Comparative Risk Apportionment*

Paan Jindapon[†], Liqun Liu[‡], and William S. Neilson[§]

March 29, 2019

Abstract: The notion of risk apportionment, introduced by Eeckhoudt and Schlesinger (2006) and generalized by Eeckhoudt et al. (2009), has proven to be useful in understanding decision making in risky environments. A decision maker with a preference for risk apportionment prefers putting two independent risk increases in separate states to combining them in a single state. In this paper, we study comparative risk apportionment, the issue of comparing the strength of risk apportionment preference between two decision makers. Under expected utility theory, we find that the (n/m)th-degree Ross more risk aversion of Liu and Meyer (2013) is a sufficient condition for comparative nth-degree risk apportionment, whereas the corresponding (n/m)th-degree Arrow-Pratt more risk aversion is a necessary condition.

Keywords: Risk increases; Risk apportionment; Risk aversion; Downside risk aversion; Comparative risk apportionment

JEL Classification: D81

*We thank Christophe Courbage, Jack Meyer, Richard Peter and seminar participants at University of Alabama, University of Tennessee, Virginia Tech, and University of Colorado for insightful discussions and comments.

[†]Department of Economics, Finance and Legal Studies, University of Alabama, E-mail: pjindapon@ua.edu

[‡]Private Enterprise Research Center, Texas A&M University, E-mail: lliu@tamu.edu

[§]Department of Economics, University of Tennessee, E-mail: wneilson@utk.edu

1 Introduction

Using 50-50 lotteries, Eeckhoudt and Schlesinger (2006) and Eeckhoudt et al. (2009) study decision makers with a preference for risk apportionment, i.e., individuals who would rather put two independent risk increases of various degrees in separate states than combine them in a single state. They show that, under expected utility theory, the preference for risk apportionment is implied by risk aversion of various degrees.[1]

The risk apportionment framework of Eeckhoudt and Schlesinger (2006) and Eeckhoudt et al. (2009) helps deepen the understanding of higher-degree risk increases. For a long time after Ekern (1980) defines the concept of an nth-degree risk increase, economists knew little about it beyond the fact that the nth-degree risk increase is a special case of the nth-degree stochastic dominance in which the first $n-1$ moments of the two random variables are kept the same. The risk apportionment framework makes it very simple to decompose a higher-degree risk increase into lower-degree risk increases, thereby providing an intuitive interpretation of nth-degree risk increases in terms of the well-understood 1st-degree risk increases (leftward shifts in the probability mass) and/or 2nd-degree risk increases (mean-preserving spreads). In addition, the risk apportionment framework facilitates a general treatment in a large category of models of decision making under risk regarding the effects of exogenous changes in the risky environment (Nocetti 2016). The risk apportionment framework has been extended to characterize preferences for disaggregating two multiplicative risks (Wang and Li 2010 and Chiu et al. 2012), to study multiattribute risk preferences (Tsetlin and Winkler 2009, Jokung 2011, Denuit and Rey 2013, and Gollier 2018), to better understand the relationship between stochastic dominance and the corresponding preferences (Courbage et al. 2018, Ebert et al. 2018, Huang et al. 2017,

[1]As the opposite of risk apportionment, Crainich et al. (2013) study preferences for combining two independent risk increases of various degrees in a single state, as opposed to putting them in separate states, and show that these preferences can be characterized by risk loving at all even degrees and risk aversion at all odd degrees. See also Ebert (2013).

Tsetlin and Winkler 2017), and to shed light on nth-degree risk aversion in non-EU models (Eeckhoudt et al. 2017).

To the best of our knowledge, however, there exists almost no analysis on how to measure the strength of the preference for risk apportionment.[2] As is well known, the research on the measure of risk aversion since Arrow (1971) and Pratt (1964) has greatly enhanced our understanding of decision making under risk, even though the notion of risk aversion had already been mathematically formalized by Daniel Bernoulli more than two hundred years earlier (Bernoulli 1954). The present paper sets out to quantify the strength of the preference for risk apportionment, extending the literature on risk apportionment to include an analysis of comparative risk apportionment. We adopt the generalized framework of nth-degree risk apportionment based on mutual aggravation of mth- and $(n-m)$th-degree risks, where $n > m \geq 1$, by Eeckhoudt et al. (2009) and define a measure of preference for nth-degree risk apportionment, called (n/m)th-degree probability premium or $p_{n/m}$. We find that, under expected utility representation of preferences, if decision makers are nth- and mth-degree risk averse, the corresponding (n/m)th-degree probability premium will be positive. Moreover, the (n/m)th-degree Ross more risk aversion of Liu and Meyer (2013) is a sufficient condition for the interpersonal comparison of the (n/m)th-degree probability premiums, whereas the corresponding (n/m)th-degree Arrow-Pratt more risk aversion is a necessary condition.

Our findings on comparative risk apportionment can be applied to the more specialized, yet more popular, nth-degree risk apportionment framework of Eeckhoudt and Schlesinger (2006). Specifically, we can use our $(n/2)$th-degree probability premium, or $p_{n/2}$, to measure nth-degree risk apportionment preference and to derive its relationship with $(n/2)$th-degree Ross more risk aversion and $(n/2)$th-degree Arrow-Pratt more risk aversion. As an

[2] Jindapon (2010) and Watt (2011) broach the issue in their efforts to characterize comparative prudence (i.e., downside risk aversion) using Eeckhoudt and Schlesinger's (2006) representation of prudence with choices among 50-50 lottery pairs.

alternative, we propose a variation of the 50-50 lottery pairs considered in Eeckhoudt and Schlesinger (2006) so that we can use the concept of $(n/1)$th-degree probability premium, or $p_{n/1}$, as a measure of nth-degree risk apportionment. As a special case when $n = 3$, we can provide a unifying treatment of the issue of comparative prudence or downside risk aversion using probability premiums previously studied by Jindapon (2010) and Watt (2011).

The paper is organized as follows. Definitions and preliminary results related to nth-degree risk aversion and nth-degree risk apportionment are given in Section 2. In Section 3, we introduce the concept of (n/m)th-degree probability premium $p_{n/m}$ and derive comparative results under expected utility theory using the generalized Ross and Arrow-Pratt measures of higher-order risk aversion. Then in Section 4, we discuss simple cases where $m = 2$ and $m = 1$ and show how these probability premiums are related to comparative nth-degree risk apportionment. We discuss comparative prudence in Section 5 and conclude in Section 6.

2 Definitions and Preliminary Results

Through out the paper, we let $[L_1, p_1; L_2, p_2]$ denote a binary compound lottery which yields lottery L_i with probability p_i for $i = 1, 2$. In this section, we first review a popular example of a preference for 3rd-degree risk apportionment (i.e., prudence or downside risk aversion) due to Eeckhoudt and Schlesinger (2006). Consider two lotteries, $A_3 = [-k + \tilde{\epsilon}, 1/2; 0, 1/2]$ and $B_3 = [-k, 1/2; \tilde{\epsilon}, 1/2]$, where $k > 0$ and $\tilde{\epsilon}$ is a nondegenerate zero-mean risk. A decision maker who prefers more to less and dislikes risk would regard both $-k$ and $\tilde{\epsilon}$ as "bads." According to Eeckhoudt and Schlesinger (2006), a decision maker who displays (a preference for) risk apportionment—a preference for putting two independent bads in separate states, as opposed to combining them in a single state—prefers lottery B_3 to lottery A_3, for all k and $\tilde{\epsilon}$.

To put it differently, the only difference between lotteries A_3 and B_3 is that the zero-mean risk $\tilde{\epsilon}$ occurs in the high-wealth state of B_3 and in the low-wealth state of A_3. According to Menezes et al. (1980), A_3 has more downside risk than B_3. Therefore, a preference for risk apportionment in this example is the same as an aversion to downside risk increases. See Figure 1.

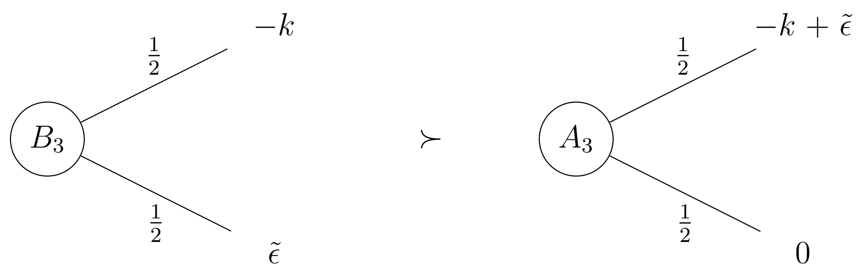


Figure 1: Preference for 3rd-degree risk apportionment

The notion of a preference for 3rd-degree risk apportionment in this example does not require the existence of expected-utility representation of a decision maker's preferences. If a decision maker has an initial wealth of w and his preferences are represented by utility function u, however, both Menezes et al. (1980) and Eeckhoudt and Schlesinger (2006) demonstrate that $Eu(w + B_3) > Eu(w + A_3)$ for all w, k, and $\tilde{\epsilon}$ if and only if $u''' > 0$. For nth-degree risk apportionment, Eeckhoudt and Schlesinger (2006) use the following definitions to represent (a preference for) nth-degree risk apportionment.

Definition 1. *(Eeckhoudt and Schlesinger 2006) Let k be a strictly positive real number, $\tilde{\epsilon}_n$, for $n \geq 2$, be a zero-mean nondegenerate random variable, and all $\tilde{\epsilon}_n$ be mutually independent. Define lotteries $B_1 = B_2 = 0$, $A_1 = -k$, and $A_2 = \tilde{\epsilon}_2$. For $n \geq 3$,*

$$A_n = [A_{n-2} + \tilde{\epsilon}_n, 1/2; B_{n-2} + 0, 1/2]$$

$$B_n = [A_{n-2} + 0, 1/2; B_{n-2} + \tilde{\epsilon}_n, 1/2]$$

Definition 2. *(Eeckhoudt and Schlesinger 2006) Preferences are said to satisfy* n*th-degree risk apportionment if* $B_n \succ A_n$ *for all* A_n *and* B_n*.*

Eeckhoudt and Schlesinger (2006) show that lottery A_n has more nth-degree risk than lottery B_n using Ekern's (1980) definition. Let $F(x)$ and $G(x)$ represent the cumulative distribution functions (CDFs) of two random variables whose supports are contained in a finite interval denoted $[a, b]$ with no probability mass at point a. This implies that $F(a) = G(a) = 0$ and $F(b) = G(b) = 1$. Let $F^{[1]}(x) = F(x)$ and $F^{[k]}(x) = \int_a^x F^{[k-1]}(y)dy$ for any integer $k \geq 2$. Similar notation applies to $G(x)$. Ekern (1980) gives the following definition.

Definition 3. *(Ekern 1980) For any integer* $n \geq 1$*,* $G(x)$ *has more* n*th-degree risk than* $F(x)$ *if*

(i) $G^{[k]}(b) = F^{[k]}(b)$ *for* $k = 1, 2, ..., n,$ *and*

(ii) $G^{[n]}(x) \geq F^{[n]}(x)$ *for all* $x \in [a, b]$ *with "*$>$*" holding for some* $x \in (a, b)$*.*

Condition (i) guarantees that the first $n - 1$ moments of $F(x)$ and $G(x)$ are held the same across the two distributions, and conditions (i) and (ii) together imply that $F(x)$ dominates $G(x)$ in nth-degree stochastic dominance. Thus, the nth-degree risk increase is a special case of nth-degree stochastic dominance in which the first $n - 1$ moments are kept the same. Also note that an increase in 1st-degree risk is equivalent to a first-degree stochastically dominated shift, that an increase in 2nd-degree risk is equivalent to a sequence of mean-preserving spreads of Rothschild and Stiglitz (1970), and that an increase in 3rd-degree risk is equivalent to a downside risk increase of Menezes et al. (1980).

Recent experimental studies have demonstrated a salient aversion to risk increases of 3rd and even higher degrees.[3] Formally, for a preference ordering denoted $\succeq$ over CDFs,

[3]For example, see Deck and Schlesinger (2010, 2014), Ebert and Wiesen (2011), Noussair et al. (2014), Grossman and Eckel (2015), Heinrich and Mayrhofer (2018).

the definition of (strict) nth-degree risk aversion, where $n \geq 2$, is given below.

Definition 4. *(Ekern 1980) Preferences are said to satisfy nth-degree risk aversion if $F(x) \succ G(x)$ for all $F(x)$ and $G(x)$ such that $G(x)$ has more nth-degree risk than $F(x)$.*

Under expected utility theory, we assume throughout that u is n-times differentiable and u' is strictly positive. We use $u^{(n)}$ to denote nth derivative of u. Key findings in Ekern (1980) and Eeckhoudt and Schlesinger (2006) can be summarized in the following theorem.

Theorem 1. *(Ekern 1980; Eeckhoudt and Schlesinger 2006) In an expected-utility framework with n-times differentiable u. All of these statements are equivalent:*

(i) *u displays nth-degree risk apportionment.*

(ii) *u exhibits nth-degree risk aversion.*

(iii) $(-1)^{n-1}u^{(n)}(x) > 0$ *for all* $x \in [a, b]$.

According to Theorem 1, nth-degree risk apportionment and nth-degree risk aversion are equivalent when the preferences of a decision maker have an expected utility representation. Therefore, the comparative nth-degree risk apportionment analysis in this paper can be reinterpreted as a comparative nth-degree risk aversion analysis, thereby complementing Liu and Neilson's (2018) analysis of the alternative approaches to comparative nth-degree risk aversion. On the other hand, nth-degree risk apportionment being equivalent to nth-degree risk aversion does not necessarily imply that the intensity measure for the former must be the same as that for the latter. For example, both prudence (preferences displaying a tendency for precautionary saving) and downside risk aversion (preferences preferring a pure risk at a higher wealth level to the same amount of risk at a lower wealth level) are characterized by a positive third derivative of the utility function in the expected-utility framework. However, the measure for the strength of prudence (Kimball 1990) is quite different from the measure for downside risk aversion (Crainich and Eeckhoudt 2008; Modica and Scarsini 2005). This discrepancy between the "direction" and "intensity" of risk preferences is emphasized by Eeckhoudt (2012).

3 General Results

A main advantage of the risk apportionment approach to describing risk preferences is that the idea of a preference for disaggregating bads can be easily applied to risk increases of any degrees, as shown in Eeckhoudt et al. (2009). In Definition 1, the difference between lotteries A_n and B_n is the location of $\tilde{\epsilon}_n$ which is either added to the better outcome B_{n-2} in B_n or the worse outcome A_{n-2} in A_n. In this section, we follow the general definition of nth-degree risk apportionment proposed in Eeckhoudt et al. (2009).

3.1 General definition of nth-degree risk apportionment

Based on Eeckhoudt et al. (2009), we define lotteries A and B as follows.

Definition 5. *For $n > m \geq 1$,*

$$A = [\tilde{y}_{n-m} + \tilde{y}_m, 1/2; \tilde{x}_{n-m} + \tilde{x}_m, 1/2]$$
$$B = [\tilde{y}_{n-m} + \tilde{x}_m, 1/2; \tilde{x}_{n-m} + \tilde{y}_m, 1/2]$$

are compound binary lotteries such that

(i) $\tilde{x}_{n-m}$ and $\tilde{y}_{n-m}$ are independent of $\tilde{x}_m$ and $\tilde{y}_m$, and

(ii) $\tilde{y}_i$ has more ith-degree risk than $\tilde{x}_i$ for $i = m, n-m$.

If the preference ordering satisfies both mth- and $(n-m)$th-degree risk aversion, we have

$$\tilde{x}_{n-m} + \tilde{x}_m \succ \left\{ \begin{array}{c} \tilde{y}_{n-m} + \tilde{x}_m \\ \\ \tilde{x}_{n-m} + \tilde{y}_m \end{array} \right\} \succ \tilde{y}_{n-m} + \tilde{y}_m.$$

Note that the two inner risks in the above rankings (i.e., $\tilde{y}_{n-m} + \tilde{x}_m$ and $\tilde{x}_{n-m} + \tilde{y}_m$) cannot be ranked without further information. Eeckhoudt et al. (2009) point out that for a decision maker to prefer B to A, he must prefer the 50-50 lottery of two "inner risks" to the 50-50 lottery of two "outer risks."[4] They further prove the following important result.[5]

Theorem 2. *(Eeckhoudt et al. 2009) Given Definition 5, A has more nth-degree risk than B.*

Using the lotteries A and B described in Definition 5, we now give a more general definition of nth-degree risk apportionment than Definition 2. See Figure 2 for an illustration.

Definition 6. *For any integer $n \geq 2$, preferences are said to satisfy nth-degree risk apportionment if $A \succ B$ for all A and B given in Definition 5 and for every positive integer $m < n$.*

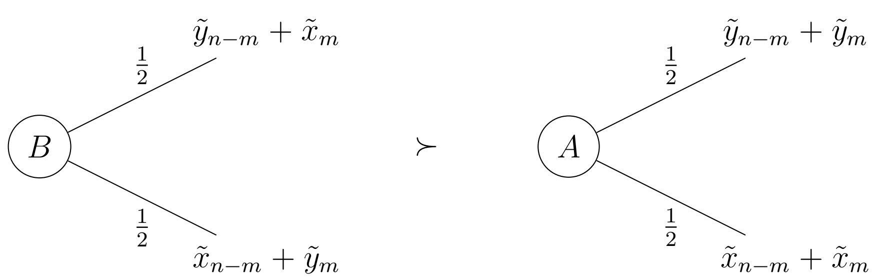


Figure 2: Preference for nth-degree risk apportionment

The preference relation defining the nth-degree risk apportionment is depicted in Figure 2. Obviously, this preference relation does not hinge on the existence of expected-utility

[4]The terminology is due to Menezes and Wang (2005).

[5]Eeckhoudt et al. (2009) present theorems both for the case where the relatively bad is an nth-degree risk increase from the relatively good, and for the case where the relatively bad is nth-degree stochastically dominated by the relatively good. For the purpose of the present paper, we only need to consider the case of risk increases.

representation. Nevertheless, when the decision maker's preferences satisfy the expected-utility axioms and can be represented by a utility function $u(x)$, the general notion of nth-degree risk apportionment is also characterized by $(-1)^{n-1}u^{(n)}(x) > 0$ for all $x \in [a, b]$. Moreover, the preference relation $B \succ A$ is equivalent to the following inequality:

$$Eu(\tilde{y}_{n-m} + \tilde{x}_m) - Eu(\tilde{y}_{n-m} + \tilde{y}_m) > Eu(\tilde{x}_{n-m} + \tilde{x}_m) - Eu(\tilde{x}_{n-m} + \tilde{y}_m). \tag{1}$$

Both sides in the above inequality represent the utility loss from an mth-degree risk increase (i.e., from $\tilde{x}_m$ to $\tilde{y}_m$), with the LHS being associated with $\tilde{y}_{n-m}$ and the RHS with $\tilde{x}_{n-m}$. Therefore, the above inequality states that the pain from an mth-degree risk increase in one asset component increases as the other asset component undergoes an $(n-m)$th-degree risk increase (i.e., from $\tilde{x}_{n-m}$ to $\tilde{y}_{n-m}$). See Denuit and Rey (2010) and Ebert et al. (2018).

Also note that, for $n \geq 3$, we can obtain lotteries A_n and B_n in Definition 1 by letting $\tilde{x}_m = 0$, $\tilde{y}_m = \tilde{\epsilon}_n$, $\tilde{x}_{n-m} = B_{n-2}$, and $\tilde{y}_{n-m} = A_{n-2}$. Thus, the lottery pair used to show nth-degree risk apportionment in Eeckhoudt and Schlesinger (2006) is a special case of A and B in Definition 5 with $m = 2$.

3.2 Probability premium measures for nth-degree risk apportionment

Using the concept of nth-degree risk apportionment given in Definition 6, we are now ready to define "(n/m)th-degree probability premium" to measure the intensity of preference for nth-degree risk apportionment illustrated in Figure 2. Like Definition 6, the following definition of $p_{n/m}$, which is illustrated in Figure 3, does not rely on the existence of expected utility representation of the preferences.

Definition 7. *Given*

$$A' = [\tilde{y}_{n-m} + \tilde{y}_m, 1/2 - p_{n/m}; \tilde{x}_{n-m} + \tilde{x}_m, 1/2 + p_{n/m}]$$
$$B' = [\tilde{y}_{n-m} + \tilde{x}_m, 1/2 - p_{n/m}; \tilde{x}_{n-m} + \tilde{y}_m, 1/2 + p_{n/m}],$$

a decision maker's (n/m)th-degree probability premium is $p_{n/m}$ such that $A' \sim B'$.

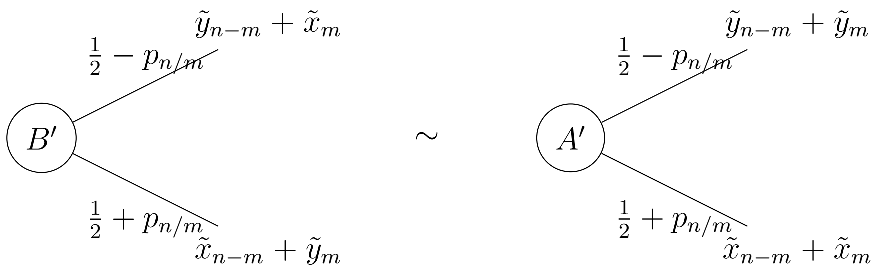


Figure 3: Probability premium $p_{n/m}$

Under expected utility theory, we can derive u's probability premium given an initial wealth w as follows.

$$\left(\frac{1}{2} - p_{n/m}\right) Eu(w + \tilde{y}_{n-m} + \tilde{y}_m) + \left(\frac{1}{2} + p_{n/m}\right) Eu(w + \tilde{x}_{n-m} + \tilde{x}_m)$$
$$= \left(\frac{1}{2} - p_{n/m}\right) Eu(w + \tilde{y}_{n-m} + \tilde{x}_m) + \left(\frac{1}{2} + p_{n/m}\right) Eu(w + \tilde{x}_{n-m} + \tilde{y}_m) \quad (2)$$

Let $\tilde{z}_m = \tilde{y}_m - \tilde{x}_m$. We define utility premium of $\tilde{z}_m$ given random initial wealth $\tilde{w}$ as

$$\Delta_{\tilde{z}_m}(\tilde{w}) = Eu(\tilde{w}) - Eu(\tilde{w} + \tilde{z}_m). \quad (3)$$

Using (2) and (3), we can write $p_{n/m}$ as

$$p_{n/m} = \frac{1}{2}\left[\frac{\Delta_{\tilde{z}_m}(w + \tilde{y}_{n-m} + \tilde{x}_m) - \Delta_{\tilde{z}_m}(w + \tilde{x}_{n-m} + \tilde{x}_m)}{\Delta_{\tilde{z}_m}(w + \tilde{y}_{n-m} + \tilde{x}_m) + \Delta_{\tilde{z}_m}(w + \tilde{x}_{n-m} + \tilde{x}_m)}\right] \quad (4)$$

and find that it is positive for any decision maker who is nth- and mth-degree risk averse.

Theorem 3. *If the decision maker is nth- and mth-degree risk averse, then $0 < p_{n/m} < 1/2$.*

Proof. See Appendix A.

Consider mixed risk averters defined by Cabellé and Pomansky (1996) as our special case. If the decision maker is mixed risk averse, then he is both mth- and nth-degree risk averse, and hence $0 < p_{n/m} < 1/2$ for every pair of A and B satisfying Definition 5, for all n and m such that $n > m \geq 1$. It is also possible for mixed risk lovers defined by Crainich et al. (2013) to have a positive $p_{n/m}$ since they are nth-degree risk averse for an odd n. So if both m and n are odd, a mixed risk lover will be both mth- and nth-degree risk averse, and his corresponding $p_{n/m}$ will be positive.

3.3 Comparative risk apportionment

In this section, we study the relationship between the interpersonal comparison of our proposed strength measure of nth-degree risk apportionment, namely the (n/m)th-degree probability premium, and two related concepts of comparative nth-degree risk aversion under expected utility theory. First, we introduce the two generalized concepts of comparative nth-degree risk aversion defined by Liu and Meyer (2013).

Definition 8. *(Liu and Meyer 2013) For $n > m \geq 1$, u is (n/m)th-degree Arrow-Pratt more risk averse than v on $[a, b]$ if*

$$\frac{(-1)^{(n-1)}u^{(n)}(x)}{(-1)^{(m-1)}u^{(m)}(x)} \geq \frac{(-1)^{(n-1)}v^{(n)}(x)}{(-1)^{(m-1)}v^{(m)}(x)} \tag{5}$$

for all $x \in [a, b]$.

Definition 9. *(Liu and Meyer 2013) For $n > m \geq 1$, u is (n/m)th-degree Ross more risk averse than v on $[a, b]$ if*

$$\frac{(-1)^{(n-1)}u^{(n)}(x)}{(-1)^{(m-1)}u^{(m)}(y)} \geq \frac{(-1)^{(n-1)}v^{(n)}(x)}{(-1)^{(m-1)}v^{(m)}(y)} \tag{6}$$

for all $x, y \in [a, b]$.

By choosing $y = x$ in Definition 9, it follows immediately that (n/m)th-degree Ross more risk averse is a stronger condition than (n/m)th-degree Arrow-Pratt more risk averse.[6] The theorem below states how the interpersonal comparison of our proposed measure of the strength of nth-degree risk apportionment–the (n/m)th-degree probability premium–is related to the above two notions of (n/m)th-degree more risk averse.

Theorem 4. *Let $p^u_{n/m}$ and $p^v_{n/m}$ be (n/m)th-degree probability premiums for decision makers u and v respectively. If both are mth- and nth-degree risk averse, then statements (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).*

(i) u is (n/m)th-degree Ross more risk averse than v.

(ii) $p^u_{n/m} \geq p^v_{n/m}$ for all w, $\tilde{x}_m$, $\tilde{y}_m$, $\tilde{x}_{n-m}$, $\tilde{y}_{n-m}$.

(iii) u is (n/m)th-degree Arrow-Pratt more risk averse than v.

Proof. See Appendix B.

[6] The notion of (n/m)th-degree Arrow-Pratt more risk averse given by Definition 8 includes many lower-degree versions as special cases: Arrow (1971) and Pratt (1964) for $n = 2$ and $m = 1$, Kimball (1990) and Chiu (2005) for $n = 3$ and $m = 2$, Crainich and Eeckhoudt (2008) for $n = 3$ and $m = 1$, and Crainich and Eeckhoudt (2011) for $n = 4$ and $m = 1, 2, 3$. Similarly, the notion of (n/m)th-degree Ross more risk averse given by Definition 9 also includes many lower-degree versions as special cases: Ross (1981) and Machina and Neilson (1987) for $n = 2$ and $m = 1$, Modica and Scarsini (2005) for $n = 3$ and $m = 1$, and Jindapon and Neilson (2007), Li (2009) and Denuit and Eeckhoudt (2010) for $n \geq 2$ and $m = 1$.

4 Special Cases of Comparative Risk Apportionment

The previous section presents our general results on comparing the strength of desire for nth-degree risk apportionment between two individuals. In this section, we focus on two specific cases. First, we consider Eeckhoudt and Schlesinger's (2006) original concept of nth-degree risk apportionment. Then, we propose an alternative lottery pair that also represents nth-degree risk apportionment but can be used for comparing the strength of desire for nth-degree risk apportionment given a larger class of utility functions.

4.1 Comparative nth-degree risk apportionment à la Eeckhoudt and Schlesinger (2006)

Since the lottery pair in Eeckhoudt and Schlesinger (2006) is a special case of the lottery pair in Definition (5) with $m = 2$, we can immediately use $p_{n/2}$ to measure the strength of desire for nth-degree risk apportionment. Specifically, by letting $\tilde{x}_m = 0$, $\tilde{y}_m = \tilde{\epsilon}_n$, $\tilde{x}_{n-m} = B_{n-2}$, and $\tilde{y}_{n-m} = A_{n-2}$, we can obtain the lottery pair given in Definition (1) and define the corresponding probability premium for nth-degree risk apportionment as follows.

Definition 10. *Given*

$$A'_n = [A_{n-2} + \tilde{\epsilon}_n, 1/2 - q_{n/2}; B_{n-2} + 0, 1/2 + q_{n/2}]$$
$$B'_n = [A_{n-2} + 0, 1/2 - q_{n/2}; B_{n-2} + \tilde{\epsilon}_n, 1/2 + q_{n/2}]$$

for $n \geq 3$, a decision maker's probability premium for nth-degree risk apportionment is $q_{n/2}$ such that $A'_n \sim B'_n$.

Thus, the probability premium for nth-degree risk apportionment, $q_{n/2}$, corresponding to Eeckhoudt and Schlesinger's definition, is a special case of our $p_{n/2}$ given in Definition

(10). See Figure 4. A decision maker with a larger $q_{n/2}$ is said to demonstrate a greater mutual aggravation from $\tilde{\epsilon}_n$ in nth-degree risk apportionment (or a stronger desire to put $\tilde{\epsilon}_n$ in its proper place).

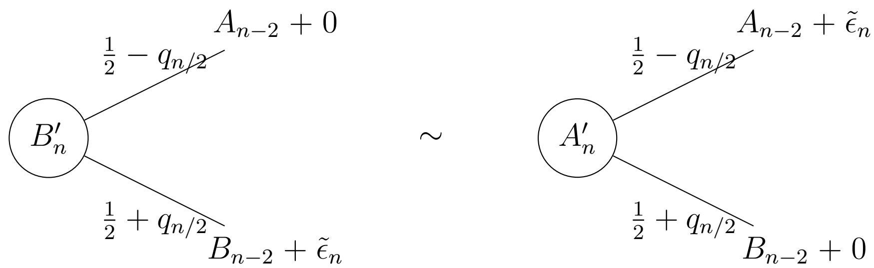


Figure 4: Probability premium $q_{n/2}$

Under expected utility theory, we have

$$q_{n/2} = \frac{1}{2}\left[\frac{\Delta_{\tilde{\epsilon}_n}(w + A_{n-2}) - \Delta_{\tilde{\epsilon}_n}(w + B_{n-2})}{\Delta_{\tilde{\epsilon}_n}(w + A_{n-2}) + \Delta_{\tilde{\epsilon}_n}(w + B_{n-2})}\right]. \tag{7}$$

Based on Theorem (4), we can immediately state the following result.

Corollary 1. *Suppose that u and v are risk averse decision makers who are also nth-degree risk averse. Then, both $q^u_{n/2}$ and $q^v_{n/2}$ are strictly positive and statements (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) for $n \geq 3$.*

(i) $\frac{(-1)^n u^{(n)}(x)}{u''(y)} \geq \frac{(-1)^n v^{(n)}(x)}{v''(y)}$ *for all $x, y \in [a, b]$.*

(ii) $q^u_{n/2} \geq q^v_{n/2}$ *for any w, $\tilde{\epsilon}_n$, A_{n-2}, and B_{n-2} defined in Definition 1.*

(iii) $\frac{(-1)^n u^{(n)}(x)}{u''(x)} \geq \frac{(-1)^n v^{(n)}(x)}{v''(x)}$ *for all $x \in [a, b]$.*

4.2 An alternative form of nth-Degree risk apportionment

Using Eeckhoudt and Schlesinger's (2006) lotteries, we can quantify the intensity of nth degree risk apportionment from the probability premium $q_{n/2}$. This measure represents the desire to mix $\tilde{\epsilon}$ which is bad with a good lottery B_{n-2} instead of a bad lottery A_{n-2}. This concept relies on the fact that decision makers are risk averse so that $\tilde{\epsilon}$ is not desirable and the corresponding probability premium will be positive. In this section, we construct another lottery pair so we can analyze the desire for mixing bad with good without the assumption of risk aversion by letting $\tilde{x}_m = 0$ and $\tilde{y}_m = -k$. Since we consider mutual aggravation from $-k$ instead of $\tilde{\epsilon}$ as in the Eeckhoudt and Schlesinger's environment, our comparison of probability premiums can be applied to any increasing utility functions. Therefore, we are allowed to compare the desire for nth degree risk apportionment between two decision makers even when they are not risk averse.

Definition 11. *Let $k_1, ..., k_n$ be strictly positive real numbers and $\tilde{\epsilon}$ be a zero-mean nondegenerate random variable. We define $\hat{B}_1 = \hat{B}_2 = 0$, $\hat{A}_1 = -k_1$, and $\hat{A}_2 = \tilde{\epsilon}$. For $n \geq 3$, we define*

$$\hat{A}_n = [\hat{A}_{n-1} - k_n, 1/2; \hat{B}_{n-1} + 0, 1/2]$$
$$\hat{B}_n = [\hat{A}_{n-1} + 0, 1/2; \hat{B}_{n-1} - k_n, 1/2]$$

Based on the above definition, if the decision maker is nth-degree risk averse, then $\hat{B}_n \succ \hat{A}_n$. We provide another definition of nth-degree probability premium based on this alternative lottery pair. See Figure 5.

Definition 12. *Given*

$$A''_n = [\hat{A}_{n-1} - k_n, 1/2 - q_{n/1}; \hat{B}_{n-1} + 0, 1/2 + q_{n/1}]$$
$$B''_n = [\hat{A}_{n-1} + 0, 1/2 - q_{n/1}; \hat{B}_{n-1} - k_n, 1/2 + q_{n/1}]$$

for $n \geq 3$, a decision maker's probability premium for nth-degree risk apportionment is $q_{n/1}$ such that $A''_n \sim B''_n$.

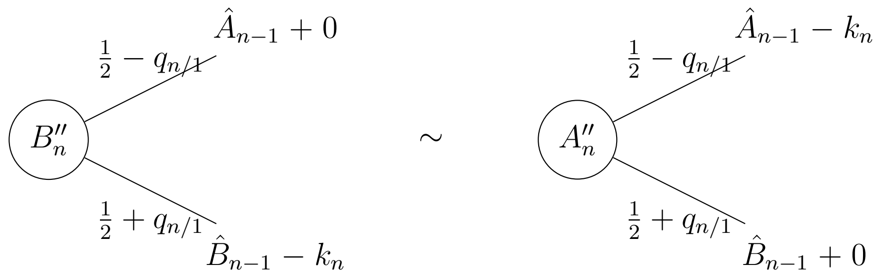


Figure 5: Probability premium $q_{n/1}$

Under expected utility theory, we have

$$q_{n/1} = \frac{1}{2}\left[\frac{\Delta_{-k_n}(w + \hat{A}_{n-1}) - \Delta_{-k_n}(w + \hat{B}_{n-1})}{\Delta_{-k_n}(w + \hat{A}_{n-1}) + \Delta_{-k_n}(w + \hat{B}_{n-1})}\right]. \tag{8}$$

Based on Theorem (4), we can immediately state the following result.

Corollary 2. *Suppose that u and v are nth-degree risk averse. Then, both $q^u_{n/1}$ and $q^v_{n/1}$ are strictly positive and statements (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) for $n \geq 3$.*

(i) $\frac{(-1)^{n-1}u^{(n)}(x)}{u'(y)} \geq \frac{(-1)^{n-1}v^{(n)}(x)}{v'(y)}$ *for all $x, y \in [a, b]$.*

(ii) $q^u_{n/1} \geq q^v_{n/1}$ *for any w, k_n, $\hat{A}_{n-1}$, and $\hat{B}_{n-1}$ defined in Definition 11.*

(iii) $\frac{(-1)^{n-1}u^{(n)}(x)}{u'(x)} \geq \frac{(-1)^{n-1}v^{(n)}(x)}{v'(x)}$ *for all $x \in [a, b]$.*

Jindapon and Neilson (2007) call the sufficient condition in part (i) of Corollary (2) u being nth-degree Ross more risk averse than v and use it to compare optimal choices of costly nth-degree risk reduction between two decision makers.

5 Prudence Probability Premiums

The focus of this section is a preference for 3rd-degree risk apportionment which is also known as prudence and downside risk aversion. A prudent decision maker prefers $B_3 =$

$[-k, 1/2; \tilde{\epsilon}, 1/2]$ to $A_3 = [-k + \tilde{\epsilon}, 1/2; 0, 1/2]$ as illustrated in Figure 1 and previously discussed in Section 2. Using Definitions 12 and 10, we obtain two prudence probability premiums $q_{3/1}$ and $q_{3/2}$ which can be illustrated in Figures 6 (a) and (b), and expressed under expected utility theory as

$$q_{3/1} = \frac{1}{2}\left[\frac{\Delta_{-k}(w+\tilde{\epsilon}) - \Delta_{-k}(w)}{\Delta_{-k}(w+\tilde{\epsilon}) + \Delta_{-k}(w)}\right] \tag{9}$$

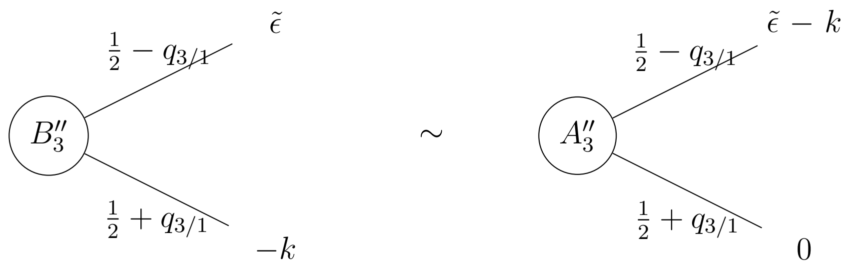
$$q_{3/2} = \frac{1}{2}\left[\frac{\Delta_{\tilde{\epsilon}}(w-k) - \Delta_{\tilde{\epsilon}}(w)}{\Delta_{\tilde{\epsilon}}(w-k) + \Delta_{\tilde{\epsilon}}(w)}\right] \tag{10}$$

respectively.

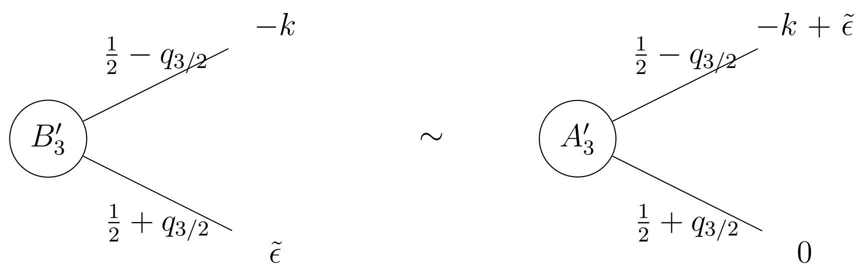
In the literature, two studies have previously used a probability premium to measure the strength of prudence or downside risk aversion; Watt (2011) proposed a probability premium concept similar to $q_{3/2}$ while Jindapon's (2010) probability premium is defined slightly different. Specifically, Jindapon's prudence probability premium is r_3 such that $A_3^* \sim B_3$ where

$$A_3^* = [-k + \tilde{\epsilon}, 1/2 - r_3; 0, 1/2 + r_3].$$

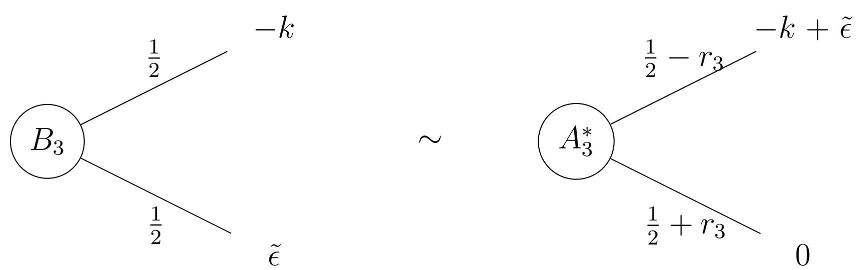
See Figure 6 (c) for an illustration of r_3. The key difference between r_3 and the first two probability premiums is that the probability of each state in B_3 that we use to derive r_3 is unchanged. To compare prudence probability premiums between two expected-utility maximizers, Jindapon (2010) identifies a sufficient condition for $r_3^u > r_3^v$. However, its application is quite limited, because his sufficient condition depends not only on the utility functions, but also on $\tilde{\epsilon}$. Watt's (2011) sufficient condition for $q_{3/2}^u > q_{3/2}^v$ is considered incomplete for the same reason. Based on our results from the previous section, we can provide a sufficient condition for comparing each probability premium concept without a restriction on $\tilde{\epsilon}$.



(a) Prudence probability premium $q_{3/1}$



(b) Prudence probability premium $q_{3/2}$



(c) Prudence probability premium r_3

Figure 6: Various concepts of prudence probability premium

Under expected utility theory, we have

$$r_3 = \frac{1}{2}\left[\frac{\Delta_{\tilde{\epsilon}}(w-k) - \Delta_{\tilde{\epsilon}}(w)}{u(w) - Eu(w-k+\tilde{\epsilon})}\right] = \frac{1}{2}\left[\frac{\Delta_{\tilde{\epsilon}}(w-k) - \Delta_{\tilde{\epsilon}}(w)}{\Delta_{-k}(w) + \Delta_{\tilde{\epsilon}}(w-k)}\right]. \tag{11}$$

Consider the ratio inside the last brackets. Each of these conditions, $u'(x) > 0$, $u''(x) < 0$, and $u'''(x) > 0$, implies $\Delta_{-k}(w) > 0$, $\Delta_{\tilde{\epsilon}}(w-k) > 0$, and $\Delta_{\tilde{\epsilon}}(w-k) - \Delta_{\tilde{\epsilon}}(w) > 0$, respectively. Thus, r_3 is positive for any prudent risk averter. Following the proof of Theorem 4, we can derive a sufficient condition for $r_3^u \geq r_3^v$ given any $\tilde{\epsilon}$. We summarize sufficient conditions for comparing prudence probability premiums between two decision makers as follows.

Corollary 3. *Suppose that u and v are prudent. Then, $q^i_{3/1}, q^i_{3/2}, r^i_3 > 0$ for $i = u, v$ and any given w, k, and $\tilde{\epsilon}$.*

(i) If $\frac{u'''(x)}{u'(y)} \geq \frac{v'''(x)}{v'(y)}$ for all $x, y \in [a, b]$, then $q^u_{3/1} \geq q^v_{3/1}$ for all w, k, and $\tilde{\epsilon}$.

(ii) If both u and v are risk averse and $-\frac{u'''(x)}{u''(y)} \geq -\frac{v'''(x)}{v''(y)}$ for all $x, y \in [a, b]$, then $q^u_{3/2} \geq q^v_{3/2}$ for all w, k, and $\tilde{\epsilon}$.

(iii) If both u and v are risk averse, $\frac{u'''(x)}{u'(y)} \geq \frac{v'''(x)}{v'(y)}$, and $-\frac{u'''(x)}{u''(y)} \geq -\frac{v'''(x)}{v''(y)}$ for all $x, y \in [a, b]$, then $r^u_3 \geq r^v_3$ for all w, k, and $\tilde{\epsilon}$.

As discussed in the previous section, the comparison between $q^u_{3/1}$ and $q^v_{3/1}$ does not need both agents to be risk averse. The sufficient condition in Corollary 3 (*i*), i.e., more (3/1)th-degree Ross more risk averse, is actually equivalent to more strongly downside risk averse defined by Modica and Scarsini (2005). Parts (*ii*) and (*iii*) provide sufficient conditions for comparing Watt's and Jindapon's probability premiums respectively. Based on the derivation of each probability premium concept in (9), (10), and (11), we can see how we obtain such sufficient conditions. Specifically, $q_{3/1}$ is $\frac{1}{2}$ times the ratio of the utility premium of a third-degree risk increase to the utility premium of a first-degree risk increase; $q_{3/2}$ is $\frac{1}{2}$ times the ratio of the utility premium of a third-degree risk increase to the utility premium of a second-degree risk increase; and r_3 is $\frac{1}{2}$ times the ratio of the utility premium of a third-degree risk increase to the utility premium of a second-degree stochastically dominated change (of which both the first-degree risk increase and the second-degree risk increase are special cases).[7]

6 Conclusion and Discussion

In this paper, we propose a concept of probability premium that can be used to compare the strength of preference for nth-degree risk apportionment between two individuals. Specifically, we define (n/m)th-degree probability premium, denoted by $p_{n/m}$, and prove that, under expected utility theory, the (n/m)th-degree Ross more risk aversion of Liu and Meyer (2013) is a sufficient condition for comparative nth-degree risk apportionment, whereas the corresponding (n/m)th-degree Arrow-Pratt more risk aversion is a necessary condition.

While there are $n-1$ ways to measure the strength of nth-degree risk apportionment by using probability premiums $p_{n/m}$ where $m = 1, 2, ..., n-1$, we can find a measure for nth-degree risk apportionment according to Eeckhoudt and Schlesinger (2006) by choosing $m = 2$. Specifically for Eeckhoudt and Schlesinger's framework, we define $q_{n/2}$ as a measure of preference for nth-degree risk apportionment, for example, we can use $q_{3/2}$ as a measure of prudence and $q_{4/2}$ as a measure of temperance. As Eeckhoudt and Schlesinger (2006) point out, the preference relation between two 50-50 lotteries—which involve two independent risks—that they use to define temperance can also be used to define proper risk aversion (Pratt and Zeckhauser 1987), risk vulnerability (Gollier and Pratt 1996) and standard risk aversion (Kimball 1993), as long as the risks in the 50-50 lotteries are given appropriate

[7] In general, the utility premium refers to the reduction in expected utility caused by a change in the random wealth variable. While it has been long-recognized that the utility premium is not interpersonally comparable, the ratio of two utility premiums is. See, for example, Crainich and Eeckhoudt (2008), Eeckhoudt and Schlesinger (2009), Denuit and Rey (2010), Menegatti (2011), and Li and Liu (2014).

reinterpretations. Therefore, the way we define the $q_{4/2}$ measure for temperance can be used to formulate measures of proper risk aversion, risk vulnerability, and standard risk aversion. Such extensions/applications of our comparative risk apportionment approach are left for future research.

References

[1] Arrow, K. J. (1971), Essays in the Theory of Risk-Bearing, Markham: Chicago, IL.

[2] Bernoulli, D. (1954), Exposition of a new theory on the measurement of risk, Econometrica 22, 23-36.

[3] Cabellé, J., A. Pomansky (1996). Mixed risk aversion. Journal of Economic Theory 71, 485-513.

[4] Chiu, W. H. (2005). Degree of downside risk aversion and self-protection. Insurance: Mathematics and Economics 36, 93-101.

[5] Chiu, W. H., L. Eeckhoudt, B. Rey (2012). On relative and partial risk attitudes: theory and implications. Economic Theory 50, 151-167.

[6] Courbage, C., H. Loubergé, B. Rey (2018). On the properties of high-order non-monetary measures for risks, Geneva Risk and Insurance Review 43, 77-94.

[7] Crainich, D., L. Eeckhoudt (2008). On the intensity of downside risk aversion. Journal of Risk and Uncertainty 36, 267-276.

[8] Crainich, D., L. Eeckhoudt (2011). Three measures of the intensity of temperance, Manuscript.

[9] Crainich, D., L. Eeckhoudt, A. Trannoy (2013). Even (mixed) risk lovers are prudent. American Economic Review 103, 1529-1535.

[10] Deck, C., H. Schlesinger (2010). Exploring higher-order risk effects. Review of Economic Studies 77, 1403-1420.

[11] Deck, C., H. Schlesinger (2014). Consistency of higher order risk preferences. Econometrica 82, 1913-1943.

[12] Denuit, M., B. Rey (2010). Prudence, temperance, edginess, and risk apportionment as decreasing sensitivity to detrimental changes. Mathematical Social Sciences 60, 137-143.

[13] Denuit, M., B. Rey (2013). Another look at risk apportionment. Journal of Mathematical Economics 49, 335-343.

[14] Denuit, M., L. Eeckhoudt (2010). Stronger measures of higher-order risk attitudes. Journal of Economic Theory 145, 2027-2036.

[15] Ebert, S. (2013). Even (mixed) risk lovers are prudent: Comment. American Economic Review 103, 1536-1537.

[16] Ebert, S., D. C. Nocetti, H. Schlesinger (2018). Greater mutual aggravation. Management Science 64, 2809-2811.

[17] Ebert, S., D. Wiesen (2011). Testing for prudence and skewness seeking. Management Science 57, 1334-1349.

[18] Eeckhoudt, L. (2012). Beyond Risk Aversion: Why, how and what's next? Geneva Risk and Insurance Review 37, 141-155.

[19] Eeckhoudt, L., R. J. A. Laeven, H. Schlesinger (2017). Risk apportionment: The dual story. Manuscript.

[20] Eeckhoudt, L., H. Schlesinger (2006). Putting risk in its proper place. American Economic Review 96, 280-289.

[21] Eeckhoudt, L., H. Schlesinger (2009). On the utility premium of Friedman and Savage. Economics Letters 105, 46-48.

[22] Eeckhoudt, L., H. Schlesinger, and I. Tsetlin (2009). Apportioning of risks via stochastic dominance. Journal of Economic Theory 144, 994-1003.

[23] Ekern, S. (1980). Increasing Nth degree risk. Economics Letters 6, 329-333.

[24] Gollier, C. (2018). A general theory of risk apportionment. Manuscript.

[25] Gollier, C., J. W. Pratt (1996). Risk vulnerability and the tempering effect of background risk. Econometrica 64, 1109-1123.

[26] Grossman, P., C. Eckel (2015). Loving the long shot: risk taking with skewed lotteries. Journal of Risk and Uncertainty 51, 195-217.

[27] Huang, R. J., L. Y. Tzeng, and L. Zhao (2017). Fractional degree stochastic dominance. Manuscript

[28] Heinrich, T., T. Mayrhofer (2018). Higher-order risk preferences in social settings, Experimental Economics 21, 434-456.

[29] Jindapon, P. (2010). Prudence probability premium. Economics Letters 109, 34-37.

[30] Jindapon, P., W. S. Neilson (2007). Higher-order generalizations of Arrow-Pratt and Ross risk aversion: A comparative statics approach. Journal of Economic Theory 136, 719-728.

[31] Jokung, O. (2011). Risk apportionment via bivariate stochastic dominance. Journal of Mathematical Economics 47, 448-452.

[32] Kimball, M. S. (1990). Precautionary saving in the small and in the large. Econometrica 58, 53-73.

[33] Kimball, M. S. (1993). Standard risk aversion. Econometrica 61, 589-611.

[34] Li, J. (2009). Comparative higher-degree Ross risk aversion. Insurance: Mathematics and Economics 45, 333-336.

[35] Li, J., L. Liu (2014). The monetary utility premium and interpersonal comparisons. Economics Letters 125, 257-260.

[36] Liu, L., J. Meyer (2013). Substituting one risk increase for another: A method for measuring risk aversion. Journal of Economic Theory 148, 2706-2718.

[37] Liu, L., W. S. Neilson (2018). Alternative approaches to comparative nth-degree risk aversion. Management Science, forthcoming.

[38] Machina, M.J., W.S. Neilson (1987). The Ross characterization of risk aversion: Strengthening and extension. Econometrica 55, 1139-1149.

[39] Menegatti , M. (2011) The risk premium and the effects of risk on agents utility. Research in Economics 65 (2), 89-94.

[40] Menezes, C., C. Geiss, J. Tressler (1980). Increasing downside risk. American Economic Review 70, 921-932.

[41] Menezes, C., X.H. Wang, Increasing outer risk, Journal of Mathematical Economics 41 (2005) 875-886.

[42] Modica, S., M. Scarsini (2005). A note on comparative downside risk aversion. Journal of Economic Theory 122, 267-271.

[43] Nocetti, D. (2016). Robust comparative statics of risk changes. Management Science 62, 1381-1392.

[44] Noussair, C., S. Trautmann, G. Van De Kuilen (2014). Higher order risk attitudes, demographics, and financial decisions. Review of Economic Studies 81, 325-355.

[45] Pratt, J. W. (1964). Risk aversion in the small and in the large. Econometrica 32, 122-136.

[46] Pratt, J. W., Zeckhouser, R. J. (1987). Proper risk aversion. Econometrica 55, 143-154.

[47] Ross, S. A. (1981). Some stronger measures of risk aversion in the small and in the large with applications. Econometrica 49, 621-663.

[48] Rothschild, M., J. Stiglitz (1970). Increasing risk I: A definition. Journal of Economic Theory 2, 225-243.

[49] Tsetlin, I., R. L. Winkler (2009). Multiattribute utility satisfying a preference for combining good with bad. Management Science 55, 1942-1952.

[50] Tsetlin, I, and R. L. Winkler (2017). Multivariate almost stochastic dominance. Journal of Risk and Insurance, forthcoming

[51] Wang, J., J. Li (2010). Multiplicative risk apportionment. Mathematical Social Sciences 60, 79-81.

[52] Watt, R. (2011). A note on greater downside risk aversion. Working Paper No. 17/2011, International Center for Economic Research

[53] Watt, R., F. J. Vazquez (2013). Allocative downside risk aversion, International Journal of Economic Theory 9: 267-277.

Appendix A: Proof of Theorem 3

Since u is mth-degree risk averse and $\tilde{x}$ has more mth-degree risk than $\tilde{y}$, then,

$$\Delta^u_{\tilde{z}_m}(w+\tilde{x}_m+\tilde{x}_{n-m}) = Eu(w+\tilde{x}_m+\tilde{x}_{n-m}) - Eu(w+\tilde{y}_m+\tilde{x}_{n-m}) > 0 \tag{12}$$

and

$$\Delta^u_{\tilde{z}_m}(w+\tilde{x}_m+\tilde{y}_{n-m}) = Eu(w+\tilde{x}_m+\tilde{y}_{n-m}) - Eu(w+\tilde{y}_m+\tilde{y}_{n-m}) > 0. \tag{13}$$

Since u is nth-degree risk averse and A has more nth-degree risk than B, then $B \succ A$. Under expected utility theory, we can write

$$\frac{1}{2}[Eu(w+\tilde{x}_m+\tilde{y}_{n-m})+Eu(w+\tilde{y}_m+\tilde{x}_{n-m})] > \frac{1}{2}[Eu(w+\tilde{y}_m+\tilde{y}_{n-m})+Eu(w+\tilde{x}_m+\tilde{x}_{n-m})] \tag{14}$$

which is equivalent to

$$\Delta^u_{\tilde{z}_m}(w+\tilde{x}_m+\tilde{y}_{n-m}) - \Delta^u_{\tilde{z}_m}(w+\tilde{x}_m+\tilde{x}_{n-m}) > 0. \tag{15}$$

Given $p_{n/m}$ in (4), we find that (12), (13), and (15) jointly imply $0 < p^u_{n/m} < 1/2$.

Appendix B: Proof of Theorem 4

Part 1. (i) $\Rightarrow$ (ii)

Given $p_{n/m}$ in (4), we can write

$$p^v_{n/m} = \frac{s}{2t} \tag{16}$$

where

$$s = \Delta^v_{\tilde{z}_m}(w+\tilde{x}_m+\tilde{y}_{n-m}) - \Delta^v_{\tilde{z}_m}(w+\tilde{x}_m+\tilde{x}_{n-m}) \tag{17}$$

and

$$t = \Delta^{v}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) + \Delta^{v}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m}). \tag{18}$$

Since v is nth- and mth-degree risk averse, both s and t are positive (see the proof of Theorem 3). Given that u is Ross more risk averse than v, we can write

$$\frac{u^{(n)}(x)}{v^{(n)}(x)} \geq \lambda \geq \frac{u^{(m)}(y)}{v^{(m)}(y)} \tag{19}$$

for all $x, y \in [a, b]$ and some $\lambda > 0$. Liu and Meyer (2013) show that this condition is equivalent to

$$u(x) = \lambda v(x) + \phi(x) \tag{20}$$

where $(-1)^{m-1}\phi^{(m)}(x) \leq 0$ and $(-1)^{n-1}\phi^{(n)}(x) \geq 0$ for all $x \in [a, b]$. By substituting (20) into (4), we can write

$$p^{u}_{n/m} = \frac{1}{2}\left[\frac{\lambda s + \Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) - \Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m})}{\lambda t + \Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) + \Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m})}\right]. \tag{21}$$

It follows from (16) and (21) that $p^{u}_{n/m} \geq p^{v}_{n/m}$ if and only if

$$\begin{aligned} t[\Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) - \Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m})] \geq \\ s[\Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) + \Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m})]. \end{aligned} \tag{22}$$

Since $(-1)^{m-1}\phi^{(m)}(x) \leq 0$ and $(-1)^{n-1}\phi^{(n)}(x) \geq 0$ for all $x \in [a, b]$, then

$$\Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) \leq 0, \tag{23}$$

$$\Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m}) \leq 0, \tag{24}$$

$$\Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) - \Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m}) \geq 0. \tag{25}$$

As a result, the inequality in (22) always holds and, therefore, $p^u_{n/m} \geq p^v_{n/m}$.

Part 2. (ii) ⇒ (iii)

Suppose that (iii) is false, i.e., $u(x)$ is not (n/m)th-degree Arrow-Pratt more risk averse than $v(x)$. Then, there exists $x \in [a, b]$ such that

$$\frac{(-1)^{n-1}u^n(x)}{(-1)^{m-1}u^m(x)} < \frac{(-1)^{n-1}v^n(x)}{(-1)^{m-1}v^m(x)}. \tag{26}$$

Since both u and v are mth- and nth-degree risk averse, the above inequality implies

$$\frac{u^{(n)}(x)}{v^{(n)}(x)} < \frac{u^{(m)}(x)}{v^{(m)}(x)}. \tag{27}$$

Due to continuity, there exists $\mu > 0$ and $[c, d] \subset [a, b]$ such that

$$\frac{u^{(n)}(y)}{v^{(n)}(y)} < \mu < \frac{u^{(m)}(z)}{v^{(m)}(z)} \tag{28}$$

for all $y, z \in [c, d]$. Define $\psi(x) = u(x) - \mu v(x)$. Differentiating yields

$$(-1)^{m-1}\psi^m(x) = (-1)^{m-1}u^m(x) - \mu(-1)^{m-1}v^m(x) > 0 \tag{29}$$

and

$$(-1)^{n-1}\psi^n(x) = (-1)^{n-1}u^n(x) - \mu(-1)^{n-1}v^n(x) < 0 \tag{30}$$

for all $x \in [c, d]$. If we pick w, k, and $\tilde{\epsilon}$ so that the support of all possible levels of final wealth is a subset of $[c, d]$, then we have

$$\Delta^{\psi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) > 0, \tag{31}$$

$$\Delta^{\psi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m}) > 0, \tag{32}$$

$$\Delta^{\psi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{y}_{n-m}) - \Delta^{\phi}_{\tilde{z}_m}(w + \tilde{x}_m + \tilde{x}_{n-m}) < 0. \tag{33}$$

It follows that the inequality in 22 is reversed so that $p^u_{n/m} < p^v_{n/m}$. Therefore, (ii) is false.