makes the asset worthless. While the loss also includes the expected discounted interest forgone on \$$p$ due to the marginal sale delay, there is no dividend disbursement delay, and hence no analogous interest forgone in (20). These demand trade-offs depend on and discipline the price through market-clearing conditions. We will use the following consequences of the marginal returns (20)–(21) for the time-0 entrant's time-t-expected-discounted-return function $R_0(t, \cdot)$ on (t, ∞) into $\mathbb{R}$:

(C1) when $p < 1$, then a selling time $s > t$ is a stationary point of $R_0(t, \cdot)$ if and only if

$$h(s) = \frac{f(s)}{1 - F(s)} = \frac{rp}{1 - p}, \tag{22}$$

i.e., the first-order condition equates the marginal noninterest net gain with the marginal interest loss;

(C2) when $p < 1$ and h (in a sense a dividend likelihood) is strictly decreasing, then a stationary point of $R_0(t, \cdot)$, if any, is a maximum and $R_0(t, \cdot)$ has a truncated- or half-bell shape, meaning that:

(a) $R_0(t, \cdot)$ is strictly increasing on $(t, \infty) \cap h^{-1}([rp/(1-p), \infty))$ and

$$\lim_{s \to t^+} R_0(t,s) = p;$$

(b) $R_0(t,\cdot)$ is strictly decreasing on $(t,\infty) \cap h^{-1}((-\infty, rp/(1-p)])$ and

$$\lim_{s\to\infty} R_0(t,s) = V_0(t);$$

(C3) when $p < 1$ and h is strictly decreasing, then every selling time $s > t$ satisfies

$$R_0(t,s) > \min\{p, V_0(t)\},$$

because $R_0(t,\cdot)$ has a (generalized) bell shape in the sense of (C2).

Optimal future selling times of potential asset-holders underpin their current buying decisions and play a key role in determining equilibrium prices. An attractive plan may be to sell in finite time to get a premium above the infinite-horizon valuation $V_0(t)$, which is possible by (C2)–(C3) if the hazard function is strictly decreasing, as when $\alpha < 1$ (see, for example, *Klugman et al.*, 2012): It suffices to have the price between the infinite-horizon valuation ($p \geq V_0(t)$) and the dividend amount ($p < 1$), which, for $t \in [0,T]$, both are necessary conditions for p to be a Harrison-Kreps equilibrium price. In this case, the expected-discounted-return function $R_0(t,\cdot)$ has a maximizing, but potentially infeasible, selling time if $t = 0$, because $\lim_{s\to 0^+} h(s) = \infty$, or if $t > 0$ and $h(t) > rp/(1-p)$. The trader would only consider holding the asset to trading time points $s \in \Delta\mathbb{Z}$ close to this maximizer.

4.2. Disagreement across Experience Levels

Proposition 3, below, says that a necessary and sufficient condition for the heterogeneity of long-term valuations is $\alpha \neq 1$. This is the belief-shape-parameter threshold at which the long-term valuation function V_0 is constant and switches from being strictly decreasing, for $\alpha < 1$, to being strictly increasing, when $\alpha > 1$. For a quick intuition about these slopes, we can use the fact that the relative rate of change of the density f at every dividend (waiting) time $\theta \in (0,\infty)$ is

$$\frac{f'(\theta)}{f(\theta)} = \frac{\alpha - 1}{\theta} - \frac{1}{\beta}. \tag{23}$$

If $\alpha < 1$, learning that the asset does not pay at θ time units from entry, which truncates the density further marginally, makes it relatively lower close to θ and higher far, through a smaller percentage decrease (23) far. Due to discounting, the valuation $V_0(\theta)$ must decline. An analogous intuition applies to the other two cases. A detailed proof is in Appendix A.

Proposition 3. *The shape of the long-term valuation function $V_0 : [0,\infty) \to \mathbb{R}$ depends on the parameter α as follows:*

(a) if $\alpha < 1$, then V_0 is strictly decreasing;

(b) if $\alpha = 1$, then V_0 is constant;

(c) if $\alpha > 1$, then V_0 is strictly increasing.

As a side note, this formulation with identical newcomers offers, at least partly, a simple model of symmetric-information belief heterogeneity that is acquired and experience-based as opposed to inborn or ability-based. Many economists have discussed the fine lines between heterogeneity of information, prior-beliefs, bounded rationality, and ability in various contexts (we mention *Aumann*, 1976; *Morris*, 1995; *Alaoui and Penta*, 2016). As *Morris* (1995) puts it, for instance, "individuals may have misinterpreted a signal at the beginning of time, and this is what gave them heterogeneous prior beliefs." In our model, traders enter the market at different points in time, are identical in terms of rationality, ability, and information, but, nevertheless, disagree across experience levels. In many contexts in the literature, making these subtle distinctions would indeed matter.

4.3. Bubble under Pessimistic Old-timers

In this example, our claimed condition for a perpetual bubble holds if and only if $\alpha < 1$, in which case the most experienced are always not the most optimistic according to Proposition 3: They are the least optimistic. In the other two scenarios, in violation of the condition, the most optimistic are either as optimistic as everyone else (when $\alpha = 1$) or the most optimistic (when $\alpha > 1$). Now we prove formally that the condition $\alpha < 1$ is sufficient for a perpetual bubble.

4.3.1. Overshooting Fundamental Valuations

Essentially, the bubble is a corollary of (C3), (11), and the fact that the hazard function h is strictly decreasing in this case ($\alpha < 1$): If p is a Harrison-Kreps equilibrium price, then

$$p \geq P_0(0, \Delta) = R_0(0, \Delta) > \min\{p, V_0(0)\} = V_0(0),$$

i.e., the equilibrium price is, on the one hand, strictly above the base, $\min\{p, V_0(0)\}$, of the (generalized) bell $R_0(0, \cdot)$ and ensures, on the other hand, that the base is $V_0(0)$. Since $V_0(0)$ is the maximum valuation by part (a) of Proposition 3, the equilibrium price must always be strictly greater than all traders' fundamental valuations, and hence bubbly, which we now state as Proposition 4.

Proposition 4. *If $\alpha < 1$ and p is a Harrison-Kreps equilibrium price, then every time point $t \geq 0$ satisfies $p > V_0(t)$.*

4.3.2. More Extreme Possibilities

We view this example as embodying a general overpricing principle under relatively pessimistic experienced investors on the one hand and illustrating just how far-reaching this bubble can be on the other hand. For this, we are going to deduce that, in addition to overshooting the fundamental valuations, the equilibrium price must be strictly above even very short-term expected discounted returns of holding the asset. Notice that we have also shown in Proposition 4 that the equilibrium price is strictly above not only the fundamental valuations $V_0([0,T])$ of any trading time's market participants, but of all departed traders. We argue through the following instructive sequence of necessary conditions for p to be a Harrison-Kreps equilibrium price, pinpointing a unique candidate for the equilibrium price along the way, when $\alpha < 1$:

(N1) there are times $t^* \in [0,T]$ and $s^* \geq t^* + \Delta$ such that $p = R_0(t^*, s^*)$, i.e., the price, of course, must coincide with some short-term expected discounted return, because holding the asset forever is unprofitable by Proposition 4;

(N2) the (generalized) bell $R_0(t^*, \cdot)$ has a maximizer $\bar{s} \in (t^*, s^*)$, is strictly increasing on $(t^*, \bar{s}]$, and is strictly decreasing on $[\bar{s}, \infty)$—i.e., the graph must spring from the point (t^*, p) upward to be able to pass through (s^*, p);

(N3) every selling time $s \in (t^*, s^*)$ satisfies $p > R_0(t^*, s)$, i.e., the truncated bell $R_0(t^*, \cdot)$ forms an arc over the interval (t^*, s^*);

(N4) $s^* = t^* + \Delta$, since otherwise $t^* + \Delta \in (t^*, s^*)$, and thus p would be strictly greater than the reservation price $P_{-t^*}(0, \Delta) = R_0(t^*, t^* + \Delta)$;

(N5) $p = \max_{t \in [0,T]} R_0(t, t + \Delta)$, because these expected discounted returns are precisely the reservation prices $P_{-t}(0, \Delta)$;

(N6) $t^* = 0$, since otherwise

$$\frac{dR_0(t^*, t^* + \Delta)}{dt} = \frac{\partial R_0(t^*, s^*)}{\partial t} + \frac{\partial R_0(t^*, s^*)}{\partial s} < 0,$$

because

$$\begin{aligned} \frac{\partial R_0(t^*, s^*)}{\partial t} &= rR_0(t^*, s^*) - h(t^*) + h(t^*) R_0(t^*, s^*) \\ &= rp - h(t^*)(1-p) \\ &< rp - h(\bar{s})(1-p) \\ &= 0 \end{aligned} \tag{24}$$

(the first two summands in (24) reflect drops in interest forgone and unconditional probability to get the dividend, while the third adjusts for further truncation) and

$$\frac{\partial R_0(t^*, s^*)}{\partial s} \leq 0;$$

(N7) every time point $t \in (0,T]$ satisfies $p > R_0(t, t + \Delta)$, because $t^* = 0$ is a unique maximizer in (N5);

(N8) every time point $t > T$ satisfies $p > R_0(t, t + \Delta)$, because $t \geq \Delta \geq \bar{s}$, and thus $R_0(t, \cdot)$, whose right-hand limit at t is p, is strictly decreasing;

(N9) all time points $t \geq 0$ and $s > t + \Delta$ satisfy $p > R_0(t, s)$, because $t + \Delta \geq \Delta \geq \bar{s}$, and thus $R_0(t, \cdot)$ is strictly decreasing on $[t + \Delta, \infty)$, which means that $p \geq R_0(t, t + \Delta) > R_0(t, s)$.

What we have shown (and include below in Proposition 5) is that the equilibrium price is strictly greater than all expected discounted returns of holding the asset for longer than the market-imposed minimum holding period Δ. In other words, the equilibrium price must always be bubbly even relative to all these (active and departed traders') short holding horizons in addition to infinite ones.

Proposition 5. *If $\alpha < 1$ and p is a Harrison-Kreps equilibrium price, then all time points $t \geq 0$ $(t > 0)$ and $s > t + \Delta$ $(s \geq t + \Delta)$, respectively, satisfy $p > R_0(t, s)$ and we have $p = R_0(0, \Delta)$.*

A useful by-product of this exercise of finding insightful necessary conditions for p to be an equilibrium price is the identification of a unique candidate for such a price, also documented in Proposition 5. It says that the equilibrium price p solves the equation

$$p = \int_0^{\Delta} e^{-r\theta} f(\theta)\, d\theta + (1 - F(\Delta))\, e^{-r\Delta} p$$

and that in this equilibrium, at every trading $t \in \Delta\mathbb{Z}$, only time-t entrant holds the asset: Past entrants are not willing to hold the asset even for the minimum holding period Δ, as the price is strictly greater than their, but not the time-t entrant's, expected discounted returns of doing so.

4.3.3. Full-blown Bubble under Frequent Trade

Full-blown bubbles are, loosely defined, those that overshoot all possible beliefs about fundamentals. This admittedly fragile notion is useful for evaluating the size of the bubble at least within our model, where the price threshold for being full-blown is \$1—the certain, but randomly timed, dividend amount. While the Harrison-Kreps equilibrium price is less than one according to (11), in the limit, as trading becomes more and more frequent, the equilibrium price approaches one by the following one-line argument: When $\alpha < 1$, letting $\Delta \to 0^+$ and the equilibrium price respond squeezes the unconstrained optimal selling time $\bar{s}$ in (N2) to zero, pushes $h(\bar{s})$ to infinity, and, to satisfy the first-order condition (22), drives the price to one. In this frequent-trade limit, the bubble is full-blown in the sense of the price being equal to the supremum of fundamental valuations over all possible beliefs. We state this limit result below as Proposition 6.

Proposition 6. *If $\alpha < 1$, then*

$$\lim_{\Delta \to 0^+} p^*(\Delta) = 1.$$

If we allowed for continuous trade, one might expect an equilibrium price to be this limit, equal to one, of "discrete" Harrison-Kreps equilibrium prices. However, this price would be too high for any trader to ever want to hold the asset to any future time point—strictly above any duration's expected discounted return, i.e. (1) or (5). It would be somewhat hard to make sense of such a continuous-trade approximation, as it would be unclear who holds the asset when one instant succeeds another. Yet, on the other hand, we could simply say that the asset changes hands continuously, just like a flying arrow passes through instants of continuous time. (A well-known Zeno's paradox is that such an arrow is never moving, because at every instant the arrow is in some fixed position; see, for example, *Hamming* (1998).)

4.4. Perpetual Valuation Switching

To emphasize that every time-τ entrant's valuation will be exceeded by some future (inexperienced) entrant's valuation arbitrarily soon after this time point τ if $\alpha < 1$, we dedicate a separate Proposition (No. 7) to this. It is an immediate corollary of part (c) of Proposition 3 and informally the main piece of our intuition behind the perpetual bubble. The previous literature, where all traders had the same experience, allowed every trader to anticipate comparable valuation switching and resulting possibilities to resell the asset for strictly more than the trader's valuation. These are basically the circumstances in which the previous literature with other kinds of belief heterogeneity predicts bubbly prices in the sense of them overshooting the most optimistic valuations. Notice that the valuation switching in our example is stronger in the sense that every time-τ entrant's valuation will be exceeded at all, not just some, future time points $\sigma > \tau$ if $\alpha < 1$.

Proposition 7 (Perpetual Switching)**.** *If $\alpha < 1$, then all time points τ and $\sigma > \tau$ satisfy $V_\sigma(\sigma) > V_\tau(\sigma)$.*

In the terminology of *Morris* (1996), Proposition 7 says that at each time point there are no optimists, defined as traders whose valuation is at least as high as all other traders' valuations from that time onwards. *Morris* (1996) coined the term perpetual switching to refer to the nonexistence of optimists after every history. In his model, perpetual (valuation) switching is equivalent to the existence of a speculative bubble. *Morris* (1996) characterizes this notion of an optimist in terms of the traders' prior beliefs about the probability that the asset pays a dividend as opposed to no dividend—i.e., even his uncertainty is very different from ours. In his setting, which also assumes away newcomers in the market, a trader is an optimist if and only if the trader's prior is monotone-likelihood-ratio (MLR) dominant among all trader's heterogeneous priors[1]. One of the goals of *Morris* (1996) was to investigate precisely how different the priors must be for a bubble to arise in a special case of the model of *Harrison and Kreps* (1978). The answer is this necessary and sufficient condition that rules out "that there is a single [MLR dominant] trader whose [prior] density is always increasing at the fastest rate". Over time, the traders' posteriors converge to one another, and the bubble fades

[1] *Werner* (2018) argues that MLR dominance yields optimists in a more abstract framework, retaining from the previous literature short-sales constraints, risk-neutral traders, and learning about the dividend.

away, whereas in our model market newcomers may add fuel to the fire and keep the bubble going forever. Our example allows for disagreement across experience levels and is optimist-free simply when the most experienced investors are not the most optimistic. The former approach is compelling for explaining temporary bubbles such as overpriced IPOs relative to long-run values (*Miller*, 1977) and requires neither bounded rationality nor market newcomers. *Morris* (1996) argues that the assumption of unexplained differences in prior beliefs at initial public offerings is hard to refute, because there have been no opportunities to learn the true data-generating process. In contrast, our sufficient condition seems more applicable to "seasoned" assets such as stocks of old firms. Bubbles due to presence of permanently overconfident traders in *Scheinkman and Xiong* (2003) are also more relevant to newly issued assets, because over time the traders could realize their overconfidence.

4.5. Optimistic Old-timers (Bubble-free Case)

Our model is rich enough to separate the wheat from the chaff—i.e., to accommodate bubble-free scenarios. In our example, the most experienced are among the most optimistic if and only if $\alpha \geq 1$ (Proposition 3). In this case, the most optimistic fundamental valuation is always $V_0(T)$. Thus, we want to show that $p = V_0(T)$ is a Harrison-Kreps equilibrium price when $\alpha \geq 1$. This would be impossible if the hazard function h were strictly decreasing, because then our bubble argument (Section 4.3.1) would apply almost word for word and yield the contradiction

$$\begin{aligned} V_0(T) = p \geq P_{-T}(0, \Delta) &= R_0(T, T + \Delta) \\ &> \min\{p, V_0(T)\} = V_0(T). \end{aligned}$$

Indeed, in this case h is increasing (see again, for example, *Klugman et al.*, 2012). The consequence is that expected discounted returns as functions of selling time are no longer (generalized) bells, but inverted bells, have minima rather than maxima in (C2), with weak monotonicites in place of strict ones. Any $p \in (V_0(T), 1)$ is already above the maximum reservation price

$$V(p) < p,$$

as these inverted bells spring from points with vertical coordinate p downward or straight, may change direction only once, and approach less optimistic valuations. Thus, a Harrison-Kreps equilibrium price, which must be at least $V_0(T) = P_{-T}(0, \infty)$, cannot be different from $V_0(T)$ and hence must be bubble-free. This is a part of our Proposition 8, below, while the rest of this proposition is due to the inverted-bell shapes and the fact that the long-term valuation function V_0 is strictly increasing for $\alpha > 1$ and constant for $\alpha = 1$ (Proposition 3). In the latter case, the beliefs f reduce to an exponential distribution, the hazard function H is constant, expected discounted returns as functions of selling time are monotone (generalized bells), and all reservation prices in (9) equal $p = V_0(T)$.

Proposition 8. *If $\alpha \geq 1$, then the price $p = V_0(T)$ is a unique (bubble-free) Harrison-Kreps equilibrium price. In this equilibrium, asset holdings at every trading time point $t \in \Delta\mathbb{Z}$ depend on the shape parameter $\alpha \geq 1$ as follows:*

(a) if $\alpha > 1$, then only the most experienced trader holds the asset;

(b) if $\alpha = 1$, then any trader can hold the asset.

A. Proof of Proposition 3

(a) Since all $t, s \in (0, \infty)$ such that $t < s$ satisfy

$$\int_t^\infty \frac{f_{\alpha,\beta}(\theta)}{1 - F_{\alpha,\beta}(t)} d\theta = \int_s^\infty \frac{f_{\alpha,\beta}(\theta)}{1 - F_{\alpha,\beta}(s)} d\theta = \int_t^\infty \frac{f_{\alpha,\beta}(\theta - t + s)}{1 - F_{\alpha,\beta}(s)} d\theta,$$

there is an $\eta \in (t, \infty)$ with

$$\frac{f_{\alpha,\beta}(\eta)}{1 - F_{\alpha,\beta}(t)} = \frac{f_{\alpha,\beta}(\eta - t + s)}{1 - F_{\alpha,\beta}(s)}.$$

Since every $\theta \in [t, \infty)$ satisfies

$$\frac{f_{\alpha,\beta}(\theta)}{1 - F_{\alpha,\beta}(t)} \cdot \frac{1 - F_{\alpha,\beta}(s)}{f_{\alpha,\beta}(\theta - t + s)} = \frac{1 - F_{\alpha,\beta}(s)}{1 - F_{\alpha,\beta}(t)} e^{\frac{s-t}{\beta}} \left(\frac{\theta}{\theta - t + s} \right)^{\alpha - 1},$$

we have

$$\frac{f_{\alpha,\beta}(\theta)}{1 - F_{\alpha,\beta}(t)} > \frac{f_{\alpha,\beta}(\theta - t + s)}{1 - F_{\alpha,\beta}(s)}$$

if $\theta < \eta$, and

$$\frac{f_{\alpha,\beta}(\theta)}{1 - F_{\alpha,\beta}(t)} \leq \frac{f_{\alpha,\beta}(\theta - t + s)}{1 - F_{\alpha,\beta}(s)}$$

if $\theta \geq \eta$. Now

$$\begin{aligned}
V_0(t) - V_0(s) &= \int_t^\infty \frac{f_{\alpha,\beta}(\theta)}{1 - F_{\alpha,\beta}(t)} e^{-r(\theta - t)} d\theta - \int_s^\infty \frac{f_{\alpha,\beta}(\theta)}{1 - F_{\alpha,\beta}(s)} e^{-r(\theta - s)} d\theta \\
&= \int_t^\infty \frac{f_{\alpha,\beta}(\theta)}{1 - F_{\alpha,\beta}(t)} e^{-r(\theta - t)} d\theta - \int_t^\infty \frac{f_{\alpha,\beta}(\theta - t + s)}{1 - F_{\alpha,\beta}(s)} e^{-r(\theta - t)} d\theta \\
&= \int_t^\infty \left(\frac{f_{\alpha,\beta}(\theta)}{1 - F_{\alpha,\beta}(t)} - \frac{f_{\alpha,\beta}(\theta - t + s)}{1 - F_{\alpha,\beta}(s)} \right) e^{-r(\theta - t)} d\theta \\
&> \left(\int_t^\infty \frac{f_{\alpha,\beta}(\theta)}{1 - F_{\alpha,\beta}(t)} d\theta - \int_t^\infty \frac{f_{\alpha,\beta}(\theta - t + s)}{1 - F_{\alpha,\beta}(s)} d\theta \right) e^{-r(\eta - t)} \\
&= 0,
\end{aligned}$$

proving that V_0 is strictly decreasing on $(0,\infty)$. Since

$$\lim_{t\to 0^+} V_0\left(t\right) = V_0\left(0\right),$$

every time point $t \geq 0$ satisfies every $s \in (0,\infty)$ satisfies $V_0\left(s\right) < V_0\left(0\right)$, completing the proof.

(b) Every $t \in [0,\infty)$ satisfies

$$V_0\left(t\right) = \frac{1}{1-\int_0^t \frac{1}{\beta} e^{-\frac{\theta}{\beta}} d\theta} \int_t^\infty \frac{1}{\beta} e^{rt} e^{-\left(r+\frac{1}{\beta}\right)\theta} d\theta = \left(\beta r + 1\right)^{-1}.$$

(c) The proof is analogous to that of (a).

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